SUPPLEMENT TO "TESTING WHETHER JUMPS HAVE FINITE OR INFINITE ACTIVITY"

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This supplement contains the technical results and proofs.

1. Preliminary results. We first show a few facts which have been stated without proof in Section 2 of the paper, like Lemmas 1 and 2 and the inclusion $\Omega_T^{ii} \subset \Omega_T^i$. We first introduce, for each $p \geq 0$, the two increasing processes:

$$H(p)_t \;=\; \int_0^t ds \int (|x|^p \wedge 1)\, F_s(dx), \qquad H'(p)_t \;=\; \sum_{s \leq t} |\Delta X_s|^p \wedge 1. \tag{A.1}$$

When $p \geq 2$ these processes are finite-valued, but not necessarily so when $p < 2$. In this case we have to be careful: if $R = \inf(t : H(p)_t = \infty)$ and $R' = \inf(t : H'(p)_t = \infty)$ then both processes $H(p)$ and $H'(p)$ are null at 0 and finite-valued on $[0, R)$ and $[0, R')$ respectively, the first continuous and the second one càdlàg with jumps not bigger than 1, and they are infinite on (R, ∞) and (R', ∞). However, if $0 < R < \infty$ or $0 < R' < \infty$ we may have $H(p)_R$ or $H'(p)_{R'}$ finite or infinite, and the left limits satisfy $H(p)_{R-} = H(p)_R$ and $H'(p)_{R'-} \leq H'(p)_{R'} \leq H'(p)_{R'-} + 1$.

Note that $H'(0)_t$ is the number of jumps over $[0, t]$, whereas $H(0)_t = \int_0^t F_s(\mathbb{R})\, ds$.

Lemma 3. *For any random time S and any $p \geq 0$, the two sets $\{H(p)_S = \infty\}$ and $\{H'(p)_S = \infty\}$ are a.s. equal.*

Proof. Set $R_n = \inf(t : H(p)_t \geq n)$ and $R'_n = \inf(t : H'(p)_t \geq n)$, and consider the sets $A = \{H(p)_S < \infty\}$ and $A' = \{H'(p)_S < \infty\}$. Observe that $H(p)_{R_n} \leq n$ and $H'(p)_{R'_n} \leq n + 1$, which entail $A = \cup_{n \geq 1} \{S \leq R_n\}$ and $A' = \cup_{n \geq 1} \{S \leq R'_n\}$. Moreover $H(p)$ is the predictable compensator of $H'(p)$, so for any stopping time V we have $\mathbb{E}(H(p)_V) = \mathbb{E}(H'(p)_V)$. We deduce that $H'(p)_{R_n} < \infty$ and $H(p)_{R'_n} < \infty$ a.s., and the almost sure equality $A = A'$ is then obvious. □

Proof of (9). If $A_t = 0$ we have $F_t(\mathbb{R}) \leq L_t \phi(1)$, whereas $F_t(\mathbb{R}) = \infty$ if $A_t > 0$. Hence $\{\overline{A}_T > 0\} = \{H(0)_T = \infty\}$, which is a.s. equal to Ω_T^i by the previous lemma, and (9) follows. □

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Proof of Lemma 1. Denote by Γ'_t the right side of (13). Then obviously

$$\Gamma_t \;=\; \inf(p : H(p)_t < \infty), \qquad \Gamma'_t \;=\; \inf(p : H'(p)_t < \infty).$$

Since by the previous lemma, for any $p \geq 0$ we have $\{H(p)_t < \infty\} = \{H'(p)_t < \infty\}$ a.s., the almost sure equality $\Gamma_t = \Gamma'_t$ is obvious. If $p > \Gamma_t$ we have $\int (|x|^p \wedge 1) F_s(x) < \infty$ for λ-almost all $s \in [0,t]$, hence (14). When the last condition holds, and if p is bigger than the right side of (14), we have $H(p)_t < \infty$, and thus $\Gamma_t \leq p$: hence (14) is an equality. □

Proof of $\Omega_T^{ii} \subset \Omega_T^i$ under Assumption 5. On the set $\Omega_T^{i,\Gamma>0}$ it is obvious that $F_t(\mathbb{R}) = \infty$ for all t smaller than T and in a λ-positive set, so $H(0)_T = \infty$. On the set $\{H(0)_T < \infty\}$, and with the notation $G(p,u)$ of (17), we have

$$\int_0^T G(p,u)_t dt \;\leq\; \int_0^T F_t([-u,u])\, dt,$$

which goes to 0 as $u \to 0$ by the dominated convergence theorem, thus $\{H(0)_T < \infty\} \subset (\Omega_T^{i,\Gamma=0})^c$. So finally $\Omega_T^{ii} \subset \{H(0)_T = \infty\}$, which by Lemma 3 equals Ω_T^i a.s. □

Proof of Lemma 2. We suppose Assumption 3, so we can obviously apply the last part of Lemma 1, plus the fact that the instantaneous BG index of F_t is $\gamma_t\, 1_{\{A_t>0\}}$, to obtain that $\Gamma_t = \lambda-$ ess sup $(\gamma_s : A_s > 0, s \in [0,t])$. A simple calculation shows, for all $w > 0$, $v \in \mathbb{R}$, $x, y \in (0,1]$ and $u \in (x,1]$,

$$\begin{array}{l} \frac{1}{w\, x^w\, (\log(1/x))^{v^-}} \Big(1 - \frac{x^w}{u^w}\Big) \;\leq\; \int_x^u \frac{(\log(1/z))^v}{z^{1+w}}\, dz \;\leq\; \frac{(\log(1/x))^{v^+}}{w\, x^w} \\ \int_x^{x(1+y)} \frac{(\log(1/z))^v}{z^{1+w}}\, dz \;\leq\; Ky \frac{(\log(1/x))^{v^+}}{w\, u^w}, \end{array}$$

whereas if $p > 0$ and $v \geq 0$ we get

$$\begin{array}{l} \int_x^1 \frac{(\log(1/z))^v}{z}\, dz \;\leq\; (\log(1/x))^{v+1} \\ \int_x^{x(1+y)} \frac{(\log(1/z))^v}{z}\, dz \;\leq\; Ky(\log(1/x))^{v+1} \\ \frac{1}{K_p}\, x^p\, (\log(1/x))^v \;\leq\; \int_0^x \frac{x^p\, (\log(1/z))^v}{z}\, dz \;\leq\; K_p\, x^p (\log(1/x))^v. \end{array}$$

Furthermore (8) implies, with $\overline{|F|}'_t$ denoting the tail function of $|F'_t|$, and for $x \in (0,1]$:

$$\overline{|F|}'_t(x) \;\leq\; L_t / x^{c\gamma_t}$$

and also, when $\gamma_t = 0$ or $A_t = 0$, and $y \in (0,1]$,

$$\overline{|F|}'_t(x) - \overline{|F|}'_t(x(1+y)) \;\leq\; L_t \phi(2x), \qquad \int_0^x |z|^p\, |F'_t|(dz) \;\leq\; L_t\, x^p\, \phi(x).$$

If we combine these results, we deduce from (5) and (6) that for $x,y\in(0,1]$ and $p>0$,

(A.2)
$$\begin{cases} x^{\gamma_t}\,\overline{F}_t(x) \;\leq\; A_t(\log(1/x))^{L_t+1}+L_t x^{(1-c)\gamma_t} \\ \overline{F}_t(x)-\overline{F}_t(x(1+y))\leq \begin{cases} \frac{KL_t\,(\log(1/x))^{L_t}}{x^{\gamma_t}}(y+x^{(1-c)\gamma_t}) & \text{if } \gamma_t>0,\ A_t>0 \\ KL_t(\log(1/x))^{\delta+1}(y+\phi(2x)) \text{ otherwise} \end{cases} \\ \gamma_t=0 \text{ or } A_t=0 \;\Rightarrow\; \int_{\{|y|\leq x\}}|y|^p\,F_t(dy) \;\leq\; K_p\,L_t\,x^p(\log(1/x))^\delta \end{cases}$$

$$\text{(A.3)}\quad \begin{cases} \gamma_t>0,\ x\leq 1/2L_t\Rightarrow x^{\gamma_t}\overline{F}_t(x)\geq \frac{A_t(1-2^{-\gamma_t})}{(\log L_t)^{L_t}}-L_t\,x^{(1-c)\gamma_t} \\ \gamma_t=0,\ x\leq 1/L_t\Rightarrow \int_{\{|y|\leq x\}}|y|^p\,F_t(dy)\geq \frac{x^p\,(\log(1/x))^\delta\,A_t}{K_p}\Big(1-\frac{\phi(x)\,L_t}{(\log(1/x))^\delta}\Big). \end{cases}$$

(A.2) implies (16) for suitable processes $L(\varepsilon)$ and $L'(p)$, and $a=(1-c)/2$ and the functions $x\mapsto\phi((2x)\wedge 1)$ and $\psi(x)=(\log(1/x))^\delta$. It remains to prove that $\Omega_T^i=\Omega_T^{ii}$ a.s., which by (9) and $\Omega_T^{ii}\subset\Omega_T^i$ a.s. amounts to $\{\overline{A}_T>0\}\subset\Omega_T^{ii}$ a.s. First, if $\Gamma_T>0$ we have $\overline{A}_T>0$ and for any $a'\in(0,1)$ the set of all $s\in[0,T]$ such that $A_s>0$ and $\gamma_s>a'\,\Gamma_T$ has positive Lebesgue measure. For such an s, we deduce from the first part of (A.3) that $x^{a'\Gamma_T}\,\overline{F}_s(x)\to\infty$ as $x\to 0$, hence $\{\overline{A}_T>0,\Gamma_T>0\}\subset\Omega_T^{i,\Gamma>0}$. On the other hand if $\Gamma_T>0$ and $\overline{A}_T>0$ the set of all $s\in[0,T]$ such that $A_s>0$ and $\gamma_s=0$ has positive Lebesgue measure. For such an s we deduce from the last part of (A.3) that $\liminf_{u\to 0}G(p,u)_s>0$, hence $\{\overline{A}_T>0,\Gamma_T=0\}\subset\Omega_T^{i,\Gamma=0}$, and this ends the proof. $\square$

2. Estimates. Throughout this section, we assume without special mention the following:

Assumption 7. *There are three constants $L>1$, $a\geq 0$ and $H\in(0,2]$, and a function ϕ as in (15), such that with the notation $\phi'(x)=x^{aH}$ when $a\in(0,1]$ and $\phi'(x)=x^H\phi(x)$ when $a=0$, we have for all ω and t:*

$$\text{(A.4)}\qquad \begin{array}{l} |b_t|\;\leq\;L,\qquad |\sigma_t|\;\leq\;L,\qquad \int(x^2\wedge 1)F_t(x)\;\leq\;L \\ x\in(0,1]\;\;\Rightarrow\;\; x^H\,\overline{F}_t(x)\;\leq\;L \\ x,y\in(0,1]\;\;\Rightarrow\;\; \overline{F}_t(x)-\overline{F}_t(x(1+y))\leq\frac{L}{x^H}\Big(y+\phi'(x)\Big). \end{array}$$

A simple calculation shows that we then have (below, K denotes a constant which may change from line to line, and may depend on the bounds for X like L,H,a above and also on the power p used in the statistics, and is denoted K_q if we want to emphasize its dependency on an additional parameter q):

$$\text{(A.5)}\qquad q>H\;\;\Rightarrow\;\;\int_{\{|x|\leq u\}}|x|^q F_t(dx)\;\leq\;K_q\,u^{q-H},$$

(A.6) $$u \in (0,1) \quad \Rightarrow \quad \int_{\{u<|x|\leq 1\}} |x| F_t(dx) \;\leq\; \begin{cases} K u^{1-H} & \text{if } H>1 \\ K\log(1/u) & \text{if } H=1 \\ K & \text{if } H<1. \end{cases}$$

We fix the sequence u_n, subject to (30) for some $\rho \in (0,1/2)$, and we may assume $u_n \leq 1$ since $u_n \to 0$. We also pick $r \in \big(0, \frac{2}{3\rho H} - \frac{2}{3}\big)$, so that

(A.7) $$\eta_n \;=\; \frac{\Delta_n}{u_n^{H(1+r)}} + \frac{\Delta_n^2}{u_n^{H(2+3r)}} \;\to\; 0.$$

Next, we introduce additional notation. We denote by $\mathbb{E}^n_{i-1}$ and $\mathbb{P}^n_{i-1}$, respectively, the conditional expectation and conditional probability with respect to $\mathcal{F}_{(i-1)\Delta_n}$. With any càdlàg process Y and any $\delta \in (0,1)$ we associate the processes

(A.8) $$Y(\delta)_t \;=\; \sum_{s\leq t} \Delta Y_s 1_{\{|\Delta Y_s|>\delta)\}}, \qquad Y'(\delta) \;=\; Y - Y(\delta)$$

and the variables

(A.9) $$\zeta(Y,p)^n_i \;=\; |\Delta^n_i Y|^p\, 1_{\{|\Delta^n_i Y|\leq u_n\}}.$$

We also define the following increasing processes (j is an integer):

(A.10) $$\begin{array}{rcl} D(j,p,n)_t &=& \sum_{s\leq t} |\Delta X_s|^p\, 1_{\{u_n^{1+r}<|\Delta X_s|\leq j u_n\}} \\ D(p,u_n)_t &=& \sum_{s\leq t} |\Delta X_s|^p\, 1_{\{|\Delta X_s|\leq u_n\}}. \end{array}$$

Lemma 4. *If $p>H$ we have*

(A.11) $$\mathbb{E}^n_{i-1}\big(\big|\zeta(X(u_n^{1+r}),p)^n_i - \Delta^n_i D(1,p,n)\big|\big) \;\leq\; K\Delta_n\, u_n^{p-H}\, \eta_n.$$

Proof. 1) The compensator of $D(j,p,n)$ is

$$\widetilde{D}(j,p,n)_t = \int_0^t dr \int_{\{u_n^{1+r}<|x|\leq j u_n\}} |x|^p F_r(dx).$$

Next, (A.5) yields

$$\widetilde{D}(j,p,n)_{t+s} - \widetilde{D}(j,p,n)_t \;\leq\; K\, s\, j^p\, u_n^{p-H}$$

for all $s,t\geq 0$. Hence for any finite stopping time S we have

(A.12) $$\mathbb{E}(D(j,p,n)_{S+s} - D(j,p,n)_S \mid \mathcal{F}_S) \;\leq\; K\, s\, j^p\, u_n^{p-H}.$$

2) The compensator of the process $N^n_t = \sum_{s\leq t} 1_{\{|\Delta X_s|>u_n^{1+r}\}}$ is $\widetilde{N}^n_t = \int_0^t \overline{F}_r(u_n^{1+r})dr$. Let $S^n_0 = R^n_0 = (i-1)\Delta_n$ and $S^n_1, S^n_2, \cdots$ (resp. $R^n_1, \cdots$) be the successive jump

times of $X(u_n^{1+r})$ after time $(i-1)\Delta_n$ (resp., and with jump size bigger than u_n). Using (A.4), we have for $j \geq 1$, and on the set $\{S_{j-1}^n < i\Delta_n\}$:

$$\begin{aligned} \mathbb{P}(S_j^n \leq i\Delta_n \mid \mathcal{F}_{S_{j-1}^n}) &\leq \mathbb{E}\left(N_{i\Delta_n}^n - N_{S_{j-1}^n}^n \mid \mathcal{F}_{S_{j-1}^n}\right) \\ &= \mathbb{E}\left(\widetilde{N}_{i\Delta_n} - \widetilde{N}_{S_{j-1}^n}^n \mid \mathcal{F}_{S_{j-1}^n}\right) \ \leq \ K\Delta_n\, u_n^{-H(1+r)}. \end{aligned} \tag{A.13}$$

Then by induction on j we see that

$$\mathbb{P}_{i-1}^n(S_j^n \leq i\Delta_n) \ \leq \ K^j\,\Delta_n^j\,u_n^{-jH(1+r)}, \tag{A.14}$$

and in the same way,

$$\mathbb{P}_{i-1}^n(R_j^n \leq i\Delta_n) \ \leq \ K^j\,\Delta_n^j\,u_n^{-jH}. \tag{A.15}$$

3) Now we consider the variable $V_n = \zeta(X(u_n^{1+r}),p)_i^n - \Delta_i^n D(1,p,n)$, which we evaluate on the sets $B_j^n = \{S_j^n \leq i\Delta_n < S_{j+1}^n\}$, which form a partition of Ω when $j = 0, 1, \cdots$. First, $V_n = 0$ on the sets B_0^n and B_1^n. Second, on the set B_2^n we have

$$|V_n| \ \leq \ K(\Delta_i^n D(2,p,n) + u_n^p\,1_{\{R_2^n \leq i\Delta_n\}})$$

(because $\Delta_i^n D(1,p,n) = 0$ when $R_2^n \leq i\Delta_n$, whereas $\zeta(X(u_n^{1+r}),p)_i^n = 0$ when $|\Delta X_{S_j^n}| \leq u_n < 2u_n < |\Delta X_{S_k^n}|$ for either $(j,k) = (1,2)$ or $(j,k) = (2,1)$). Therefore

$$\mathbb{E}_{i-1}^n\left(|V_n| 1_{B_2^n}\right) \ \leq \ K\mathbb{E}_{i-1}^n\left(\Delta_i^n D(2,p,n)\,1_{\{S_2^n \leq i\Delta_n\}}\right) + K u_n^p\,\mathbb{P}_{i-1}^n(R_2^n \leq i\Delta_n).$$

The variable under the first conditional expectation in the right is smaller than

$$(D(2,p,n)_{i\Delta_n} - D(2,p,n)_{S_1^n})1_{\{S_1^n \leq i\Delta_n\}} + (D(2,p,n)_{S_i^n} - D(2,p,n)_{(i-1)\Delta_n})1_{\{S_2^n \leq i\Delta_n\}},$$

and by conditioning first w.r.t. $\mathcal{F}_{S_1^n}$ we deduce from (A.12) and (A.13) and (A.14) that its conditional expectation w.r.t. $\mathcal{F}_{(i-1)\Delta_n}$ is smaller than $K\Delta_n^2\,u_n^{p-H(2+r)}$. If we use also (A.15), we arrive at

$$\mathbb{E}_{i-1}^n\left(|V_n| 1_{B_2^n}\right) \ \leq \ K\Delta_n^2\,u_n^{p-H(2+r)}. \tag{A.16}$$

Finally $|V_n| \leq u_n^p(1+j)$ on the set B_j^n, hence by (A.14) we get

$$\sum_{j\geq 3} \mathbb{E}_{i-1}^n(|V_n|\,1_{B_j^n}) \ \leq \ 2u_n^p \sum_{j\geq 3} j\left(K\Delta_n\,u_n^{-H(1+r)}\right)^j \ \leq \ K\Delta_n^3\,u_n^{p-3H(1+r)},$$

by virtue of (A.7) (which yields that for any $K > 0$ we have $K\Delta_n u_n^{-H(1+r)} \leq 1/2$ for all n large enough, whereas $\sum_{j\geq 3} j/2^{j-3} < \infty$). This and (A.16) and $V_n = 0$ on $B_0^n \cup B_1^n$ readily yield (A.11). $\square$

Lemma 5. *We have, for all $w \in (0,1/4)$:*

$$\begin{aligned} &\mathbb{P}^n_{i-1}\big(u_n(1-w) \le |\Delta^n_i X(u_n^{1+r})| \le u_n(1+w)\big) \\ &\le K\Delta_n\, u_n^{-H}\Big(w + \phi'(u_n) + \eta_n + \Delta_n u_n^{-H(1+r)}\big(\tfrac{w}{u_n^r}\big)^{\frac{1}{H+2}}\Big). \end{aligned} \tag{A.17}$$

Proof. For $x \in \mathbb{R}$ we set $A_n(x) = \{y : |y| > u_n^{1+r}, u_n(1-w) \le |x+y| \le u_n(1+w)\}$, and also $A(z,z') = \{y : z \le |y| \le z(1+z')\}$ when $z, z' > 0$. A simple calculation, using the fact that $w \in \big(0, \frac{1}{4}\big)$, shows that for all $\alpha \in (0,1)$:
(A.18)

$$\begin{aligned} &|x| \le \tfrac{u_n}{2} \;\Rightarrow\; A_n(x) \subset A(z_n(x), 8w) \;\text{ for some } z_n(x) \in \big[\tfrac{u_n}{4}, 2u_n\big] \\ &A_n(x) \cap \big\{y : |y| \le \tfrac{u_n}{2}\big\} \subset A\big(z_n(x), \tfrac{2w}{\alpha}\big) \cup A(u_n^{1+r}, \alpha u_n^{-r}) \;\text{ for some } z_n(x) \in [\alpha u_n, u_n]. \end{aligned}$$

The compensator of the process $D(z,z')_t = \sum_{s \le t} 1_{A(z,z')}(\Delta X_s)$ is $\widetilde{D}(x)^n_t = \int_0^t F_s(A(z,z'))\,ds$. Therefore, the last part of (A.4) yield, for any finite stopping time S:

$$\mathbb{E}(D(z,z')_{S+t} - D(z,z')_S \mid \mathcal{F}_S) \;\le\; \frac{Kt}{z^H}\,(z' + \phi'(z)). \tag{A.19}$$

Below, we use the notation of the previous proof, and denote by R'^n_j the successive jump times of $X(u_n^{1+r})$ after $(i-1)\Delta_n$, and whose absolute size is bigger than $u_n/4$. Exactly as for (A.15), we have

$$\mathbb{P}^n_{i-1}(R'^n_j \le i\Delta_n) \;\le\; K^j\,\Delta_n^j\,u_n^{-jH}. \tag{A.20}$$

Let $G_n = \{u_n(1-w) \le |\Delta^n_i X(u_n^{1+r})| \le u_n(1+w)\}$. First, we have

$$\mathbb{P}^n_{i-1} L\big(G_n \cap (\cup_{j\ge 3} B^n_j)\big) \;\le\; \mathbb{P}^n_{i-1}(S^n_3 \le i\Delta_n) \;\le\; K\Delta_n^3 u_n^{-3H(1+r)}. \tag{A.21}$$

Next, $G_n \cap B^n_1 \subset A_n(0)$, hence (A.18) and (A.19) with $z = z_n(0) \ge u_n/4$ yield

$$\mathbb{P}^n_{i-1}(G_n \cap B^n_1) \;\le\; \mathbb{E}^n_{i-1}(\Delta^n_i D(z_n(0), 8w)^n) \;\le\; K\Delta_n u_n^{-H}(w + \phi'(u_n)) \tag{A.22}$$

The analysis of $G_n \cap B^n_2$ is more difficult. We have $G_n \cap B^n_2 \cap \{R'^n_1 > i\Delta_n\} = \emptyset$, hence $G_n \cap B^n_2 \subset C^n_1 \cup C^n_2 \cup C^n_3$, where

$$\begin{aligned} C^n_1 &= \{S^n_2 \le i\Delta_n,\ |\Delta X_{S^n_1}| \le u_n/4,\ \Delta X_{S^n_2} \in A_n(\Delta X_{S^n_1})\} \\ C^n_2 &= \{S^n_2 \le i\Delta_n,\ R'^n_1 = S^n_1,\ |\Delta X_{S^n_1}| > u_n/4,\ |\Delta X_{S^n_2}| \le u_n/4,\ \Delta X_{S^n_1} \in A_n(\Delta X_{S^n_2})\} \\ C^n_3 &= \{R'^n_2 \le i\Delta_n\}. \end{aligned}$$

(A.18) and (A.19) yield on the sets $\{S^n_1 < i\Delta_n\}$ and $\{R'^n_1 < i\Delta_n\}$ respectively (since $\phi'(\alpha u_n)$ for $\alpha \le 2$ and $\phi'(u_n^{1+r})$ are bounded):

$$\begin{aligned} \mathbb{P}^n_{i-1}(C^n_1 \mid \mathcal{F}_{S^n_1}) &\le \mathbb{E}^n_{i-1}\Big(\sup_{z \in [u_n/4, 2u_n]} \mathbb{E}(D(z,8w)_{i\Delta_n} - D(z,8w)^n_{S^n_1} \mid \mathcal{F}_{S^n_1})\Big) \\ &\le K\frac{\Delta_n}{u_n^H}(w + \phi'(2u_n)) \le K\frac{\Delta_n}{u_n^H} \end{aligned}$$

$$\begin{aligned}\mathbb{P}^n_{i-1}(C^n_2 \mid \mathcal{F}_{R'^n_1}) \;=\; & \mathbb{E}^n_{i-1}\Big(\sup_{z\in[\alpha u_n, u_n/4]} \mathbb{E}(D(z,2w/\alpha)_{i\Delta_n} - D(z,2w/\alpha)^n_{R'^n_1} \mid \mathcal{F}_{R'^n_1}) \\ & \qquad + \mathbb{E}(D(u_n^{1+r},\alpha/u_n^r)_{i\Delta_n} - D(u_n^{1+r},\alpha/u_n^r)^n_{R'^n_1} \mid \mathcal{F}_{R'^n_1})\Big) \\ \le\; & K\Delta_n\Big(\frac{w}{\alpha^{H+1}u_n^H} + \frac{\alpha}{u_n^{r+H(1+r)}} + \frac{1}{u_n^{H(1+r)}}\Big)\end{aligned}$$

These expressions vanish on the sets $\{S^n_1 \ge i\Delta_n\}$ and $\{R'^n_1 \ge i\Delta_n\}$ respectively, so by (A.14) and (A.20), we get

$$\text{(A.23)}\quad \mathbb{P}(G_n\cap B^n_2) \;\le\; K\frac{\Delta_n^2}{u_n^{H(2+r)}}\Big(1+\frac{\alpha}{u_n^r}+\frac{u_n^{Hr}\,w}{\alpha^{H+1}}\Big) \;\le\; K\frac{\Delta_n^2}{u_n^{H(2+r)}}\Big(1+\big(\frac{w}{u_n^r}\big)^{\frac{1}{H+2}}\Big),$$

where the last inequality follows upon taking $\alpha=\alpha_n=(wu_n^{r(H+1)})^{1/(H+2)}$.

Putting together (A.21), (A.22) and (A.23), plus the property $G_n\cap B^n_0=\emptyset$, we get the result. □

The next lemma is an auxiliary result on an Itô semimartingale X satisfying

$$\text{(A.24)}\qquad B_t=C_t=0,\qquad |\Delta X_t|\le u\le 1,\qquad \int |x|^\gamma F_t(\omega,dx)\le \alpha<\infty$$

(use the notation (2), here $\gamma\in[0,2]$). Many versions of this type of results are scattered in the literature, but the following seems to be new.

Lemma 6. *Under (A.24), for all $s,t\ge 0$ and $p\ge\gamma\vee 1$ and u as in (A.24) we have*

$$\text{(A.25)}\qquad \mathbb{E}\Big(\sup_{r\le s}|X_{t+r}-X_t|^p \mid \mathcal{F}_t\Big) \le \begin{cases} K_p\, s\,\alpha\, u^{p-\gamma} & \text{if } p\le 2\\ K_p\big(s\alpha u^{p-\gamma}+s^{p/2}\alpha^{p/2}u^{p-\gamma p/2}\big) & \text{if } p\ge 2\end{cases}$$

Proof. We set $Z(w)_t=\sum_{s\le t}|\Delta X_s|^w$, which by assumption is locally integrable with compensator $\widetilde{Z}(w)$ satisfying $\widetilde{Z}(w)_{t+s}-\widetilde{Z}(w)_t\le s\alpha u^{w-\gamma}$, as soon as $w\ge\gamma$. Then $\frac{q}{w}\le v\le 1$ implies

$$\text{(A.26)}\quad \mathbb{E}((Z(w)_{t+s}-Z(w)_t)^v \mid \mathcal{F}_t)\le \mathbb{E}(Z(wv)_{t+s}-Z(wv)_t \mid \mathcal{F}_t) \;\le\; s\alpha u^{wv-\gamma}.$$

When $v>1$ we can write $Z(w)=Z(w)-\widetilde{Z}(w)+\widetilde{Z}(w)$ and use Burkholder-Davis-Gundy inequality to obtain (when $w\ge\gamma$):

$$\mathbb{E}((Z(w)_{t+s}-Z(w)_t)^v\mid\mathcal{F}_t) \;\le\; K_v\Big(\mathbb{E}((Z(2w)_{t+s}-Z(2w)_t)^{v/2}\mid\mathcal{F}_t)+(s\alpha u^{w-\gamma})^v\Big).$$

This and (A.26) show that if $2^n<v\le 2^{n+1}$ for some $n\ge 1$, then

$$\mathbb{E}((Z(w)_{t+s}-Z(w)_t)^v\mid\mathcal{F}_t) \;\le\; K_v\Big(s\alpha u^{wv-\gamma}+\sum_{j=0}^n\big(s\alpha u^{2^jw-\gamma}\big)^{v/2^j}\Big).$$

Now, when $1 \leq v' \leq v'$ we obviously have $x^{v/v'} \leq x + x^v$ for any $x \geq 0$. Therefore we deduce from the above that

$$\text{(A.27)} \qquad v \geq 1 \ \Rightarrow \ \mathbb{E}((Z(w)_{t+s} - Z(w)_t)^v \mid \mathcal{F}_t) \ \leq \ K_v\big(s\alpha u^{wv-\gamma} + s^v \alpha^v u^{wv-\gamma v}\big).$$

Coming back to the problem at hand, another use of Burkholder-Davis-Gundy inequality yields that the left side of (A.25) is smaller than $K_p\,\mathbb{E}(Z(2)_{t+s} - Z(2)_t)^{p/2} \mid \mathcal{F}_t)$. Then the result readily follows from (A.26) and (A.27). □

Lemma 7. *When $p > 2$ and $\rho < \rho_1(p) = \frac{p-2}{2p}$, for all $\gamma > 0$ we have*

$$\text{(A.28)} \qquad \mathbb{E}^n_{i-1}\left(\left|\zeta(X,p)^n_i) - \zeta(X(u_n^{1+r}),p)^n_i\right|\right) \ \leq \ K\Delta_n\, u_n^{p-H}\,\eta(p)_n,$$

where

(A.29)

$$\begin{cases} \eta(p)_n \ = \ u_n^H \phi'(u_n) + \sum_{j=1}^4 (u_n)^{x_j-\gamma}, \\ x_1 = \frac{1}{\rho} - H(1+r),\ x_2 = \frac{2}{\rho} - H(2+3r),\ x_3 = r - \frac{Hr}{p},\ x_4 = \frac{p-2}{2p\rho} + \frac{H}{p} - 1 \end{cases}$$

If $X^c_t = \int_0^t \sigma_s dW_s$ of X vanishes identically, the same holds when $p > 1 \vee H$ and when $p = 2$ if $H = 2$, provided $\rho < \frac{p-1}{p}$, and with x_4 above substituted with $x_4 = \frac{p-1}{p\rho} + \frac{H}{p} - 1$.

Proof. 1) We have

$$X - X(u_n^{1+r}) \ = \ X_0 + X^c + M(n) + B(n),$$

where

$$B(n)_t = \int_0^t \Big(b_s - \int_{\{u_n^{1+r} < |x| \leq 1\}} xF_s(dx)\Big)\,ds,$$
$$M(n)_t = \int_0^t \int_{\{|x| \leq u_n^{1+r}\}} x\,(\mu - \nu)(ds,dx).$$

Observe that by (A.4) and (A.6), $|\Delta^n_i B(n)|$ is smaller than $K\Delta_n$ if $H < 1$, than $K_z \Delta_n u_n^{-z}$ for any $z > 0$ when $H = 1$, and than $K\Delta_n u_n^{(1-H)(1+r)}$ if $H > 1$. Then, because of (A.7),

$$|\Delta^n_i B(n)|^q \ \leq \ K_q\big(\Delta_n^q + \Delta_n u_n^{(q-H)(1+r)}\big)$$

for all $q > 1$. Next, the Itô semimartingale $M(n)$ satisfies (A.24) with $u = u_n^{1+r}$ and $\gamma = 2$ and $\alpha = K u_n^{(2-H)(1+r)}$ by (A.4) when $H = 2$ and (A.5) when $H < 2$. Then Lemma 6 yields

$$\mathbb{E}^n_{i-1}(|\Delta^n_i M(n)|^q) \ \leq \ K_q \Delta_n u_n^{(q-H)(1+r)}$$

(use (A.7) again when $q > 2$), for $q \geq 2$ and also for $q > H \vee 1$. Moreover $\mathbb{E}(|\Delta^n_i X^c|^q) \leq K_q \Delta_n^{q/2}$ for any $q > 0$ by classical estimates (and because σ_t is bounded). Then, setting $\kappa = 0$ if X^c is identically 0 and $\kappa = 1$ otherwise, we deduce that for $q \geq 2$ or $q > H \vee 1$:

$$\mathbb{E}(|\Delta_i^n(X - X(u_n^{1+r})|^q) \leq K_q \left(\kappa\Delta_n^{q/2} + (1-\kappa)\Delta_n^q + \Delta_n u_n^{(q-H)(1+r)}\right). \tag{A.30}$$

2) Next, we use the following estimate, for $u, w, \varepsilon \in (0,1)$ and $p > 1$:

$$\begin{aligned} &\big||x+y|^p 1_{\{|x+y|\leq u\}} - |x|^p 1_{\{|x|\leq u\}}\big| \\ \leq\ & \varepsilon|x|^p 1_{\{|x|\leq u\}} + \frac{K_p}{\varepsilon^{p-1}}\,|y|^p + u^p\,1_{\{|y|>uw\}} + u^p\,1_{\{u(1-w)\leq|x|\leq u(1+w)\}}. \end{aligned}$$

We have $\mathbb{E}_{i-1}^n(\zeta(X(u_n^{1+r}),p)_i^n) \leq K\Delta_n u_n^{p-H}$ by Lemma 4 and (A.7) and the (easy) fact that $\mathbb{E}_{i-1}^n(\Delta_i^n D(1,p,n)) \leq K\Delta_n u_n^{p-H}$. Then, using (A.30) and Markov's inequality, plus (A.17), we deduce that if $p > H \vee 1$ or if $p = 2$ the inequality (A.28) holds with

$$\begin{aligned} \eta(p)_n = \varepsilon &+ \frac{\kappa\Delta_n^{p/2-1}}{\varepsilon^{p-1}\,u_n^{p-H}} + \frac{(1-\kappa)\Delta_n^{p-1}}{\varepsilon^{p-1}\,u_n^{p-H}} + \frac{u_n^{pr-Hr}}{\varepsilon^{p-1}} + \frac{\Delta_n\,w^{1/(H+2)}}{u_n^{H(1+r)+r/(H+2)}} \\ &+ \frac{\kappa\Delta_n^{q/2-1}}{w^q\,u_n^{q-H}} + \frac{(1-\kappa)\Delta_n^{q-1}}{w^q\,u_n^{q-H}} + \frac{u_n^{qr-Hr}}{w^q} + w + u_n^H\phi'(u_n) + \eta_n, \end{aligned}$$

where $q \geq 2$ and $\varepsilon \in (0,1)$ and $w \in (0,1/4)$ are arbitrary. Taking advantage of (30) and (A.7), we deduce that in fact (A.28) holds with

$$\begin{aligned} \eta(p)_n = u_n^H\phi'(u_n) &+ u_n^{\frac{1}{\rho}-H(1+r)} + u_n^{\frac{2}{\rho}-H(2+3r)} + \varepsilon \\ &+ u_n^{pr-Hr}\varepsilon^{-p+1} + w + u_n^{qr-Hr}w^{-q} + u_n^{\frac{1}{\rho}-H(1+r)-\frac{r}{H+2}}\,w^{\frac{1}{H+2}} \\ &+ \kappa\Big(u_n^{\frac{p-2}{2\rho}+H-p}\varepsilon^{-p+1} + u_n^{\frac{q-2}{2\rho}+H-q}w^{-q}\Big) \\ &+ (1-\kappa)\Big(u_n^{\frac{p-1}{\rho}+H-p}\varepsilon^{-p+1} + u_n^{\frac{q-1}{\rho}+H-q}w^{-q}\Big). \end{aligned} \tag{A.31}$$

It remains to choose ε, w and q. We first take

$$\begin{aligned} \varepsilon &= u_n^{\frac{p-\kappa-1}{(\kappa+1)p\rho}+\frac{H}{p}-1} + u_n^{r-Hr/p} \\ w &= u_n^{\frac{(H+2)(q(1-(1+\kappa)\rho)+(1+\kappa)(2\varpi H+\rho Hr-2))-(1+\kappa)\rho r}{(1+\kappa)\rho(1+q(H+2))}} + u_n^{r-\frac{(H+2)(1-\rho H)}{\rho(1+q(H+2))}}. \end{aligned}$$

It follows that $\eta(p)_n \leq u_n^H + \phi'(u_n) + \sum_{j=1}^4 u_n^{x_j} + \sum_{j=5}^8 u_n^{x(q)_j}$, where

$$x(q)_5 = \tfrac{(H+2)(q(1-\kappa\varpi)+\kappa(2\varpi H+\varpi Hr-2))-\kappa\varpi r}{\kappa\varpi(1+q(H+2))}, \quad x(q)_6 = r - \tfrac{(H+2)(1-\varpi H)}{\varpi(1+q(H+2))}$$
$$x(q)_7 = \tfrac{q(H+2)(1-\varpi H(1+r))-\varpi Hr}{\varpi(1+q(H+2))}, \quad x(q)_8 = \tfrac{q(1+\kappa H+2\kappa-\kappa\varpi(1-r+H(H+2)(1+r)))-\kappa(1-\varpi H)}{\kappa\varpi(1+q(H+2))}.$$

Now, for $\gamma>0$ arbitrary, we can choose q large enough to have $x(q)_j\geq x'_j-\gamma$ for $j=5,6,6,8$, where

$$x'_5=\frac{1}{(1+\kappa)\rho},\quad x'_6=r,\quad x'_7=\frac{1-\rho H(1+r)}{\rho},\quad x'_8=\frac{3+(1+\kappa)H+2\kappa-(1+\kappa)\rho(1-r+H(H+2)(1+r))}{(1+\kappa)\rho(H+2)}.$$

Since $\rho<\frac{1}{2}$ and $\kappa\in\{0,1\}$, we have $x'_5>x_4$ and $x'_6\geq x_3$ and $x'_8>x'_7=x_1$. The result follows. □

Lemma 8. *Under the same condition on p and ρ as in Lemma 7, we have*

$$\mathbb{E}\left(\sup_{s\leq t}\ |B(p,u_n,\Delta_n)_s-D(p,u_n)_s|\right)\ \leq\ K\,t\,u_n^{p-H}\,\eta(p)_n. \tag{A.32}$$

Proof. With the notation (A.10), we see that $D(p,u_n)-D(1,p,n)$ is increasing and admits the compensator $\int_0^t ds\int_{\{|x|\leq u_n^{1+r}\}}|x|^p\,F_s(dx)$. Therefore (A.5) yields

$$\mathbb{E}^n_{i-1}\left(|\Delta_i^nD(p,u_n)-\Delta_i^nD(1,p,n)|\right)\ \leq\ K\Delta_n\,u_n^{(p-H)(1+r)}.$$

This allows to deduce from (A.11) and (A.28) that

$$\mathbb{E}^n_{i-1}\left(|\zeta(X,p)^n_i)-\Delta_i^nD(p,u_n)|\right)\ \leq\ K\Delta_nu_n^{p-H}\eta(p)_n,$$

which immediately yields (A.32). □

Lemma 9. *Let $p>2$.*

a) Assume $a=0$ and (30) with $\rho<\rho_1(p)$. There is $H_0(p,\rho)>0$ such that, for any $H\in(0,H_0(p,\rho)]$, we have $u_n^{-H}\eta(p)_n\to0$ for a suitable choice of r and $\gamma>0$.

b) Assume $a>0$ and (30) with $\rho<\rho_1(p)$. There is $\chi(a,p)\in(0,1/2)$ such that for all $H\in[0,2]$ we have $\eta(p)_n\leq Ku_n^{H\chi(a,p)}$ for a suitable choice of r and $\gamma>0$.

c) Assume $a>1/2$ and (30) with $\rho<\rho_2(p)$. Then for all $H\in[0,2]$ we have $u_n^{-H/2}\eta(p)_n\to0$ for a suitable choice of r and $\gamma>0$.

Proof. The proof is based on the form (A.29) of $\eta(p)_n$.

a) When $a=0$, we have $\phi'(u_n)=\phi(u_n)\to0$. Then, with $\rho<\rho_1(p)$ and r arbitrary in $\left(0,\frac{2}{3\rho H}-\frac{2}{3}\right)$, it is enough to show that if H is smaller that some number $H_0(p,\rho)>0$, then $x_j>H$ for all j. If $H\to0$ we observe that x_j/H tends to a limit bigger than 1 for all $j=1,2,3,4$, so the result is obvious.

b) Since $u_n^H\phi'(u_n)=u_n^{aH}$ when $a>0$, the result will follow with $\chi(a,p)=a\wedge\chi'$ if, when $\rho=\rho_1(p)$, we can find $\chi'=\chi'(p)$ such that $x_j\geq H\chi'$ for all j and $H\in(0,2]$ and a suitable choice of r. We then have $x_4=H/p$, whereas $x_j\geq H\chi'$ for $j=1,2,3$ is implied by $2+2r+2\chi'\leq1/\rho$ and $4+6r+2\chi'\leq2/\rho$ and $r\geq2\chi'+2r/p$. Since $p>2$, we can choose $r>0$ and $\chi'>0$ small enough for these to hold, hence the result.

c) We assume $a>1/2$, so again $u_n^H\phi'(u_n)=u_n^{aH}$. Hence, taking $\rho=\rho_2(p)$, it suffices to show that for any $H\in(0,2]$ we can find r such that $x_j\geq H/2$ for all j. We take $r=Hp/(2p-2H)$, and checking $x_j\geq H/2$ for $j=1,2,3,4$ is a simple matter. □

3. Limiting results for $B(p, u_n, \Delta_n)$. We denote by $\stackrel{\text{u.c.p.}}{\longrightarrow}$ the convergence in probability, locally uniformly in time. We set

$$A(p)_t \;=\; m_p \int_0^t |\sigma_s|^p\, ds. \tag{A.33}$$

Proposition 1. *Under Assumption 1, and if (30) holds with some $\rho < 1/2$, we have*

$$B(2, u_n, \Delta_n) \;\stackrel{u.c.p.}{\longrightarrow}\; A(2) \tag{A.34}$$

and for any $p > 2$,

$$\Delta_n^{1-p/2}\, B(p, u_n, \Delta_n)_T \;\stackrel{\mathbb{P}}{\longrightarrow}\; A(p)_T \qquad \text{on the set } \Omega_T^f. \tag{A.35}$$

If moreover we have Assumption 2 and if $k \geq 2$ is an integer and $p \geq 2$, the 2-dimensional variables

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p/2} B(p, u_n, \Delta_n)_T - A(p)_T,\, \Delta_n^{1-p/2} B(p, u_n, k\Delta_n)_T - k^{p/2-1} A(p)_T\right),$$

stably converge in law, in restriction to the set Ω_T^f, to a limit which is defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and which, conditionally on $\mathcal{F}$, is a centered Gaussian variable with variance-covariance matrix

$$\frac{1}{m_{2p}} \begin{pmatrix} (m_{2p} - m_p^2) A(2p)_T & (m_{k,p} - k^{p/2} m_p^2) A(2p)_T \\ (m_{k,p} - k^{p/2} m_p^2) A(2p)_T & k^{p-1}(m_{2p} - m_p^2) A(2p)_T \end{pmatrix}. \tag{A.36}$$

Proof. 1) We first prove (A.34) and the last claim. We set $\Gamma = \{(\omega, t) : F_t(\omega, \mathbb{R}) < \infty\}$. By localization we may suppose that the processes b_t, σ_t and $\int_{\{|x|\leq 1\}} |x| F_t(dx)$ restricted to the set Γ are bounded. Since $t \mapsto F_t$ can be modified on a λ-null set, we can replace F_t by the null measure for all $t \leq R = \inf(s : H(0)_s = \infty)$ (notation (A.1)) such that $F_t(\mathbb{R}) = \infty$, and so $b'_t = b_t - 1_{\{t \leq R\}} \int_{\{|x|\leq 1\}} x F_t(dx)$ is a bounded process. Therefore the process $Y = X' + Z$ where

$$X'_t \;=\; X_0 + \int_0^t b'_s ds + \int_0^t \sigma_s dW_s, \qquad Z_t = \sum_{s \leq t\wedge R} \Delta X_s \tag{A.37}$$

is well defined. Moreover, the following is obvious:

$$t \leq R \quad \Rightarrow \quad X_t \;=\; Y_t. \tag{A.38}$$

We associate with X' the processes $B'(p, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X'|^p$. Since X' is continuous and satisfies Assumption 1, we know (see for example (11) and Theorem 2 of Aït-Sahalia and Jacod (2009a)) that (A.35) and under Assumption 2 the stable convergence in the last claim hold true, if we substitute $B(p, u_n, k'\Delta_n)_T$ with $B'(p, k'\Delta_n)$ (where $k' = 1$ or $k' = k$). Assumption 2 is slightly weaker than in Aït-Sahalia and Jacod (2009a), but the extension to the present case is straightforward. Hence the desired results will hold if we prove that

$$\mathbb{P}(\Omega_T^f \cap \{B(p, u_n, k'\Delta_n)_T \neq B'(p, k'\Delta_n)_T) \;\to\; 0. \tag{A.39}$$

To see this we observe that there are integers $n_1(\omega)$ and $n_2(\omega)$ with the following properties: $n \geq n_1$ implies $|\Delta_i^n X'| \leq u_n$ for all $i \leq [T/\Delta_n]$ (by the boundedness of b' and σ and in view of Lévy's modulus of continuity and of the assumption (30) for some $\rho < 1/2$); and, when $n \geq n_2$, the (finitely many) jumps of Z have a modulus bigger than $2u_n$ and the time separating any two of them is bigger than $k'\Delta_n$. Furthermore on Ω_T^f we have $R \geq T$, so $X_t = Y_t$ for all $t \leq T$ by (A.38). Hence on the set Ω_T^f and if $n \geq n_1 \vee n_2$ we have $\Delta_i^n X 1_{\{|\Delta_i^n X| \leq u_n\}} = \Delta_i^n X'$ for all $i \leq [T/\Delta_n]$: this readily implies (A.39). 2) Now we turn to (A.34). It is enough to prove that $B(2, u_n, \Delta_n)_t \xrightarrow{\mathbb{P}} A(2)_t$ for each fixed t. For any $\varepsilon > 0$, and if n is large enough for having $2\varepsilon\sqrt{\Delta_n} < u_n < \varepsilon$ (recall (30) with $\rho < 1/2$) we have

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]} f_\varepsilon(\Delta_i^n X/\sqrt{\Delta_n}) \;\leq\; B(2, u_n, \Delta_n)_t \;\leq\; \sum_{i=1}^{[t/\Delta_n]} g_\varepsilon(\Delta_i^n X), \tag{A.40}$$

where $f_\varepsilon(x) = x^2 h_\varepsilon(x)$ and $g_\varepsilon(x) = x^2 h_\varepsilon(x)$ and $h_\varepsilon(x) = (1 - |x|/2\varepsilon)^+ \wedge 1$. Now by Theorems 2.2 and 2.4 of Jacod (2008), we have

$$\begin{aligned} &\textstyle\sum_{i=1}^{[t/\Delta_n]} g_\varepsilon(\Delta_i^n X) \;\xrightarrow{\mathbb{P}}\; A(2)_t + \int_0^t \int g_\varepsilon(x) F_t(dx), \\ &\textstyle\Delta_n \sum_{i=1}^{[t/\Delta_n]} f_\varepsilon(\Delta_i^n X/\sqrt{\Delta_n}) \;\xrightarrow{\mathbb{P}}\; \int_0^t \rho_{\sigma_s}(f_\varepsilon) ds, \end{aligned} \tag{A.41}$$

where ρ_a denotes the normal law $\mathcal{N}(0, a^2)$. Then the two right sides of (A.41) converge to $A(2)_t$ as $\varepsilon \to 0$, and the result readily follows from (A.40). □

Now we turn to the behavior of the processes $B(p, u_n, \Delta_n)$ on the set Ω_T^{ii}, under Assumption 5. First, we consider the processes

$$\widetilde{D}(p, u_n)_t \;=\; \int_0^t ds \int_{\{|x| \leq u_n\}} |x|^p F_s(dx) \tag{A.42}$$

Proposition 2. *Under Assumptions 1 and 5, and if $p > 2$ and (30) holds with $\rho < \rho_1(p)$, there is an $\varepsilon > 0$ such that for all $T > 0$,*

$$u_n^{\Gamma_T(1-\varepsilon)-p} \sup_{t \leq T} |B(p, u_n, \Delta_n)_t - \widetilde{D}(p, u_n)_t| \;\xrightarrow{\mathbb{P}}\; 0 \quad \text{on} \quad \{\Gamma_T > 0\}. \tag{A.43}$$

Proof. We take ε arbitrary in $(0, \chi)$, where $\chi = \chi(a, p)$ is defined in Lemma 9, with a as in Assumption 5. We pick α in the (non-empty) interval $(\frac{1-\chi}{1-\varepsilon}, 1)$, and also an integer $m > (2/\alpha) \vee (1/\varepsilon)$, and we set $z_1 = 2 - 1/m$ and $z_j = 2\alpha^{j-1}$ for $j \geq 2$. The intervals $I_j = (z_{j+1}, z_j]$ for $j \geq 1$ form a partition of $[0, 2 - 1/m]$, and since $\Gamma_T < 2$ and m is arbitrarily large, for getting (A.43) on $\{\Gamma_T > 0\}$ it is enough to prove it separately on each set $\Omega_T^j = \{\Gamma_T \in I_j\}$. Below, $j \geq 1$ is fixed. The process Γ_t is optional and increasing (not necessarily càdlàg, though). Recalling (3), we can thus set

$$\begin{aligned} X'_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t 1_{\{\Gamma_{s-} \leq z_j\}} \int x 1_{\{|x| \leq 1\}} (\mu - \nu)(ds, dx) \\ + \int_0^t 1_{\{\Gamma_{s-} \leq z_j\}} \int x 1_{\{|x| > 1\}} \mu(ds, dx) \end{aligned} \tag{A.44}$$

This process obviously satisfies Assumption 5 with the same constant a and function ϕ and ψ as X, and with a global BG index $z_j \wedge \Gamma_t$ instead of Γ_t. Moreover, on the set Ω_T^j we have $X'_t = X_t$ for all $t \leq T$. Hence it is enough to prove the result for X' instead of X. Or, in other words, we can assume that X itself satisfies Assumption 5 with $\Gamma_t \leq z_j$ for all t. Next we pick $v \in (1, \alpha\frac{1-\varepsilon}{1-\chi})$, and we set $H = z_j v$ when $j < \infty$ and $H = \varepsilon$ when $j = \infty$. Then by Assumption 5 we see that the first two properties in (16) hold, if $\Gamma_t + \varepsilon$ is substituted with H, with a locally bounded process L_t instead of $L(\varepsilon)_t$. Hence, by a standard localization argument, it suffices to prove the result when the process satisfies Assumption 7, with H as before and $a > 0$ as in Assumption 5. For any $q > H$ the compensator of $D(q, u_n)$ is $\widetilde{D}(q, u_n)$, which is bounded by (A.5) for any t. The quadratic variation of the square-integrable martingale $D(p, u_n) - \widetilde{D}(p, u_n)$ is $D(2p, u_n)$. Therefore (A.5) again yields, together with Doob's inequality:

$$\mathbb{E}\Big(\sup_{s\leq T} |D(p,u_n)_s - \widetilde{D}(p,u_n)_s|^2\Big) \;\leq\; 4\mathbb{E}(\widetilde{D}(2p,u_n)_T) \;\leq\; KTu_n^{2p-H}. \tag{A.45}$$

Combining this with (A.32) and using Cauchy-Schwarz inequality yield

$$\mathbb{E}\Big(\sup_{s\leq T} |B(p,u_n,\Delta_n)_s - \widetilde{D}(p,u_n)_s|\Big) \;\leq\; KT\,u_n^p\Big(\eta(p)_n u_n^{-H} + u_n^{-H/2}\Big). \tag{A.46}$$

Our choice of ρ implies by Lemma 9 that we can choose the number r in such a way that $\eta(p)_n \leq K u_n^{H\chi}$. Then, recalling $z_{j+1}/z_j \geq 1/\alpha$ and $\chi < 1/2$, we deduce from our choices of α and v that, if $z_{j+1} < \Gamma_T \leq z_j$, both $-H/2$ and $H\chi - H$ are bigger than $-\Gamma_T(1-\varepsilon)$. Then (A.43) in restriction to the set Ω_T^j readily follows from (A.46). ☐

Proposition 3. *Under Assumptions 1 and 5, and if $p > 2$ and (30) holds with $\rho < \rho_1(p)$, for all $T > 0$ we have*

$$\frac{1}{u_n^p} \sup_{t\leq T} |B(p,u_n,\Delta_n)_t - D(p,u_n)_t| \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \quad \text{on} \quad \{\Gamma_T = 0\}. \tag{A.47}$$

Moreover, if $\psi(0) = \infty$ we have

$$\frac{1}{u_n^p\,\psi(u_n)} \sup_{t\leq T} |B(p,u_n,\Delta_n)_t - \widetilde{D}(p,u_n)_t| \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \quad \text{on} \quad \{\Gamma_T = 0\}. \tag{A.48}$$

Proof. The scheme of the proof is essentially the same as in the previous proposition. We define a new process X' by (A.44), in which we take $z_j = 0$. This process has a global BG index identically 0, and it satisfies Assumption 5 with the same functions ϕ and ψ and, say, $a = 1$ (a is indeed irrelevant in this case). Moreover, on the set $\{\Gamma_T = 0\}$ we have $X'_t = X_t$ for all $t \leq T$. Hence it is enough to prove the two convergences (A.47) and (A.48) on the whole set Ω, when the process X satisfies Assumptions 1 and 5 and $\Gamma_t = 0$ identically. By a standard localization argument, it suffices to prove the results when X satisfies Assumption 7, with $H = H(p, \rho)$ (see Lemma 9) and $a =$

0, and also when the processes $G(q,u)_t = \frac{1}{u_n^q\,\psi(u)}\int_{\{|x|\leq u\}}|x|^q\,F_t(dx)$ are bounded by the same constant L for $q=p$ and $q=2p$ when $u\in(0,1]$. Then by Lemma 9-(a) we can choose the number r in such a way that $u_n^{-H}\eta(p)_n\to 0$, and (A.47) then readily follows from Lemma 8. In view of (A.47), (A.48) will hold as stated, as soon as it holds with $B(p,u_n,\Delta_n)$ substituted with $D(p,u_n)$. To see that this is true, we use the estimate $G(2p,u)_t\leq L$, so (A.45) holds with the right side substituted with $4LTu_n^{2p}\psi(u_n)$. Thus the sequence $\frac{1}{u_n^p\sqrt{\psi(u_n)}}\sup_{t\leq T}|D(p,u_n)_t-\widetilde{D}(p,u_n)_t|$ is tight, and (A.48) follows. □

The asymptotic behavior of $B(p,u_n,\Delta_n)$ is of course not completely specified by these two propositions, but they will be enough for our purposes, in restriction to the set $\{\Gamma_T>0\}$, and also on $\{\Gamma_T=0\}$ when $\psi(0)=\infty$. When $\psi(0)=1$, however, we need more. The next result holds as soon as $u_n\to 0$.

Proposition 4. *Under Assumption 5 with $\psi(0)=1$, the variables $u_n^{-p}\,D(p,u_n)_T$ are bounded away from 0 in probability, in restriction to the set $\Omega_T^{i,\Gamma=0}$, that is*

$$\lim_{\varepsilon\to 0}\ \limsup_n\ \mathbb{P}\Big(\{u_n^{-p}D(p,u_n)_T\leq\varepsilon\}\cap\Omega_T^{i,\Gamma=0}\Big)\ =\ 0. \tag{A.49}$$

Proof. If (A.49) fails we can extract a subsequence of u_n, which we still denote with the same index n, such that for some sequence $\varepsilon_n\to 0$ and some $\eta>0$ we have

$$\mathbb{P}\Big(\{u_n^{-p}D(p,u_n)_T\leq\varepsilon_n\}\cap\Omega_T^{i,\Gamma=0}\Big)\ \geq\ \eta.$$

Now, let

$$H_t^n=\sum_{s\leq t}1_{\{\varepsilon_n^{1/p}u_n<|\Delta X_s|\leq u_n\}},$$

whose compensator is

$$\widetilde{H}_t^n=\int_0^t ds(\overline{F}_s(\varepsilon_n^{1/p}u_n)-\overline{F}_s(u_n)).$$

Observe that if $u_n^{-p}D(p,u_n)_T\leq\varepsilon_n$ then $H_T^n=0$. Therefore it suffices to prove that the next property brings forth a contradiction:

$$\mathbb{P}\Big(\{H_T^n=0\}\cap\Omega_T^{i,\Gamma=0}\Big)\ \geq\ \eta. \tag{A.50}$$

Fix s and ω for a moment, and suppose that $a\leq G(p,u)_s\leq a'$ for two constants $0<a<a'$ and all $u\in(0,1]$. Since $\psi\equiv 1$, and for $0<x,v\leq 1$, we deduce

$$\overline{F}_s(vx)-\overline{F}_s(v)\ \geq\ \frac{1}{v^p}\int_{\{vx<|y|\leq v\}}|y|^p\,F_s(dy)=G(p,v)_s-x^pG(p,vx)_s\geq y:=a-x^pa',$$

and $y>0$ if we take x small enough. Now, the interval $(\varepsilon_n^{1/p}u_n,u_n]$ contains exactly $\frac{1}{p}\left[\log(\varepsilon)/\log(x)\right]$ disjoint intervals of the form $(vx,v]$, and the previous minoration yields $\overline{F}_s(\varepsilon_n^{1/p}u_n)-\overline{F}_s(u_n))\geq\frac{y}{p}\,[\log(\varepsilon_n)/\log(x)]$, hence $\overline{F}_s(\varepsilon_n u_n)-\overline{F}_s(u_n))\to\infty$. Therefore, in view of (20) and the last part of (16), and applying Fatou's Lemma, we deduce

$$\widetilde{H}^n_T\ \to\ \infty\quad\text{on the set}\ \ \Omega^{i,\Gamma=0}_T. \tag{A.51}$$

Now, set $S_n=\inf(t:H^n_t>0)$. We have $H^n_{S_n}\leq 1$, so $\mathbb{E}(\widetilde{H}^n_{S_n})=\mathbb{E}(H^n_{S_n})\leq 1$ and thus $\mathbb{P}(\widetilde{H}^n_{S_n}>A)\leq 1/A$. Therefore

$$\begin{aligned}\mathbb{P}(\{S_n\geq T\}\cap\Omega^{i,\Gamma=0}_T)&\leq\mathbb{P}(\widetilde{H}^n_{S_n}>A)+\mathbb{P}(\Omega^{i,\Gamma=0}_T\cap\{\widetilde{H}^n_T\leq A\})\\ &\leq\frac{1}{A}+\mathbb{P}(\Omega^{i,\Gamma=0}_T\cap\{\widetilde{H}^n_T\leq A\})\end{aligned}$$

So, by choosing first A large and second n large, and taking advantage of (A.51), we deduce that $\mathbb{P}(\{S_n\geq T\}\cap\Omega^{i,\Gamma=0}_T)\to 0$. Since $H^n_T=0$ when $S_n>T$ we see that this contradicts (A.50), and we are done. □

Finally, we prove much more precise limiting results under Assumption 6. As seen before, the behavior of $D(p,u_n)$ is key, and we start by studying these processes. We need a law of large numbers and a central limit theorem, and a 4-dimensional one for the latter. Taking $p'>p$ and $\gamma>1$ and writing $u'_n=\gamma u_n$, we introduce 4-dimensional processes D^n, $\widetilde{D}^n$, B^n and $\overline{D}$ with respective components

(A.52)

$$\begin{array}{lll} D^{n,1}=u_n^{\beta-p}D(p,u_n), & \widetilde{D}^{n,1}=u_n^{\beta-p}\widetilde{D}(p,u_n), & B^{n,1}=u_n^{\beta-p}B(p,u_n,\Delta_n),\\ D^{n,2}=u_n^{\beta-p'}D(p',u_n), & \widetilde{D}^{n,2}=u_n^{\beta-p'}\widetilde{D}(p',u_n), & B^{n,2}=u_n^{\beta-p'}B(p',u_n,\Delta_n),\\ D^{n,3}=u_n'^{\,\beta-p}D(p,u'_n), & \widetilde{D}^{n,3}=u_n'^{\,\beta-p}\widetilde{D}(p,u'_n), & B^{n,3}=u_n'^{\,\beta-p}B(p,u'_n,\Delta_n),\\ D^{n,4}=u_n'^{\,\beta-p'}D(p',u'_n), & \widetilde{D}^{n,4}=u_n'^{\,\beta-p'}\widetilde{D}(p',u'_n), & B^{n,4}=u_n'^{\,\beta-p'}B(p',u'_n,\Delta_n),\end{array}$$

$$\overline{D}^1=\overline{D}^2=\frac{\beta\overline{A}}{p-\beta},\qquad\overline{D}^3=\overline{D}^4=\frac{\beta\overline{A}}{p'-\beta}. \tag{A.53}$$

Proposition 5. *Assume Assumption 6, and suppose $p>\beta$.*

a) We have $D^n\stackrel{u.c.p.}{\longrightarrow}\overline{D}$.

b) If further $\beta'<\beta/2$, the 4-dimensional processes $u_n^{-\beta/2}(D^n-\overline{D})$ stably converge in law to a limit which is a continuous process, defined on an extension of $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, and which conditionally on $\mathcal{F}$ is a centered Gaussian martingale with covariance (or quadratic variation) process

$$\widetilde{C}\ =\ \begin{pmatrix}\frac{\beta\overline{A}}{2p-\beta} & \frac{\gamma^{\beta-p}\beta\overline{A}}{2p-\beta} & \frac{\beta\overline{A}}{p+p'-\beta} & \frac{\gamma^{\beta-p'}\beta\overline{A}}{p+p'-\beta}\\ \frac{\gamma^{\beta-p}\beta\overline{A}}{2p-\beta} & \frac{\gamma^{\beta}\beta\overline{A}}{2p-\beta} & \frac{\gamma^{\beta-p}\beta\overline{A}}{p+p'-\beta} & \frac{\gamma^{\beta}\beta\overline{A}}{p+p'-\beta}\\ \frac{\beta\overline{A}}{p+p'-\beta} & \frac{\gamma^{\beta-p}\beta\overline{A}}{p+p'-\beta} & \frac{\beta\overline{A}}{2p'-\beta} & \frac{\gamma^{\beta-p'}\beta\overline{A}}{2p'-\beta}\\ \frac{\gamma^{\beta-p'}\beta\overline{A}}{p+p'-\beta} & \frac{\gamma^{\beta}\beta\overline{A}}{p+p'-\beta} & \frac{\gamma^{\beta-p'}\beta\overline{A}}{2p'-\beta} & \frac{\gamma^{\beta}\beta\overline{A}}{2p'-\beta}\end{pmatrix}. \tag{A.54}$$

Proof. a) We easily deduce from (22) that if $u > 0$ and $q > \beta$,

$$\left| \int_{\{|x|\leq v\}} |x|^p F_t(dx) - A_t \frac{\beta v^{q-\beta}}{p-\beta} \right| \ \leq\ K L_t v^{q-\beta'}, \tag{A.55}$$

which, together with the definition (A.42) and $\beta' < \beta$, gives us

$$q > \beta,\ v_n \to 0 \quad \Rightarrow \quad v_n^{\beta-q} \widetilde{D}(q, v_n) \ \stackrel{\text{u.c.p.}}{\longrightarrow}\ \frac{\beta \overline{A}}{q-\beta}. \tag{A.56}$$

Observe that $Y^n = v_n^{\beta-q}(D(q,v_n) - \widetilde{D}(q,v_n))$ is a martingale with predictable bracket $v_n^{2\beta-2q}\widetilde{D}(2q, v_n)$, which goes to 0 by (A.56), hence another application of (A.56) yields

$$q > \beta,\ v_n \to 0 \quad \Rightarrow \quad v_n^{\beta-q} D(q, v_n) \ \stackrel{\text{u.c.p.}}{\longrightarrow}\ \frac{\beta \overline{A}}{q-\beta}. \tag{A.57}$$

This implies (a).

b) Observe that $\sup_{s\leq t} \|u_n^{-\beta/2}(\widetilde{D}^n_s - \overline{D}_s\| \to 0$ by (A.55), as soon as $\beta' < \beta/2$. Hence for (b) it remains to prove that $M^n = u_n^{-\beta/2}(D^n - \widetilde{D}^n)$ stably converges in law to a continuous process M, defined on an extension of $(\Omega, F, (F_t)_{t\geq 0}, P)$, and which conditionally on F is a centered Gaussian martingale with covariance process (A.54).

Now, M^n is a 4-dimensional martingale, whose quadratic covariation process C^n is

$$\begin{array}{ll}
C^{n,11} = u_n^{\beta-2p} D(2p, u_n), & C^{n,22} = \gamma^\beta {u'_n}^{\beta-2p} D(2p, u'_n), \\
C^{n,33} = u_n^{\beta-2p'} D(2p', u_n), & C^{n,44} = \gamma^\beta {u'_n}^{\beta-2p'} D(2p', u'_n), \\
C^{n,12} = \gamma^{\beta-p} u_n^{\beta-2p} D(2p, u_n), & C^{n,13} = u_n^{\beta-p-p'} D(p+p', u_n), \\
C^{n,14} = \gamma^{\beta-p'} u_n^{\beta-p-p'} D(p+p', u_n), & C^{n,23} = \gamma^{\beta-p} u_n^{\beta-p-p'} D(p+p', u_n), \\
C^{n,24} = \gamma^\beta {u'_n}^{\beta-p-p'} D(p+p', u'_n), & C^{n,34} = \gamma^{\beta-p'} u_n^{\beta-2p'} D(2p', u_n).
\end{array}$$

Then $C^n_t \stackrel{\mathbb{P}}{\longrightarrow} \widetilde{C}_t$ for all t by (A.57). This implies that the sequence (M^n) converges in law to a continuous limit M which a local martingale having quadratic covariation process $\widetilde{C}$.

At this stage, we are left to show that

$$\langle M^n, N \rangle_t \ \stackrel{\mathbb{P}}{\longrightarrow}\ 0, \tag{A.58}$$

for any bounded martingale N in a set N which generates all martingales (in the sense of stochastic integrals). A choice of N consists in all M that are orthogonal to $\mu - \nu$, and those which are of the form $\Psi * (\mu - \nu)$ for Ψ predictable and vanishing on the set $\{(\omega, t, x) :\ |x| \leq \varepsilon\}$ for some $\varepsilon > 0$. Now $\langle M^n, N\rangle$ vanishes if N in the first class above, and also for those in the second class as soon as $u'_n < \varepsilon$: then (A.58) obtains. $\square$

We now apply this to derive results on the processes $B(p,u_n,\Delta_n)$; recalling the notation (A.52) and (A.53), the following result is an obvious consequence of Lemmas 8 and 9 and Proposition 5.

Proposition 6. *Assume Assumptions 1 and 6, and let* $p'>p>2$.
a) If (30) holds with $\rho\leq\rho_1(p)$, *we have* $B^n\stackrel{u.c.p.}{\longrightarrow}\overline{D}$.
b) If (30) holds with $\rho\leq\rho_2(p)$, *and* $\beta'<\beta/2$, *the* 4*-dimensional processes* $u_n^{-\beta/2}(B^n-\overline{D})$ *stably converge in law to a limit which is a continuous process, defined on an extension of* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$, *and which conditionally on* $\mathcal{F}$ *is a centered Gaussian martingale with covariance given by (A.54).*

4. Consistency results. In this section we prove Theorems 1 and 4, and we give a description of the asymptotic behavior of the variance estimates V_n and V'_n given by (37) and (44).

Proof of Theorem 1. By virtue of Lemma 2, we can assume Assumptions 1, and also 5 for (b). Take $p>2$. We have (A.35) and also (upon substituting Δ_n with $k\Delta_n$):

$$\Delta_n^{1-p/2}\,B(p,u_n,k\Delta_n)_t\ \stackrel{\mathbb{P}}{\longrightarrow}\ k^{p/2-1}A(p)_T$$

on the set Ω_T^f. Since further $A(p)_T>0$ on the set Ω_T^W, (34) follows from (33). The proof of (35) is more involved. Let $\varepsilon>0$ be such that (A.43) holds, and let $a'\in(1-\varepsilon,1)$. Recall that

$$\int_{\{|x|\leq u_n\}}|x|^pF_t(dx)\ =\ p\int_0^{u_n}y^{p-1}\overline{F}_t(y)\,dy. \tag{A.59}$$

Therefore if $Z_t^n=\inf_{x\in(0,u_n]}x^{a'\Gamma_T}\overline{F}_t(x)$, and in view of (A.42), we get

$$\widetilde{D}(p,u_n)_T\ \geq\ u_n^{p-a'\Gamma_T}\frac{p}{p-a'\Gamma_T}\int_0^TZ_t^n\,dt$$

(note that $p>a'\Gamma_T$ because $a'\Gamma_T<2$). Now, (19) implies that on the set $\Omega_T^{i,\Gamma>0}$ the sequence Z_t^n, which increases in n, goes to $+\infty$ on a subset of $[0,T]$ with positive λ-measure. Then $\int_0^TZ_t^ndt$ increases to $+\infty$, and we deduce from the previous minoration that for n large enough (depending on ω), we have

$$\widetilde{D}(p,u_n)_T\ \geq\ u_n^{p-a'\Gamma_T}\qquad\text{on }\ \Omega_T^{i,\Gamma>0}. \tag{A.60}$$

By our choice of a' we have $\Gamma_T(1-\varepsilon)<a'\,\Gamma_T$. Then (A.43) and (A.60) imply that for all n large enough,

$$B(p,u_n,\Delta_n)_T\ =\ \widetilde{D}(p,u_n)_T(1+Y_n),\qquad\text{where }\ Y_n\ \stackrel{\mathbb{P}}{\longrightarrow}\ 0\quad\text{on }\ \Omega_T^{i,\Gamma>0}, \tag{A.61}$$

and in exactly the same way we obtain, again for n large enough,

$$B(p,u_n,k\Delta_n)_T\ =\ \widetilde{D}(p,u_n)_T(1+Y'_n),\quad\text{where }\ Y'_n\ \stackrel{\mathbb{P}}{\longrightarrow}\ 0\quad\text{on }\ \Omega_T^{i,\Gamma>0}. \tag{A.62}$$

At this stage, (35) restricted to $\Omega_T^{i,\Gamma>0}$ readily follows. Second, recalling $\widetilde{D}(p,u_n)_T = u_n^p\psi(u_n)\int_0^T G(p,u_n)_s ds$, we deduce from (20) that

$$\liminf_n \frac{1}{u_n^p\,\psi(u_n)}\,\widetilde{D}(p,u_n)_T \;>\; 0 \qquad \text{on } \; \Omega_T^{i,\Gamma=0}. \tag{A.63}$$

Hence when $\psi(0)=\infty$, we can apply (A.48) to obtain (A.61), and (A.62) as well, on the set $\Omega_T^{i,\Gamma=0}$ instead of $\Omega_T^{i,\Gamma>0}$, and (35) restricted to $\Omega_T^{i,\Gamma=0}$ follows when $\psi(0)=\infty$. Finally, suppose that $\psi(0)=1$. We apply (A.47) for Δ_n and $k\Delta_n$, to obtain that

$$B(p,u_n,j\Delta_n)_T = D(p,u_n)_T + u_n^p\,Y_n^j \quad \text{on } \; \{\Gamma_T=9\}, \quad \text{where } \; Y_n^j \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \tag{A.64}$$

for $j=1$ and $j=k$. We have $|S_n-1|\leq(|Y_n^1|+|Y_n^k|)/|\varepsilon+Y_n^k|$ if $u_n^{-p}D(p,u_n)_T>\varepsilon$, hence for all $\varepsilon,\eta>0$ we have

$$\limsup_n \mathbb{P}(\{|S_n-1|>\eta\}\cap\Omega_T^{i,\Gamma=0}) \;\leq\; \limsup_n \mathbb{P}(\{u_n^{-p}D(p,u_n)_T\leq\varepsilon\}\cap\Omega_T^{i,\Gamma=0}).$$

We deduce from (A.49) that (35) holds on the set $\Omega_T^{i,\Gamma=0}$, and in view of (18) this finishes the proof. □

Proof of Theorem 4. In view of (40), and since $A(p)_T>0$ and $A(p')_T>0$ a.s. on Ω_T^W, (42) is a trivial consequence of (A.35). For (41), it suffices to prove it under Assumption 6, on the set $\Omega_T^{i\beta}$, and this follows from (a) of Proposition 5. □

We end this section with the asymptotic behavior of the variances V_n and V_n' defined in (37) and (44). We have two exponents $p'>p>2$, and we assume that (A.15) holds with $\rho=\rho_1(p)$, which is smaller than $\rho_1(p')$. First, under Assumption 1, we deduce from Proposition 1 that

$$\frac{V_n}{\Delta_n} \;\stackrel{\mathbb{P}}{\longrightarrow}\; V \;:=\; N(p,k)\,\frac{A(2p)_T}{(A(p)_T)^2} \qquad \text{in restriction to } \; \Omega_T^f\cap\Omega_T^W, \tag{A.65}$$

$$\begin{aligned}\frac{V_n'}{\Delta_n} \stackrel{\mathbb{P}}{\longrightarrow} 2\gamma^{2p'-2p}\Big(&(1-\gamma^{-p})\frac{A(2p)_T}{A(p)_T^2}+(1-\gamma^{-p'})\frac{A(2p')_T}{A(p')_T^2}\\ &-(2-\gamma^{-p}-\gamma^{-p'})\frac{A(p+p')_T}{A(p)_T A(p')_T}\Big) \qquad \text{in restriction to } \; \Omega_T^f\cap\Omega_T^W.\end{aligned} \tag{A.66}$$

The behavior of V_n on the set $\Omega_T^{\beta\delta}$ under Assumption 5 is more difficult to establish and is given in the next proposition:

Proposition 7. *Under Assumptions 1 and 5, and if (30) holds with $\rho=\rho_1(p)$, we have*

$$V_n \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \quad \text{in restriction to the set } \begin{cases}\Omega_T^{ii} & \text{if } \; \psi(0)=\infty\\ \Omega_T^{i,\Gamma>0} & \text{if } \; \psi(0)=1.\end{cases} \tag{A.67}$$

Proof. Exactly as in the proof of Theorem 1 above, for all n large enough we have

$$B(p,u_n,\Delta_n)_T=\widetilde{D}(p,u_n)_T(1+Y_n),\ B(2p,u_n,\Delta_n)_T=\widetilde{D}(2p,u_n)_T(1+Y'_n), \tag{A.68}$$

and Y_n and Y'_n go to 0 in probability in restriction to the set $\Omega_T^{i,\Gamma>0}$. We also have (A.60) and $\widetilde{D}(2p,u_n)_T\leq K_\eta L(\eta)_T u_n^{2p-\Gamma_T-\eta}$ for any $\eta>0$, by (A.59). Then for n large enough we have on the set $\Omega_T^{i,\Gamma>0}$:

$$V_n \;\leq\; K\frac{1+|Y''_n|}{(1+Y_n)^2}\;L(\eta)_T\,u_n^{(2a'-1)\delta_T-\eta}.$$

Since $L(\eta)_T<\infty$ for all $\eta>0$ and $2a'>1$, we deduce (A.67) in restriction to $\Omega_T^{i,\Gamma>0}$. Next, we still have (A.68) with Y_n and Y'_n going to 0 in probability in restriction to the set $\Omega_T^{i,\Gamma=0}$, when $\psi(0)=\infty$. Moreover if $\Gamma_T=0$ we also have $\widetilde{D}(2p,u_n)_T\leq L'(2p)_T u_n^{2p}\psi(u_n)$ by (16). Therefore, denoting by Z the left side of (A.63), for all n large enough we have on the set $\Omega_T^{i,\Gamma=0}$:

$$Z>0,\qquad V_n\;\leq\;K\frac{1+|Y''_n|}{(1+Y_n)^2}\;\frac{1}{Z\psi(u_n)}\;L'(2p)_T.$$

Hence if $\psi(0)=\infty$, we deduce (A.67) in restriction to $\Omega_T^{i,\Gamma=0}$. ☐

Finally, we deduce from (A.56) and under Assumptions 1 and 6 that, in restriction to the set $\Omega_T^{i\beta}$:

$$\frac{V'_n}{u_n^\beta}\stackrel{\mathbb{P}}{\longrightarrow}V'=\frac{\gamma^{2p'-2p}}{\beta\overline{A}_T}\Big(\frac{(p-\beta)^2}{2p-\beta}(1+\gamma^\beta-2\gamma^{\beta-p})+\frac{(p'-\beta)^2}{2p'-\beta}(1+\gamma^\beta-2\gamma^{\beta-p'})$$
$$-\frac{2(p-\beta)(p'-\beta)}{p+p'-\beta}(1+\gamma^\beta-\gamma^{\beta-p}-\gamma^{\beta-p'})\Big). \tag{A.69}$$

5. Central Limit Theorems and Theorems 3 and 6. In this section, we prove Theorems 2 and 3, resp. 5 and 6, and the proof is the same for both pairs of theorems, and for the first pair it essentially reduces to Theorems 5 and 6 of Aït-Sahalia and Jacod (2009b). Moreover it is enough to prove the results under Assumptions 5 instead of 3 and 6 instead of 4, according to the case. There are three steps:

Proof. 1) We need a CLT for S_n and S'_n under the null hypothesis. For this, we apply Propositions 1 and 6, and after some (elementary) calculations we obtain the following stable convergence in law, under Assumptions 1, and respectively 2 and 6:

$$\frac{1}{\sqrt{\Delta_n}}\,(S_n-k^{p/2-1})\;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\;U\quad\text{in restriction to }\Omega_T^f\cap\Omega_T^W \tag{A.70}$$

where U is F-conditionally centered Gaussian with variance V given by (A.65), and

$$\frac{1}{u_n^{\beta/2}}\,(S'_n-\gamma^{p'-p})\;\stackrel{\mathcal{L}-(s)}{\longrightarrow}\;U'\quad\text{in restriction to }\Omega_T^{i\beta} \tag{A.71}$$

where U' is F-conditionally centered Gaussian with variance V' given by (A.69).

2) We combine the stable convergence in law in (A.70) and (A.71) with the convergence in probability in (A.65) and (A.69), to deduce Theorems 2 and 5. In turn, these results immediately yield that in both theorems, the asymptotic level of the test equals a.

3) It remains to prove that in both cases the asymptotic power equals 1. In case of Theorem 3 we see by (A.67) and also (35) that $(S_n - k^{p/2-1})/\sqrt{V_n} \to -\infty$ on the set Ω_T^{ii} (when $\psi(0) = \infty$), whereas in case of Theorem 6 we deduce from (A.66) and also (35) that $(S'_n - \gamma^{p'-p})/\sqrt{V'_n} \to -\infty$ on the set Ω_T^f. So in both cases the asymptotic power is obviously 1. □

6. In the presence of noise.

Proof of Theorem 7. 1) In view of the definition of $\overline{S}_n$ and $\overline{S}'_n$, it is clearly enough to show the following convergence in probability, with f is as in (47):

$$\text{(A.72)} \qquad \frac{\Delta_n}{u_n^{p+1}}\,\overline{B}(p,u_n,\Delta_n)_t \ \stackrel{\mathbb{P}}{\longrightarrow}\ \frac{2f(0)}{p+1}\,t$$

for the appropriate values of ρ in (30), in connection with the value of p.

2) We start with the behavior of the truncated power variations when there is only noise. That is, we denote by $\widetilde{B}(p,u_n,\Delta_n)$ the process defined by (32), when we substitute $X_{i\Delta_n}$ with $\varepsilon_{i\Delta_n}$, that is

$$\text{(A.73)} \qquad \widetilde{B}(p,u_n,\Delta_n)_t \ = \ \sum_{i=1}^{[t/\Delta_n]} |\varepsilon_i^n - \varepsilon_{i-1}^n|^p\, 1_{\{|\varepsilon_i^n - \varepsilon_{i-1}^n| \le u_n\}}.$$

The aim of this step is to show the following convergence result:

$$\text{(A.74)} \qquad \frac{\Delta_n}{u_n^{p+1}}\,\widetilde{B}(p,u_n,\Delta_n)_t \ \stackrel{\mathbb{P}}{\longrightarrow}\ \frac{2f(0)}{p+1}\,t.$$

Let $\widetilde{B}^+(p,u_n,\Delta_n)_t$ and $\widetilde{B}^-(p,u_n,\Delta_n)_t$ be the variables defined by the right side of (A.73), except that the sum is extended over all indices i that are even, resp. odd. It is then sufficient to prove that, suitably normalized, both these variables converge to half the limit in (A.74), and it suffices to prove the result for, say, $\widetilde{B}_+(p,u_n,\Delta_n)$.

The summands $\zeta_i^n = |\varepsilon_i^n - \varepsilon_{i-1}^n|^p\, 1_{\{|\varepsilon_i^n - \varepsilon_{i-1}^n| \le u_n\}}$ for i even are i.i.d. for each n, with a law depending on n. Moreover if $G(x) = \mathbb{E}\big(|\varepsilon_1 - \varepsilon_0|^p 1_{\{|\varepsilon_1 - \varepsilon_0| \le x\}}\big)$, we have $\mathbb{E}(\zeta_i^n) = G(u_n)$ and $\mathbb{E}((\zeta_i^n)^2) \le u_n^p G(u_n)$. The result then readily follows from standard properties of i.i.d. triangular arrays, plus $\Delta_n/u_n \to 0$ (because $\rho \le 1$) and the convergence $G(x)/x^{p+1} \to 2f(0)/(p+1)$ as $x \to 0$, which is a consequence of (47).

3) In view of (A.74), it is enough to prove that

$$\text{(A.75)} \qquad \frac{\Delta_n}{u_n^{p+1}}\,\Big|\overline{B}(p,u_n,\Delta_n)_t - \widetilde{B}(p,u_n,\Delta_n)_t\Big| \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

Since Assumption 1 holds, by a standard localization procedure we may in fact suppose that

$$\mathbb{E}(|\Delta_i^n X|^2) \;\leq\; K\Delta_n. \tag{A.76}$$

Observe that if for some reals x, y we have $|x| \leq u_n/2$ and either $|x+y| \leq u_n < |y|$ or $|y| \leq u_n < |x+y|$, then $|y - u_n| \leq |x|$, and thus

$$\begin{aligned}\big||x+y|^p\, 1_{\{|x+y|\leq u_n\}} - |y|^p\, 1_{\{|y|\leq u_n\}}\big| &\leq K\big(u_n^p\big(1_{\{|x|>u_n/2\}} + 1_{\{|y-u_n|\leq|x|\leq u_n\}}\big) + u_n^{p-1}|x|\big)\\ &\leq K\Big(u_n^{p-2}|x|^2 + \frac{u_n^p|x|^{1/2}}{|y-u_n|^{1/2}}\, 1_{\{|y|\leq 2u_n\}} + u_n^{p-1}|x|\Big)\end{aligned}$$

for any $q > 0$. Then, taking (A.76) and (47) into account, the latter implying $\int_{-2u_n}^{2u_n} \frac{f(y)}{|y-u_n|^{1/2}}\, dy \leq K u_n^{1/2}$, we deduce that the expectation of the left side of (A.75) is smaller than

$$Kt\Big(\frac{\Delta_n}{u_n^3} + \frac{\Delta_n^{1/4}}{u_n^{1/2}} + \frac{\Delta_n^{1/2}}{u_n}\Big).$$

If $\rho < \frac{1}{3}$, the above goes to 0, and the proof is complete. □