Understanding Mid-Latitude Jet Variability and Change using Rossby Wave Chromatography: Methodology

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ABSTRACT
Rossby Wave Chromatography (RWC) is implemented in a linearized barotropic model as a tool to diagnose and understand the interaction between the mid-latitude jet and the eddies. Given the background zonal-mean flow and the space-time structure of the baroclinic wave activity source, RWC calculates the space-time structure of the upper tropospheric eddy momentum fluxes. Using the convergence of the vertical EP flux in the upper troposphere as the wave source, RWC reproduces the main features of a GCM’s mean state and response to external forcing. When coupled to the zonal-mean zonal wind and a simple model of wave activity source phase speed changes RWC also simulates the temporal evolution of the GCM’s internally generated zonal-mean zonal wind anomalies. Because the full space-time structure of the baroclinic wave activity source is decoupled from the background flow, RWC can be used to isolate and quantify the dynamical mechanisms responsible for 1) the poleward shift of the mid-latitude jet and 2) the feedbacks between the eddy momentum fluxes and the background flow in general.

1. Introduction
This is the first in a series of papers on understanding and quantifying the fundamental dynamical mechanisms responsible for the eddy momentum flux response to zonal-mean zonal wind anomalies. Since the effect of the eddies on the zonal-mean in the extratropics is well known, the above theory will help provide an explanation for the structure of mid-latitude jet variability and the response of the mid-latitude jet to external forcing. In this paper, we develop a model designed to cleanly separate dynamical mechanisms. The model is based on the idea that a theory for eddy fluxes can be usefully divided into two components: 1) a theory for the space-time structure of wave activity propagating into the upper troposphere from baroclinic instability and 2) a theory of the meridional propagation of the waves in the upper troposphere that determine the pattern of wave dissipation and the resulting mean flow deceleration (Held and Hoskins (1985); Held and Phillips (1987); Randel and Held (1991)). In this paper, we focus on developing a quantitative model for part two of this eddy closure, which Held and Phillips (1987) gave the name Rossby Wave Chromatography (RWC). We also propose a simple model for the changes in the phase speed spectrum of the wave activity propagating into the upper troposphere from baroclinic instability.

The dynamical model for our implementation of RWC is the forced, linearized barotropic vorticity equation. Forced barotropic and shallow water models have been used before to understand zonal-mean zonal wind variability and change (Vallis et al. (2004); Chen et al. (2007); Barnes et al. (2010); Barnes and Hartmann (2011); Kidston and Vallis (2012)). Unlike these studies, which specify the forcing directly, we specify the baroclinic wave activity source and find the forcing that is consistent with the wave source. The method essentially amounts to specifying the covariability between the vorticity forcing and the vorticity, where the vorticity related to the forcing by the linearized barotropic vorticity equation. The initial motivation for our method came from experiments designed to test the ideas of Chen et al. (2007): changes in wave phase speeds are responsible for the poleward shift of the jet in response to external forcing that acts to make the jet stronger. We found that prescribing changes in the phase speed of the forcing led to unintended increases or decreases in wave amplitude depending on whether the new
phase speed is closer to the resonance frequency of the free modes of the barotropic model. This made it impossible to separate the contributions of phase speed and wave amplitude to the latitudinal shift of the jet (Chen et al. (2007) found that increasing the wave amplitude also led to a poleward shift, for example). Another problem with forcing the vorticity equation directly is that there is no good way to specify the space-time structure of the forcing based on diagnostics from a General Circulation Model (GCM) or observations. It is easy to see how a control run of barotropic model with forcing at high versus low phase speeds will be most sensitive to external forcing at different locations solely due to the choices involving the random forcing. By taking our approach, the RWC model can be compared directly to a GCM or to observations using the convergence of the vertical EP flux (Edmon et al. (1980)) in the upper troposphere as the wave source, and moreover the forcing amplitude is automatically tuned so that resonance is not an issue. DelSole (2001) also specified the magnitude of the wave activity source at each latitude in a linearized barotropic model. The temporal structure of the forcing, however, was assumed to be white noise, so the space-time structure of the wave activity source is not necessarily realistic. More importantly, specifying the temporal structure makes it impossible to change the phase speed of the waves independently of changes in the background zonal wind. In our approach, on the other hand, the entire phase speed-latitudewavenumber structure of the wave activity source is specified and can be manipulated to better understand the dynamics.

In this paper, we develop and test a quantitative model of RWC. In later papers we will use RWC to separate and understand dynamical mechanisms (Lorenz (2013a); Lorenz (2013b)). In sections 2 and 3, we describe the GCM experiments and the implementation of RWC in a linearized barotropic model. Next we compare the RWC simulations with the GCM (section 4). In section 5, we explore the factors that control changes in the phase speed spectrum of the wave activity source and develop a simple quantitative model of the changes in phase speed. In section 6, we couple the RWC model to a simple barotropic model of zonal-mean zonal wind anomalies and show how the coupled RWC model reproduces the response of the GCM to reduced friction and the evolution of the GCM’s internally generated variability. We end with a brief summary.

2. GCM experiments and diagnostics

The GCM used in this series of papers is the grid point model described by Held and Suarez (1994). The horizontal resolution is 2.5x2 degrees in longitude and latitude, with 20 equally spaced sigma levels in the vertical. The model uses an Arakawa C-Grid and an explicit leapfrog scheme with a 4.8-minute time step. Details of the dynamical core are described in Suarez and Takacs (1995). All model experiments were run for 4500 days from an isothermal initial condition. Only the final 4000 days were used for analysis. The instantaneous zonal and meridional winds ($u$, $v$), potential temperature ($\theta$) and surface pressure ($p_s$) were archived every 8 hours for analysis. Because the boundary conditions and dynamics are symmetric about the equator, we average the results of both hemispheres together to decrease sampling noise.

The standard idealized forcing given in Held and Suarez (1994) is used as the control run. To understand the response to external forcing we perturb the parameters of the control run in six ways. Following Robinson (1997) and Chen et al. (2007), we decrease the near-surface Rayleigh friction but only the component acting on the zonal-mean $u$, since this isolates the essential processes important for the poleward shift. We also run a separate integration with friction reduced everywhere. Following Haigh et al. (2005), Williams (2006) and Lorenz and DeWeaver (2007), we increase the tropopause height by decreasing the thermal equilibrium temperature of the stratosphere. Following Gerber and Vallis (2007), we increase 1) the pole-to-equator temperature difference of the thermal equilibrium profile and 2) the rate of relaxation to thermal equilibrium by reducing the thermal relaxation profile everywhere by a prescribed factor. Finally, we increase tropical static stability parameter in Held and Suarez (1994). The details of the six experiments and the acronyms used to reference the individual experiments are given in Table 1.

All of the above perturbations have the direct effect of increasing the strength of the jet and therefore, consistent with Kidston and Vallis (2012), they also shift the jet poleward (Fig. 1). The experiments with more poleward shift per degree of strengthening have larger negative anomalies on the equatorward flank of the time-mean jet (shaded) relative to the positive anomalies on the poleward side. Experiments in this class include TSTRAT, $\Delta \theta_{\text{TROP}}$, FRIC and possibly ZFRIC. For a given amount of strengthening, $\Delta T'_y$ and RAD have a relatively small amount of poleward shift. We discuss these runs in more detail below.
Table 1: Perturbed GCM experiments.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Control Value</th>
<th>Perturbed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZFRIC</td>
<td>decrease friction on zonal mean $u$</td>
<td>$k_f = 1 \text{ day}^{-1}$</td>
<td>$k_f = 0.8 \text{ day}^{-1}$ on zonal mean $u$</td>
</tr>
<tr>
<td>TSTRAT</td>
<td>decrease stratosphere temp.</td>
<td>$T_{strat} = 200 \text{ K}$</td>
<td>$T_{strat} = 195 \text{ K}$</td>
</tr>
<tr>
<td>$\Delta T_y$</td>
<td>increase pole-to-equator temp. diff.</td>
<td>$\Delta T_y = 60 \text{ K}$</td>
<td>$\Delta T_y = 70 \text{ K}$</td>
</tr>
<tr>
<td>$\Delta \theta_z, TROP$</td>
<td>increase tropical static stability</td>
<td>$\Delta \theta_z = 10 \text{ K}$</td>
<td>$\Delta \theta_z = 13 \text{ K}$</td>
</tr>
<tr>
<td>RAD</td>
<td>decrease thermal relaxation time scale</td>
<td>$k_a = 1/40 \text{ day}^{-1}$ and $k_s = 1/4 \text{ day}^{-1}$</td>
<td>$k_a = 1/35 \text{ day}^{-1}$ and $k_s = 1/3.5 \text{ day}^{-1}$</td>
</tr>
<tr>
<td>FRIC</td>
<td>decrease friction</td>
<td>$k_f = 1 \text{ day}^{-1}$</td>
<td>$k_f = 0.8 \text{ day}^{-1}$</td>
</tr>
</tbody>
</table>

Figure 1: The climatological zonal-mean zonal wind in the control run (shaded) and the change in zonal-mean zonal wind for the six experiments in Table 1 (contours) (m s$^{-1}$). The x-axis is latitude (degrees) and the y-axis is pressure (hPa).

To apply a barotropic model of the eddies to diagnose the multi-level GCM, we must prescribe a single level background zonal wind and absolute vorticity field. To this end, we calculate the vertical Empirical Orthogonal Functions (EOFs) of $v$ over longitude and time for each latitude. The values of the first $v$ EOF define the relative weights for averaging the multi-level zonal-mean zonal wind ($\bar{u}$) and $\beta$ ($= a^{-1} \partial_\phi (f + \zeta)$ where $\phi$ is the latitude) into a single number for each latitude. Note that longitude and time are treated the same in the calculation of the EOFs. The leading $v$ EOF peaks at the tropopause and its structure is more barotropic at high latitudes than at low latitudes. The variations in latitude and pressure of the local $v$ EOF scale height are approximately proportional to $f/N$, where $f$ is the Coriolis force and $N$ is the buoyancy frequency. We calculate the multi-level $\beta$ first and then we apply the vertical weighting. Unless explicitly stated otherwise, this same vertical weighting is applied to all GCM diagnostics that involve a single vertical level of $\bar{u}$.

Space-time cross-spectral analyses of eddy fluxes are calculated using the method of Randel and Held (1991) except that phase speed spectra are given in terms of angular phase speed ($c_\omega$) instead of phase speed. The temporal spectral analysis is performed over 64 day chunks (= 192 times given the 8 hour sampling time) that overlap by 32 days. The resulting frequency spectrum is further smoothed with a running mean over five adjacent frequency bands. Typically, the stationary phase speed/frequency is not considered because the effects of strong stationary wave sources dominate the observed phase speed spectra. In our GCM with axisymmetric boundary conditions, however, the stationary frequency (phase speed) is not enhanced relative to adjacent frequency bands and we therefore consider the full phase speed spectrum.

For all figures in this paper, angular phase speed is given in terms of velocity at 45° latitude in m s$^{-1}$. In other
words, our angular phase speed \(c\) is related to the angular phase speed in \(\text{rad s}^{-1}\) \((c_\omega)\) by 
\[c = c_\omega a / \cos(45°),\]
where \(a\) is the radius of the earth. The resolution in phase speed for all analysis and figures is \(2 \text{ m s}^{-1}\).

## 3. Implementing Rossby wave chromatography

The starting point for our implementation of RWC, is the barotropic vorticity equation on a sphere linearized about a background zonal-mean zonal wind, \(\bar{u}\), and absolute vorticity gradient, \(\beta = a^{-1} \partial_\phi (f + \hat{\zeta})\),
\[
\frac{\partial \zeta'}{\partial t} + \frac{\bar{u}}{a \cos \phi} \frac{\partial \zeta'}{\partial \lambda} + \beta v' - \nu \nabla^2 \zeta' = F',
\]
where \(\zeta', v'\) and \(F'\) are the eddy relative vorticity, meridional velocity and vorticity forcing, \(\phi\) is the latitude, \(\lambda\) is the longitude, \(a\) is the radius of the earth, \(\nu\) is the diffusion coefficient. If the eddy variables are written in the form
\[
\eta' = \eta(\phi) \exp(i m (\lambda - c_\omega t)),
\]
where the unprimed \(\eta\) is a complex amplitude that depends only on \(\phi, m\) is the integer zonal wavenumber, and \(c_\omega\) is the angular velocity, then
\[
im \left( \frac{\bar{u}}{a \cos \phi} - c_\omega \right) \zeta + \beta v - \nu \nabla^2 \zeta = F,
\]
where of course it is straightforward to write \(v\) and \(\nabla^2 \zeta\) in terms of \(\zeta\) if desired. The wave activity equation can be found by taking the complex conjugate of (3), multiplying by \(\cos \phi \zeta' / (2\beta)\) and taking the real part
\[
\frac{\cos \phi}{2\beta} \Re(\zeta F^*) - \frac{\nu \cos \phi}{2\beta} \Re(\nabla^2 \zeta^*) = \frac{\cos \phi}{2\beta} \Re(\zeta F^*).
\]
The factor of one half comes from the fact that the zonal mean of two waves with complex amplitude \(\eta\) and \(\xi\) is \(\Re(\eta \xi^*)/2\). The first term on the left is the divergence of the wave activity flux \((- \cos \phi \Re(\bar{uv}^*)/2\)), the second term is the wave activity sink/source from diffusion and the term on the right is the wave activity source, which we want to prescribe. This source term is set equal to the convergence of the vertical component of the wave activity flux
\[
\frac{\cos \phi}{2\beta} \Re(\zeta F^*) = - \frac{\partial}{\partial p} \left( \frac{f \cos \phi \Re(\theta v^*)}{2} \frac{\partial}{\partial \theta} \right).
\]
In this paper, the analysis is performed as a layer average from \(\sigma_t = 0.125\) to \(\sigma_b = 0.525\), so (5) becomes, after transforming to \(\sigma = (p/p_e)\) coordinates and rearranging,
\[
\Re(\zeta F^*) = \frac{\beta}{\sigma_b - \sigma_t} \left[ \Re \left( \frac{f}{\partial_\theta} \frac{\partial}{\partial p} \right) \right] - \left[ \Re \left( \frac{f}{\partial_\theta} \frac{\partial}{\partial p} \right) \right] \equiv S.
\]
Note we have defined the new (real) variable \(S\) to be the right hand side of (6).

Specifying the source of wave activity amounts to specifying the covariance between \(\zeta\) and \(F\). Discretizing (6) and writing in component form with a latitude index, \(j\), and writing (3) as a matrix equation in \(\zeta\) and \(F\) as \(\zeta_j = \sum_k A_{jk} F_k\), we need to solve
\[
S_j = \Re(\zeta F^*_j)
\]
\[
= \Re\left(\sum_k A_{jk} F_k F^*_j\right)
\]
where we do not follow the repeated index summing convention and all indices run from 1 to \(n\), the number of latitudes. Equation (7) is under-constrained because \(F_j\) are complex. One can solve (7) if one provides additional constraints, for example, one can specify that all \(F_j\) are real. Equation (7) assumes that the forcing \((F_j)\) is coherent at all latitudes (i.e. the forcing at high latitudes always has the same phase and amplitude relationship to forcing at low latitudes). This assumption works satisfactorily for many \(m\) and \(c_\omega\), but the solution is subject to noise in \(F_j\) at some \(m\) and \(c_\omega\). Therefore, we assume that we have a large number, \(T\), of possibly independent “realizations” of the forcing, \(F_{jt}\), where \(t\) is an index over the realizations. The individual realizations are analogous to the partitioning of data into separate chunks when doing cross spectral analysis. For each realization, we assume
\[
\zeta_{jt} = \sum_k A_{jk} F_{kt}.
\]
The total “flux” that we need to set equal to \(S\) is the average over all realizations
\[
S_j = \frac{1}{T} \Re\left(\sum_{kt} A_{jk} F_{kt} F^*_j\right)
\]
\[
= \Re\left(\sum_k A_{jk} \frac{1}{T} \sum_t F_{kt} F^*_j\right)
\]
\[
= \Re\left(\sum_k A_{jk} C_{kj}\right),
\]
where \(C_{kj}\) is a complex covariance matrix. We close the problem by specifying the correlation structure \(C_{kj}\) so that
found in Appendix A. 

\[ g \] solutions are highly oscillatory in latitude and we find that complex solutions, where \( g \) is the latitude. Almost all solutions are highly oscillatory in latitude and we find that by simply choosing a smooth and broad initial guess we can converge on the relevant smooth \( g \). The details can be found in Appendix A.

Once \( g \) is found the momentum flux \( (u'v')_j \), for example, is

\[ (u'v')_j = \frac{1}{2} \Re \left( \sum_{kl} U_{jk} V_{kl} \hat{C}_{kl} g_j g_k \right), \]

where \( U_{jk} \) and \( V_{jk} \) are matrices that relate the forcing, \( F_k \), to the zonal and meridional wind, respectively. All other fluxes can be calculated in a similar way. The above method gives the solution for a particular wavenumber, \( m \), and angular velocity, \( c_\omega \). The total solution is simply the sum of the fluxes over all \( m \) and \( c_\omega \).

The RWC model is applied to the full sphere. In all cases, the background flow that forces the model is symmetric, therefore we only show the results in the Northern Hemisphere. The matrix \( A_{jk} \) and the calculation of \( u \) from the stream-function, \( \psi \), are all approximated using simple, centered second-order finite differences on the same grid as the GCM. The boundary conditions are \( \psi = 0 \) and \( \zeta = 0 \) at the poles. There are two free parameters: the diffusivity, \( \nu \), and the latitudinal scale in the correlation matrix, \( b \). The same \( \nu \) and \( b \) were used for all included wavenumbers, \( m \), and angular phase speeds, \( c_\omega \). Only \( m \) up to 20 are considered. The model was fit by trial-and-error by simply sweeping through parameter space and using the parameters that gave the smallest error for the momentum flux after integrating over \( m \) and \( c_\omega \). The conclusions of this paper are not dependent on the size of these parameters, which mostly determine the amplitude of the fluxes. For example, both \( b = 0 \) (no coherence) and \( b = \infty \) (perfect coherence) lead to the same conclusions regarding mechanisms.

4. Application of Rossby wave chromatography to GCM

In this section, we look at the performance of RWC in simulating the momentum fluxes, \( \overline{u'v'} \), given the background zonal-mean flow \( (\bar{u} \text{ and } \beta = a^{-1}\partial_\phi (f + \bar{\zeta})) \) and the space-time structure of the wave activity source (= convergence of the vertical component of EP flux (Edmon et al. (1980))). Next, we look at the ability of RWC to simulate the changes in \( \overline{u'v'} \) in response to external forcing given the changes in background flow and in the wave activity source. The interpretation of the changes seen here is reserved for later papers. For all comparisons, the GCM momentum fluxes are averaged from \( \sigma = 0.125 \) to \( \sigma = 0.525 \) exactly like the convergence of the vertical component of EP flux used to force the RWC model.

As mentioned above, the momentum fluxes integrated over all wavenumbers, \( m \), and angular phase speeds, \( c_\omega \), were used to tune the two parameters of RWC (Fig. 2a). The largest errors are poleward of 50° where RWC does not simulate enough negative \( \overline{u'v'} \) (i.e. there is too little poleward propagating wave activity). In addition the main positive center of \( \overline{u'v'} \) is shifted slightly poleward in the RWC model. The latitude/phase speed spectrum (Randel and Held (1991)) of the wave activity source, which is specified in RWC, peaks on the equatorward flank of the jet but decays much more quickly to zero on the equator side than on the pole side of the jet (Fig. 2b). The latitude/phase speed spectrum of \( \overline{u'v'} \) in the GCM shows that poleward (equatorward) momentum (wave activity) fluxes dominate–this is a fundamental asymmetry caused by the spherical geometry. The momentum fluxes are weighted more toward lower phase speeds compared to the wave source. The RWC \( \overline{u'v'} \) reproduces the general features in the GCM \( \overline{u'v'} \) but the distribution in latitude and phase speed tends to be less broad than the GCM (Fig. 2de). In particular, the waves in the RWC model do not propagate
The change in the total integrated $\overline{u'v'}$ for each of the six GCM experiments is compared with RWC in Fig. 3. RWC correctly simulates the variability in general structure (monopole versus weak dipole) and the variability in amplitude among the six experiments. In general, the RWC model simulation is too narrow and the nodal line is too far poleward when the $\overline{u'v'}$ changes are of the dipole type. Interestingly, when the RWC model is coupled to $\overline{u}$ these biases are significantly reduced (3a) (see section 6). The RWC model also tends to under-estimate the magnitude of the $\overline{u'v'}$ changes.

The change in the latitude/phase speed spectrum of $\overline{u'v'}$ for each of the six GCM is compared with RWC in Fig. 4. The most obvious change is an increase in $\overline{u'v'}$ across the jet axis. In some experiments there are significant negative $\overline{u'v'}$ changes and these tend to occur at lower phase speeds than the positive changes. In general, RWC gets all the features of the full GCM and one would have no problem matching the correct RWC simulation with the corresponding GCM if the figures were randomly scrambled. The RWC model has a negative bias just poleward of the critical level on the poleward flank of the jet. This feature is also present in the GCM but its amplitude is below the contour interval. Like the control run latitude/phase speed spectrum, the changes in the RWC spectra tend to be too sharp in latitude and phase compared to the GCM.

While there are clearly significant biases in the RWC model, the RWC simulation reproduces all the general features of the GCM and we believe is similar enough to the GCM that the mechanisms operating in the RWC model are likely the same mechanisms operating in the GCM. Similarly, we believe that mechanisms that do not operate in the RWC likely do not operate in a significant way in the GCM.

**5. Change in wave source phase speeds**

Below we couple the RWC model to $\overline{u}$. Because the wave activity source can potentially change with the background flow, we need to be able to model the changes in wave activity source given the background flow. In this section, we propose a very simple model of the changes in wave activity source phase-speed spectra in response to changes in $\overline{u}$. We consider only the phase speed changes in this paper because the focus is on the simpler case of nearly barotropic anomalies. Based on the results below,
we believe that to first order the dynamics related to mechanical forcing and the internal variability can be described without considering changes in wave activity source magnitude. In future work, we will couple the RWC model to the zonal-mean circulation in the latitude-pressure plane and consider the effects of changes in source magnitude.

a. Diagnosing GCM

First, we want to come to a general understanding of the changes in the phase speed of the wave activity source magnitude in the GCM. Equation (3) leads to the following dispersion relation:

\[ c = \frac{\bar{u}}{a \cos \phi} - \frac{\beta a \cos \phi}{m^2 + l^2}, \quad (13) \]

where \( l \) is the meridional wave number. While the changes in \( \bar{u} \) and \( \beta \) can be estimated directly from the zonal-mean flow the changes in \( m^2 + l^2 \) cannot. In this paper we estimate the total wavenumber squared in the GCM using

\[ m^2 + l^2 = \frac{\zeta^2}{\psi'^2}, \quad (14) \]

where \( \zeta' \) and \( \psi' \) are the eddy relative vorticity and streamfunction, and the caret denotes averaging in longitude and weighting in the vertical by the scheme in Section 2.

The percent change in \( m^2 + l^2 \) and \( \beta \) for the GCM runs ZFRIC and \( \Delta T_y \) is shown in Fig. 5a. The GCM run \( \Delta T_y \) was chosen because it has the biggest scale changes. The changes in \( \beta \) dominate over the changes in scale, and moreover the change in scale for ZFRIC is nearly zero. The changes in scale are small enough that (13) can be linearized about the control run and the contribution of the individual terms evaluated. Fig. 5b shows the decomposition for the \( \Delta T_y \) run. The advection term dominates the phase speed changes and the \( \beta \) term dominates over the scale term.

So far we have estimated the changes in wave phase speed based on the dispersion relation but we have not looked directly at the actual changes in phase speed. To quantify the actual phase speed change in the GCM, we use linear regression to relate the change in wave source to the mean and derivative of the mean wave source:

\[ \Delta S = a_0 + a_1 S + a_2 \left( -\frac{\partial S}{\partial c} \right), \quad (15) \]

where \( a_0, a_1 \) and \( a_2 \) are constants. For infinitesimal perturbations, \( a_2 \) is the shift of the spectrum in phase speed. The actual phase speed changes are about the same magnitude as the phase speeds estimated from the background flow, but the actual phase speed change has a broader distribution in latitude particularly on the equatorward flank (Fig 5c, for ZFRIC). These results suggest that (13) could be used in a parameterization of the effect of \( \bar{u} \) on phase speeds and that a more quantitatively accurate model might involve smoothing the background flow and/or the predicted phase speed changes in latitude.
Figure 4: a) The change in the latitude/phase speed spectrum of eddy momentum flux in the GCM for ZFRIC (m s$^{-1}$). The gray line is the critical level. The x-axis is angular phase speed in m s$^{-1}$ at 45$^\circ$ (see section 2). b) as in a) but for RWC. c) as in a) but for TSTRAT. d) as in b) but for TSTRAT. e) as in a) but for $\Delta T_y$. f) as in b) but for $\Delta T_y$. g) as in a) but for $\Delta T_{TROP}$. h) as in b) but for $\Delta T_{TROP}$. i) as in a) but for RED. j) as in b) but for RAD. k) as in a) but for FRIC. l) as in b) but for FRIC.

b. Simple model of phase speed changes

We now describe a simple model of the phase speed changes based on the dispersion relation (13). We found above that changes in the horizontal wavenumber are small, therefore we assume that $m^2 + l^2$ is constant, so that the phase speed in a perturbed state (subscript 2) can be related to that in the control (subscript 1) by:

$$c_{\omega 2} = \bar{u}_2 \cos \phi + \frac{\beta_2}{\beta_1} \left( c_{\omega 1} - \frac{\bar{u}_1}{a \cos \phi} \right).$$  (16)

We found above that the actual wave source phase speed changes appear to be smoothed in latitude relative to predictions based on the dispersion relation. Therefore, for our simple model of the phase speed change, we smooth $\bar{u}_1$, $\bar{u}_2$, $\beta_1$ and $\beta_2$ with a gaussian kernel, $\exp(-\phi^2/a_c^2)$, where $a_c$ is 17$^\circ$ latitude, and then we apply (16). The predicted phase speed change is then used to shift the phase speed spectrum of the control run in a conservative way by mapping the power in each phase speed bin to the two bins closest to the new predicted phase speed. To compare the simple model to the GCM, we remove the component of the GCM’s change involving changes in source magnitude by rescaling the perturbed wave activity source so that the sum over phase speed is the same as the control for each
The difference in wave activity source phase-speed spectrum between the rescaled perturbed run minus the control is shown for the reduced zonal-mean friction run (Fig. 6a) and the increased pole-to-equator temperature gradient run (Fig. 6c). The simple model does a reasonable job predicting the magnitude and structure of the phase speed changes (Fig. 6b and Fig. 6d). The largest differences are on the poleward flank of the jet. One unsatisfactory aspect of this simple model is the arbitrary smoothing parameter, $a_c$. While this smoothing parameter impacts the phase speed changes seen in Fig. 6, we will see that, due to the integrating nature of phase speed changes, this smoothing parameter has amazingly little impact on the momentum fluxes when we use these predicted wave sources to force the RWC model (Lorenz (2013b)).

6. Coupling the RWC model to the Zonal Wind

a. Response to reduced friction

In this section, we describe how the RWC model is coupled to $\bar{u}$ in a way that is consistent with the GCM.
We also compare the evolution and equilibrium response of the coupled RWC to the GCM. In this paper, RWC is coupled to a one layer version of \( \bar{u} \):

\[
\frac{dz}{dt} = \alpha h - \frac{z}{\tau} + F_z, \tag{17}
\]

where \( z \) is the modelled one-layer \( \bar{u} \) anomaly, \( h \) is the eddy vorticity flux anomaly in the RWC model when \( z \) is added to the climatological background wind, \( \alpha \) and \( \tau \) are constants that need to be determined and \( F_z \) is the external forcing. Following Lorenz and Hartmann (2001), we estimate the parameters \( \alpha \) and \( \tau \) through a cross spectral analysis of the analogs of \( z \) and \( h \) in the GCM. Unlike Lorenz and Hartmann (2001), who consider vertical averages, the \( z \) is the \( \bar{u} \) weighted as described in section 2 and \( h \) is the eddy vorticity flux averaged from \( \sigma = 0.125 \) to \( \sigma = 0.525 \). While vertical averaging makes more sense from the perspective of the momentum budget, the RWC model involves \( \bar{u} \) and \( \bar{u}\bar{v}' \) that are weighted as above. In practice, the budget represented by (17) is nearly as closed as before: the cross spectral coherence drops by only 0.01 when we follow the RWC weighting. The constants \( \alpha \) and \( \tau \), however, do change with the weighting (\( \alpha \) is nearly one in Lorenz and Hartmann (2001)). For the control run, \( \alpha = 0.60 \) and \( \tau = 9.36 \) days (see Appendix B).

We now design a coupled RWC experiment that is analogous to the ZFRIC GCM experiment. The latitudinal profile of \( F_z \) in this experiment is the profile of the mechanical Rayleigh damping integrated though the depth of the GCM. In the climatology, the average \( \bar{u} \) in the Rayleigh friction layer peaks at 10 m s\(^{-1}\). Therefore a 20% reduction in the zonal-mean component of friction (i.e. the ZFRIC integration) is analogous to a \( F_z \) that leads to a 2 m s\(^{-1}\) increase in the vertically averaged wind at the jet core. To convert this to “z”, we also need the typical ratio of the RWC weighted \( \bar{u} \) to the vertically averaged \( \bar{u} \), which is 1.16 (see Appendix B). Therefore, for the coupled RWC integration of the ZFRIC GCM run, \( F_z = 1.16 \cdot 2/\tau \) m s\(^{-1}\) day\(^{-1}\). For this particular integration we also want to use a different value for \( \tau \); \( \tau_2 = 11.07 \) days. This value is calculated from the ZFRIC run and is only used in (17) not in the determination of \( F_z \). Note that \( \tau_2 < \tau/0.8 \), presumably because \( \bar{u} \) anomalies are more barotropic in the ZFRIC GCM run.

The coupled model is integrated for 200 days from the initial condition \( z = 0 \). We found that by smoothing \( \zeta \bar{v}'_{RWC} \) in latitude, we could take a significantly larger time step (= 8 hours) without numerical instability but with negligible impact on the simulation. Therefore we apply a simple 1-2-1 smoothing filter to \( \zeta \bar{v}' \cos^{2} \phi \) before forcing \( z \). The smoother operates on \( \zeta \bar{v}' \cos^{2} \phi \) instead of \( \zeta \bar{v}' \) because then it is easy to satisfy the constraint that \( \zeta \bar{v}' \cos^{2} \phi \) must integrate to zero over \( \phi \).

The response of the coupled RWC model is quite similar to the GCM response to a 20% reduction in the zonal-mean component of friction (Fig. 7ab). RWC tends to reduce (increase) \( \bar{u} \) too much on the equatorward (poleward) side of 50°. For \( \bar{u}\bar{v}' \), RWC exhibits a poleward shift of the response relative to the GCM and it exaggerates the amplitude of the \( \bar{u}\bar{v}' \) decreases in the subtropics relative to the \( \bar{u}\bar{v}' \) increases in the mid-latitudes.

In Fig. 8, we show the anomalies in \( \bar{u} \) and \( \bar{u}\bar{v}' \) for the spin-up of the RWC model to reduced friction conditions. The zonal wind initially becomes stronger over the first few days and then begins shifting poleward almost immediately (Fig. 8a). The adjustment of \( \bar{u} \) to its eventual amplitude, however, takes place on a longer time scale. The presence of at least two time scales is more evident in the momentum flux (Fig. 8b). For example, the reductions in \( \bar{u}\bar{v}' \) in the subtropics occur quickly over the first 15 days and then actually weaken slightly afterwards. The \( \bar{u}\bar{v}' \) increases on the poleward flank of the jet, on the other hand, gradually equilibrate over the entire 100 day time.
Period. This time scale contrast is also evident in the “average” spin-up of the GCM to an instantaneous reduction in the zonal-mean component of friction. Here the “average” GCM spin-up is an average of 800 100-day integrations with initial conditions taken in 5 day increments from the control run. We also average the results from the southern and northern hemispheres together which implies we have effectively 1600 spin-ups. The presence of two GCM time scales is most evident in the $u'v'$ evolution although the long time scale is longer in the GCM and the short time scale is shorter in the GCM (Fig. 8d). Comparison of the GCM and RWC $u$ evolution is hampered by the negative $u$ bias in the subtropics of the coupled RWC model. The fact that the RWC model reproduces the presence of the two time scales implies that the explanation does not involve 1) the dynamics of the zonal-mean circulation, 2) multiple time scales of adjustment to different diabatic processes or 3) differing time scales among different eddy processes. Instead, the different time scales represent different amplitudes and/or sign of the eddy momentum flux feedback for the $z$ “modes” of the system. For example, suppose the RWC model is linearized: $h = Bz$, where $B$ is a matrix. Let $\lambda_j$ be the eigenvalues of the matrix $B$, then the time scales for the adjustment of the different eigenvectors or “modes” is $(1/\tau - \alpha \lambda_j)^{-1}$. The larger the positive feedback, $\lambda_j$, the larger the time scale.

The difference in time-scales implies that the magnitude of the eddy feedbacks are biased in the RWC. For EOF1 anomalies, the de-correlation time in the RWC is 22 days compared to 35 days in the GCM. This bias could be due to a “baroclinic feedback” or it could be simply a result of biases in our implementation of RWC: RWC tends to underestimate the response even when it is given the full change in the wave activity source spectrum (Fig. 3). Another source of bias might be the fact that the eddies respond instantaneously to changes in the background $\bar{u}$ in our implementation of RWC.

b. Internal variability

In this section, we confirm that the coupled RWC model can reproduce aspects of the internal variability of the GCM. Lorenz and Hartmann (2001) and Lorenz and Hartmann (2003) argue that the variability in the eddy momentum flux can be usefully partitioned into a random component and a component that is proportional to the $\bar{u}$ anomaly. To avoid the complications of characterizing the random component of $u'v'$, we only consider the evolution of $\bar{u}$ after the $\bar{u}$ anomalies have been set up. To this end, we
to perform a one-point lag regression analysis on the GCM $\bar{u}$ anomalies (e.g. Feldstein (1998)). The one-point lag regression of pentad $\bar{u}$ anomalies on the $\bar{u}$ anomaly at 22$^\circ$ is shown in Fig. 9a. The positive lags shown represent the typical future evolution of $\bar{u}$ following anomalies at 22$^\circ$. After one pentad the amplitude of the lag 0 pattern has weakened considerably. After that point the anomalies slowly migrate poleward as in Feldstein (1998). Switching to a base point of 34$^\circ$, we still see the rapid decay after one pentad but the anomalies remain stationary afterwards instead of propagating poleward.

To simulate this evolution in the coupled RWC model, (17) is integrated as an initial value problem with $F_z = 0$. The initial conditions are the $\bar{u}$ anomalies at lag 0 in Fig. 9ab. The RWC model reproduces the poleward propagation versus the persistence of the anomalies associated with the base points 22$^\circ$ (Fig. 9c) and 34$^\circ$ (Fig. 9d). However, the RWC model does not simulate the rapid transient decrease in amplitude seen in the GCM.

It turns out that the rapid transient decay is due to a transient negative eddy feedback. To see this consider a large number of initial value problems taken from a control run of the GCM but with the zonal-mean state modified to be like either the positive of negative phase of EOF1. After averaging over 1460 initial value problems and both hemispheres and taking the difference between the positive and negative phase of EOF1 we get Fig. 10 (This is a standard Held and Suarez (1994) run but with the resolution half that of all other experiments shown here). The vertically averaged $\bar{u}$ anomaly decays rapidly over the first few days and then afterwards is maintained at a relatively constant amplitude. The vertically averaged $\bar{u}$ anomaly initially acts to damp the $\bar{u}$ anomalies and then reverses sign and maintains the anomalies. Watterson (2002) found similar behavior in a barotropic model initialized with EOF1-type anomalies. Looking at the momentum budget in more detail it is clear that $\bar{u}'v'$ “explains” the behavior of the $\bar{u}$ anomalies (not shown). This transient effect is not a result of nonlinearities in the response of $u'$ to $\bar{u}$ amplitude as experiments with half amplitude EOF1 anomalies demonstrate (not shown). Instead we believe this is a transient negative response to any changes in $\bar{u}$. The EOF1 example shown is the most dramatic case because the eddy forcing reverses in time. The dynamics of this transient negative feedback is the subject of future work. Note that this negative feedback is evident in the lag correlations in, for ex-
Figure 10: a) Average response of $\bar{u}$ in the GCM to positive EOF1 anomalies in the zonal-mean state minus the average response of $\bar{u}$ to negative EOF1 anomalies in the zonal-mean state ($\text{m s}^{-1}$). The average response is an average over 1460 initial value problems with prescribed zonal-mean state but with eddy fields taken at regular intervals from a long control run. b) Same as a) but for $u'v'$ ($\text{m}^2\text{s}^{-2}$).

Figure 11: a) Estimate of the change in amplitude at lag 2 pentads relative to the lag 0 one-point regression plot of $\bar{u}$ in the GCM (thin solid). The change in amplitude is shown as a function of the latitude of the base point ($x$-axis). The change in amplitude for the coupled RWC model at 5 days and scaled by a factor of 0.74 (see text) for initial conditions taken from the lag 0 one-point regression plots of the GCM (thick dotted). b) Estimate of the shift in latitude at lag 2 pentads relative to the lag 0 one-point regression plot of $\bar{u}$ in the GCM (thin solid). The shift in latitude is shown as a function of the latitude of the base point ($x$-axis). The shift in latitude for the coupled RWC model at 5 days for initial conditions taken from the lag 0 one-point regression plots of the GCM (thick dotted).

ample, Lorenz and Hartmann (2001) and Lorenz and Hartmann (2003), but they attributed this negative effect to the random burst of momentum flux immediately preceding the peak $\bar{u}$ anomaly rather than to background $\bar{u}$ itself.

Since the RWC model is a steady state model, we do not expect it to capture the above transient negative feedback. However, it appears that the transient feedback has a similar effect on all $\bar{u}$ anomalies so we can simply scale the RWC model $\bar{u}$ anomalies to account for the transient negative feedback. First, we need to quantify the degree of amplitude change and poleward shift. Let $u(\phi_j, t)$ be the latitudinal profile of the $\bar{u}$ anomalies at lag $t$ and latitude grid points $\phi_j$ for a particular base point. The relative amplitude ($= A$) and latitudinal shift ($= \phi_0$) at lag $t$ relative to lag 0 are the values of $A$ and $\phi_0$ that minimize

$$\sum_j (u(\phi, t) - Au(\phi - \phi_0, 0))^2.$$ \hspace{1cm} (18)

We minimize (18) using the Levenberg-Marquardt algorithm under the assumption that linear interpolation holds between grid points and that $u(\phi - \phi_0, 0) = 0$ when $\phi - \phi_0$ is outside of the domain. The relative amplitude and latitude shift of the GCM pattern at lag 2 pentads relative to lag 0 for all base points is shown in Fig. 11. The amplitude is most persistent for base points nearly in phase with the two dominant centers of action of EOF1 ($32^\circ$ and $52^\circ$, not shown). The amplitude peak is relatively flat because the EOF1 structure dominates the one-point regressions of all base points close to $32^\circ$ and $52^\circ$. There are relative minima at the latitude of the time-mean jet ($42^\circ$) and at two locations in the subtropics. The rate of propagation in latitude varies quite sharply in latitude with a sawtooth-type pattern (Fig. 11b). At points equatorward (poleward) of the poleward center of action of EOF1 ($52^\circ$), $\bar{u}$ anomalies tend to propagate poleward (equatorward) although there are some exceptions. Most notably, anomalies just poleward of the equatorward center of action of EOF1 tend to
propagate “back” equatorward toward this persistent center of action.

We analyze the RWC model in the same way except we compare \( t = 0 \) to \( t = 5 \) days because the RWC model evolves faster than the GCM. Also, to take into account the rapid transient decay in the GCM we scale the RWC model amplitude by 0.74. After these modifications, the RWC model does a good job simulating the internal variability of the GCM, particularly for the relative amplitude (Fig. 11). We believe that the RWC model is close enough to the GCM that we can learn useful information about the GCM’s dynamics through the analysis of the RWC model. In future work, we plan to characterize the “random component” of \( \hat{u}' \hat{v}' \) variability and to better understand and characterize the transient negative feedback.

7. Summary

Rossby Wave Chromatography (RWC) (Held and Phillips (1987)) is implemented in a linearized barotropic model. By RWC, we mean that we calculate the space-time (i.e. phase speed-latitude-wavenumber) structure of the upper tropospheric eddy momentum fluxes given the background zonal-mean flow and the space-time structure of the baroclinic wave activity source (convergence of vertical EP flux (Edmon et al. (1980)) in the upper troposphere). Unlike other studies of the forced barotropic vorticity equation, which specify the vorticity forcing directly (Vallis et al. (2004); Barnes et al. (2010); Barnes and Hartmann (2011); Kidston and Vallis (2012)), we specify the wave source and find the forcing that is consistent with the wave source. The method essentially amounts to specifying the covariability between the vorticity forcing and the vorticity, where the vorticity related to the forcing by the linearized barotropic vorticity equation. Some advantages of this technique are: 1) the model can be compared directly to a GCM or to observations using the convergence of the vertical EP flux (Edmon et al. (1980)) in the upper troposphere. Unlike other studies of the forced barotropic vorticity equation, 2) prescribing observed wave activity sources eliminates almost all choices regarding the space-time structure of the forcing, 3) the wave activity source is more closely related to the momentum fluxes (i.e. the meridional wave activity flux) than the vorticity forcing, and 4) the wave activity source and the background flow are decoupled so that one can be changed without impacting the other.

The RWC model reproduces the important features of the mean momentum fluxes of the GCM and the responses of the momentum fluxes to poleward shifted jets in response to changes in friction, pole-to-equator temperature gradient, tropopause height, radiative relaxation time scale or tropical static stability. We also develop a simple model of the changes in the phase speeds of the wave activity sources. This simple phase speed model and the RWC model are coupled to a one-layer version of \( \bar{u} \) anomalies. The coupled model successfully simulates the poleward shift in response to reduced friction and the evolution of GCM’s internal variability as diagnosed using one point lag regressions of \( \bar{u} \) (Feldstein (1998)). In parts II and III, we will use RWC to understand the dynamical mechanisms responsible for mid-latitude jet variability and change.

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APPENDIX A

Finding the forcing from the wave activity source

In this Appendix, the procedure for solving (11) is described. Since there are \( 2^n \) complex solutions, where \( n \) is the number of latitudes, we also describe numerical experiments intended to understand the character of the solutions. It turns out we have a good sense of the structure of all \( 2^n \) solutions and the relevant solution is simply the smoothest one with the least zero crossings.

We solve the system of quadratic equations using MINPACK (Moré et al. (1980)), which solves a system of nonlinear equations using by a modification of the Powell hybrid method. The form of the (real) system we need to solve is

\[
Q_j = \sum_k B_{jk} g_j g_k - S_j = 0,
\]

where \( B_{jk} \) and \( S_j \) are given, \( g_j \) is to be determined, all indices run from 1 to \( n \) and we do not follow the repeated index summing convention. The Jacobian matrix for the minimization algorithm is

\[
\frac{\partial Q_j}{\partial g_l} = B_{jl} g_j + \sum_k B_{lk} g_k.
\]

If the matrix \( B_{jk} \) is diagonal then the equations are uncoupled and the system is easy to solve:

\[
g_j = \pm \sqrt{S_j / B_{jj}}.
\]
Our initial condition for the numerical algorithm is based on (A3) but with a few modifications to make the algorithm more robust. Let $b_j$ be the diagonal elements of $B_{jk}$ and let $\epsilon = 10^{-4}$, then if $S_j > 0$ and $b_j > = \epsilon \max(b)$ then

$$f_j = \sqrt{S_j/b_j}. \quad \text{(A4)}$$

Note we take the positive solution of (A3) in (A4) for all cases. If $S_j < 0$ then $f_j = 0$ and if $b_j < \epsilon \max(b)$ then $b_j$ is set equal to $\epsilon \max(b)$. We find the algorithm is more robust if we add a constant positive offset to the initial condition. Therefore, we choose an initial $g_j$ that is proportional to $\tilde{g}_j$:

$$\tilde{g}_j = f_j + \frac{1}{n} \sum_j f_j. \quad \text{(A5)}$$

Finally, the initial $g_j$ is scaled by a factor $\gamma$ so that it has about the correct amplitude. If

$$\tilde{S}_j = \sum_k B_{jk} \tilde{g}_j \tilde{g}_k, \quad \text{(A6)}$$

then

$$\gamma = \frac{\sum_j \tilde{S}_j S_j}{\sum_j \tilde{S}_j^2}. \quad \text{(A7)}$$

To summarize, the initial $g_j$ is

$$g_j = \gamma \tilde{g}_j. \quad \text{(A8)}$$

We find that for all $m$ and $c$ with important wave activity sources the algorithm converges to a solution. Therefore, plots of the wave activity source derived from the $g_j$ are indistinguishable from the actual source, $S_j$. When $\beta$ gets very close to zero, however, the solution does not converge in the vicinity of the small $\beta$ region. It is possible that a resolution to this issue can be found by allowing the coherence in the forcing, $\tilde{C}_{kj}$, to vary in space in the low $\beta$ region. While $\beta$ is small on the poleward flank of the jet in our experiments, in none of the experiments shown here was it small enough for this to be an issue.

In the case of diagonal $B_{jk}$ it is easy to understand the structure of the $2^n$ solutions. In this case the solutions for each latitude are uncoupled but the sign of the solution is indeterminate. Therefore, there are two possible outcomes at each latitude (positive or negative) and $n$ independent latitudes implying there are the $2^n$ solutions. The individual solutions all have the same absolute value but have different numbers and locations of zero crossings. Similar behavior appears in our more general case when instead of initializing with all $g_j > 0$ we randomly change the sign of $g_j$ at some latitudes. An example for $m = 7$ and $c = 8$ m s$^{-1}$ is shown in Fig. 12. With highly oscillatory initial $g_j$, the algorithm converges to a valid solution with the same phase as the initial $g_j$. We have confirmed this result by running a large number of tests on all wavenumbers and phase speeds and with different frequency oscillations in $g_j$. When there are significant wave activity sources at a given latitude (the source magnitude is at least 1% of the maximum source for that $m$ and $c$), the sign of the final $g_j$ is the same as the initial $g_j$ 99.4% of the time. Since the most physical solution is likely the solution with the fewest oscillations, it appears that by initializing with all...
\(g_j > 0\) we are converging toward the most relevant solution in almost all cases (we have confirmed that the RWC simulation of the GCM is worse if we take a meridional wavenumber one solution to (A1)).

APPENDIX B

Finding the constants in the one-layer zonal wind equation

In this appendix, we determine the constants \(\alpha\) and \(\tau\) in (17). For this calculation, \(z\) is the principal component of the first EOF of the vertically weighted average \(\bar{u}\). The weights for the vertical averaging are described in section 2. The time series \(h\) is the vertically averaged eddy vorticity flux anomalies from \(\sigma = 0.125\) to \(\sigma = 0.525\) projected onto the above EOF. Because we are analyzing the internal variability, \(F_z = 0\) in this case.

First, it is most convenient to write (17) in the form:

\[
k_1 \frac{dz}{dt} + k_2 z = h, \tag{B1}
\]

where \(k_1\) and \(k_2\) are constants. Taking the Fourier spectrum of (B1):

\[
(2\pi ik_1 \nu + k_2) Z = H, \tag{B2}
\]

where capital letters denote the Fourier transform and \(\nu\) is the frequency. Multiplying by the complex conjugate of \(Z\) and rearranging:

\[
2\pi ik_1 \nu + k_2 = \frac{HZ^*}{ZZ^*}. \tag{B3}
\]

\(HZ^*\) is the cross spectrum and \(ZZ^*\) is the power spectrum. The real part of their ratio averaged from \(\nu = 0\) to \(0.25\) days\(^{-1}\) defines the constant \(k_2\). The slope in \(\nu\) of the imaginary part of their ratio defines \(2\pi k_1\) (the least squares slope is fit from \(\nu = 0\) to \(0.25\) days\(^{-1}\) ). Finally, \(\tau = k_1/k_2\) and \(\alpha = k_1^{-1}\).

The remaining constant is the ratio (\(\gamma\)) of the RWC weighted \(\bar{u} (= z)\) to the vertically averaged \(\bar{z} (= \bar{z})\), where \(z\) and \(\bar{z}\) are the first principal components of the corresponding variables. We define \(\gamma\) as the average real part of \(ZZ^*/ZZ^*\) from \(\nu = 0\) to \(0.25\) days\(^{-1}\).

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