A lattice Boltzmann study on the drag force in bubble swarms

J. J. J. GILLISSEN\(^1\), S. SUNDARESAN\(^2\) AND H. E. A. VAN DEN AKKER\(^1\)

\(^1\) Department of Multi-Scale Physics, J.M. Burgers Centre for Fluid Mechanics, Delft University of Technology, Prins Bernhardlaan 6, 2628 BW Delft, Netherlands.

\(^2\) Department of Chemical Engineering, Princeton University, Princeton, NJ 08544, USA.

(Received 8 March 2011)

Lattice Boltzmann and immersed boundary methods are used to conduct direct numerical simulations of suspensions of mass-less, spherical gas bubbles driven by buoyancy in a three dimensional periodic domain. The drag coefficient \(C_D\) is computed as a function of the gas volume fraction \(\phi\) and the Reynolds number \(Re = 2RU_{slip}/\nu\) for 0.03 \(\leq \phi \leq \) 0.5 and 5 \(\leq Re \leq \) 2000. Here \(R\), \(U_{slip}\) and \(\nu\) denote the bubble radius, the slip velocity between the liquid and the gas phases and the kinematic viscosity of the liquid phase, respectively.

The results are rationalized by assuming a similarity between the \(C_D(Re_{eff})\)-relation of the suspension and the \(C_D(Re)\)-relation of an individual bubble, where the effective Reynolds number \(Re_{eff} = 2RU_{slip}/\nu_{eff}\) is based on the effective viscosity \(\nu_{eff}\) which depends on the properties of the suspension. For \(Re \leq 100\) we find \(\nu_{eff} \approx \nu/(1 - 0.6\phi^\#)\), which is in qualitative agreement with previous proposed correlations for \(C_D\) in bubble suspensions. For \(Re \geq 100\) on the other hand, we find \(\nu_{eff} \approx RU_{slip}\phi\), which is explained by considering the turbulent kinetic energy levels in the liquid phase. Based on these findings a correlation is constructed for \(C_D(Re, \phi)\).

A modification of the drag correlation is proposed to account for effects of bubble deformation, by the inclusion of a correction factor based on the theory of Moore (1965).

1. Introduction

We examine the drag force acting on a statistically homogeneous and statistically steady assembly of monodisperse gas bubbles rising in a liquid under the action of buoyancy.

In experiments breakup and coalescence of bubbles result in a dynamic distribution of bubble sizes, and inhomogeneous distributions of bubbles arise due to wall effects. Furthermore at large gas volume fraction, the homogeneous bubbly flow transits to the heterogeneous bubbly flow, which is characterized by a strong coupling between large scale fluid circulation and bubble coalescence (Harteveld et al. 2003). In addition small amounts of impurities in the carrying liquid can have large effects on the bubble dynamics. For instance Maxworthy et al. (1996) used mixtures of triply distilled water and glycerin and measured a 30% larger drag coefficient than Duineveld (1995) who performed experiments in ‘hyper clean water’.

In simulations the above mentioned complexities can be excluded which provides a way to gain insight into the essential dynamics. In this work we use numerical simulations of
highly idealized systems and use the results to construct a model for the drag coefficient in bubble swarms.

In the present analysis we ignore effects of walls and polydispersivity. In this way the problem can be reduced to relating the drag coefficient $C_D$:

$$C_D = \frac{\frac{4}{3}R(1 - \phi)g}{U_{\text{slip}}^2},$$  \hspace{1cm} (1.1)$$
to three independent, dimensionless parameters. These parameters are chosen to be the gas volume fraction $\phi$, the bubble Reynolds number:

$$Re = \frac{U_{\text{slip}}2R}{\nu},$$  \hspace{1cm} (1.2)$$
and the bubble Weber number:

$$We = \frac{\rho U_{\text{slip}}^22R}{\sigma}.$$

(1.3)

Here $g$ is the gravitational acceleration, $\rho$ is the liquid mass density, $U_{\text{slip}}$ is the velocity difference between the gas and the liquid, $\nu$ is the liquid kinematic viscosity, $\sigma$ is the surface tension of the gas-liquid system and $R = (3V_B/4\pi)^{\frac{1}{3}}$ is the bubble equivalent radius, where $V_B$ is the bubble volume. The relation for the drag coefficient (Eq. 1.1) is derived in the Appendix.

For uniform flow around a spherical bubble with free-slip boundary conditions at the bubble surface, Mei & Klausner (1992) proposed the following $C_D(Re)$-correlation that asymptotically matches the theoretical cases of $Re \ll 1$ and $Re \gg 1$ as well as accurately approximates numerical simulation data in the intermediate regime:

$$C_D = \frac{16}{Re} \left\{ 1 + \left[ \frac{8}{Re} + \frac{1}{2} \left( 1 + 3.315Re^{-\frac{1}{2}} \right) \right]^{-1} \right\}^{-1}.$$

(1.4)

Departure from the spherical shape occurs when the hydrodynamic forces acting on the bubble surface exceed the surface tension forces, i.e. when $We \gtrsim 1$. For air bubbles rising in water, significant departure from the spherical shape to the ellipsoidal shape is expected when the bubble radius exceeds a value of $R \approx 0.5$ mm (Duineveld 1995), corresponding to an aspect ratio $\chi \approx 1.2$, a Weber number of $We \approx 1$, and a Reynolds number of $Re \approx 260$, respectively. At larger $We \approx 2.5$ when the bubble aspect ratio exceeds $\chi \approx 1.7$ a recirculation wake develops (Blanco & Magnaudet 1995). In the air-water system this corresponds to $R \approx 0.7$ mm and $Re \approx 520$. By increasing $We$ even further, the bubble shape transits from ellipsoidal to spherical cap (Maxworthy et al. 1996). Assuming that the bubble shape is ellipsoidal and that there is no recirculating wake, Moore derived a relation between $We$ and the bubble aspect ratio $\chi$ (Eq. (1.5) in Moore (1965)) as well as the leading order correction $G$ to the drag coefficient as a function of $\chi$ (Eq. (2.12) in Moore (1965)). Combining these relations gives a relation between $G$ and $We$, which for $We < 3.5$ can accurately be approximated by:

$$G = \left[ 1 - \left( \frac{We}{4} \right)^{1.16} \right]^{-0.92}.$$

(1.5)

Including this factor into Eq. (1.4) gives the following expression for $C_D$ in terms of $Re$
and $We$ for an individual deformable gas bubble:

$$C_D = \left[1 - \left(\frac{We}{4}\right)^{1.16}\right]^{-0.92} \frac{16}{Re} \left[1 + \frac{8}{Re} + \frac{1}{2} \left(1 + 3.315Re^{-\frac{1}{2}}\right)^{1.16}\right]^{-1}. \quad (1.6)$$

Eq. (1.6) is valid in the absence of a recirculating wake, which for the air-water system is up to $We \approx 2.5$. Within this range Eq. (1.6) is within 15% of the experimental data of Duineveld (1995). While the $C_D$ for an individual rising gas bubble can be captured reasonably well by a relation such as Eq. (1.6), there is to date no equivalent relation for bubble swarms.

The aim of the present paper is to propose a $C_D$-relation that captures and thereby provides insight into the dynamics of rising bubble swarms over a wide range of $\phi$ and $Re$. The complexity of bubble swarms at large $Re$ stems from the combination of bubble deformation and random velocity fluctuations in the interstitial liquid, referred to as pseudo-turbulence.

We tackle this complexity in two steps. First we ignore bubble deformation and conduct direct numerical simulations (DNS) of spherical gas bubbles. From the simulation data we derive a $C_D(Re, \phi)$-relation where the role of the pseudo-turbulence is incorporated via an effective viscosity. Second we extend the model to account for bubble deformation by including the distortion factor $G$ (Eq. 1.5) into the expression for $C_D$. The final result constitutes a $C_D(Re, \phi, We)$-relation. This relation is tested against experimental data from the literature.

Previous DNS studies on bubble suspensions (see for instance: Esmaeeli & Tryggvason (2005); Van Sint Annaland et al. (2006)) focused on shape deformations and higher order velocity statistics, but the relation between $C_D$, $\phi$, $Re$ and $We$ has not yet been systematically explored. To our knowledge the widest parameter range that has been explored using DNS is due to Yin & Koch (2008), who studied systems of spherical gas bubbles for $5 < Re < 20$ and $0 < \phi < 0.25$. Sangani & Didwania (1993) used potential flow simulations to analyze spherical bubble suspensions at $Re \approx 500$. These simulations predicted a strong tendency of bubbles to cluster in horizontal planes. The effect of fluid vorticity, which is excluded in their potential flow calculations, on such clustering remains unexplored.

## 2. Numerical Method

### 2.1. Governing Equations

In this work we simulate the buoyancy-driven rise of spherical gas bubbles in a three dimensional periodic volume $V$. Gravity acts on the system in the negative $x$-direction. The volume is decomposed into a volume containing the liquid phase $V_L$ with a mass density $\rho$ and a volume containing the gas phase $V_G$ with a negligible mass density; $V = V_L + V_G$. The gas phase consists of $N_B$ spherical bubbles of radius $R$. Elastic, hard-sphere collisions of the bubbles are assumed (Chen et al. 1998). To illustrate the computational setup we have provided in Fig. 1 contour plots of the fluid velocity in vertical cross-sections of the computational domain.

The flow inside the liquid phase is described by the continuity equation:

$$\nabla \cdot \mathbf{u} = 0, \quad \forall \mathbf{x} \in V_L, \quad (2.1a)$$

and the incompressible Navier-Stokes equation:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\phi \rho \mathbf{g} \mathbf{e}_x + \nabla \cdot (-\rho \mathbf{\delta} + 2\mu \mathbf{S}), \quad \forall \mathbf{x} \in V_L. \quad (2.1b)$$
Figure 1. Fluid velocity component normal to the plane of view $u_y$ for $\phi = 0.11$ and $Re \approx 5$ (a), $Re \approx 50$ (b), $Re \approx 500$ (c) and $Re \approx 2000$ (d). Black and white correspond to negative $u_y$ and positive $u_y$, respectively. For clarity we set $u_y = 0$ in the bubbles. For $Re \approx 5$, $Re \approx 50$ and $Re \approx 500$ the linear domain sizes is $L/R = 21\frac{1}{3}$, while for $Re \approx 2000$ the linear domain sizes is $L/R = 10\frac{1}{3}$. For clarity we show this latter case on an equal domain size as the other cases, by displaying four periodic images of the same field. Gravity is pointing down. The circles in the figure indicate the cross-sections of the bubbles with the plane.

Here $t$ is time, $u$ is the liquid velocity, $p$ is the dynamic pressure, $S = \frac{1}{2}[(\nabla u)^T + \nabla u]$ is the rate of strain tensor, $\mu = \rho \nu$ is the liquid viscosity, $\nu$ is the liquid kinematic viscosity, $g$ is the gravitational acceleration, $e_x$ is the unit vector in the $x$-direction and $\delta$ is the unit tensor. The body force term $-\phi\rho g e_x$ in Eq. (2.1b) is composed of gravity and hydrostatic pressure and is derived in the Appendix. In the present numerical method Eq. (2.1b), modified as described below, is solved over the entire domain covering both the liquid phase as well as the gas phase. Since the mass density inside the gas phase is different to that is being used in Eq. (2.1b), the flow solution is considered physical meaningful only in the liquid phase.

The liquid flow field is subjected to periodic boundary conditions at the domain boundaries and free-slip and no-penetration boundary conditions on the moving, spherical bubbles. Denoting the surface of the bubble by $S$ and the bubble velocity by $u_B$, the no-penetration condition is written as:

$$(u - u_B) \cdot n = 0 \quad \forall x \in S, \quad (2.2)$$
and the zero tangential stress condition is written as:

\[ 2\mu S \cdot n \cdot (\delta - nn) = 0. \quad \forall x \in S. \]  \hspace{1cm} (2.3)

Here \( n \) is the unit outward normal vector on \( S \). The no-penetration and the zero tangential stress conditions are enforced by adding a force field \( F \) and a stress field \( \sigma \) to the r.h.s. of Eq. (2.1b). This approach is an extension of the method of Uhlmann (2005). This immersed boundary method involves interpolations from the Eulerian grid that is used to compute the fluid field onto a Lagrangian grid that is used to describe the particle surface. Although physically irrelevant, information from the inside of the particle is used to perform the interpolations. This is a well-established method for enforcing boundary conditions on solid particles, which has previously been used to simulate turbulent flows of particle suspensions (Uhlmann 2008; Lucci et al. 2010).

In addition to adding \( F \) and \( \sigma \) to Eq. (2.1b) we also modify the body force term in Eq. (2.1b) to ensure that the gas bubbles experience the correct buoyancy force. Therefore \( \forall x \in V_G \) the term \( -\phi \rho g e_x \) is replaced by \( (1 - \phi) \rho g e_x \). The resulting equation of fluid motion therefore reads:

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = \left( G_L \Psi(x) + G_G (1 - \Psi(x)) \right) e_x + F + \nabla \cdot ( -p \delta + 2\mu S + \sigma ), \]  \hspace{1cm} (2.4a)

where \( G_L \) and \( G_G \) are body forces due to gravity and hydrostatic pressure in the liquid phase and the gas phase:

\[ G_L = -\phi \rho g, \quad G_G = (1 - \phi) \rho g, \]  \hspace{1cm} (2.4b)

and \( \Psi \) is a marker function being 0 inside the gas phase and 1 inside the liquid phase.

The fluid equation of motion (Eq. 2.4) is discretized on a cubic and homogeneous mesh. The locations of the grid points are given by:

\[ x_{i,j,k} = \Delta x (i e_x + j e_y + k e_z), \]

where \( \Delta x \) is the grid spacing and \( i, j \) and \( k \) are integers. The domain sizes and number of grid points in the \( x, y \) and \( z \) direction are denoted \((L_x, L_y, L_z)\) and \((N_x, N_y, N_z)\), respectively.

### 2.2. Boundary Conditions

#### 2.2.1. No-Penetration

The no-penetration condition (Eq. 2.2) is enforced by the addition of the force field \( F \) to Eq. (2.4a). This force field is localized around the bubble surface \( S \) and has a direction normal to the bubble surface \( n \). The force \( F \) is defined such that it drives the normal component of the fluid velocity \( u \) at the bubble surface to the normal component of the bubble velocity \( u_B \). The force field is constructed by means of a set of \( N \) control points which are distributed equidistantly over \( S \). The fluid velocity on these control points is interpolated from the values at the grid points. On each control point a force is computed that counteracts the normal component of the difference between the interpolated fluid velocity and \( u_B \). This force is then distributed to the neighboring grid points using the following weights:

\[ K(x - y) = \prod_{i=1}^{3} \delta \left( \frac{x_i - y_i}{\Delta x} \right). \]  \hspace{1cm} (2.5a)

Here \( y \) is the location of the control point, \( x \) is the location of a grid cell and \( \delta(x) \) is a regularized delta-function, which is smoothed over three grid cells \( \Delta x \) (Uhlmann 2005; Uhlmann 2008; Lucci et al. 2010).
Figure 2. Sketch of the bubble normal unit vector $\mathbf{n}$ and the bubble tangent unit vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ at spherical coordinates $\theta$ and $\phi$.

Roma et al. 1999):

$$\delta(x) = \begin{cases} \frac{1}{3} \left(1 + \sqrt{-3x^2 + 1}\right) & \text{if } |x| < \frac{1}{2} \\ \frac{1}{6} \left(5 - 3x - \sqrt{-3(1-x)^2 + 1}\right) & \text{if } \frac{1}{2} < |x| < \frac{3}{2} \\ 0 & \text{if } \frac{3}{2} < |x| \end{cases}$$ (2.5b)

In contrast to the weights corresponding to Lagrangian interpolation, these weights provide a smooth temporal variation of the hydrodynamic forces while the bubble moves continuously w.r.t. the fixed Eulerian grid.

The no-penetration force, due to a single control point at position $\mathbf{y}_\alpha$, takes the form of a smoothed delta function, which is centered around $\mathbf{y}_\alpha$, and whose strength is proportional to $(\mathbf{u}_\alpha - \mathbf{u}_B) \cdot \mathbf{n}_\alpha$. Here $\mathbf{u}_\alpha$ and $\mathbf{n}_\alpha$ are the fluid velocity and bubble normal at control point $\alpha$. Taking all $N$ control points on $S$ into account the no-penetration force field $\mathbf{F}$ reads:

$$\mathbf{F}(x) = \sum_{\alpha=1}^{N} F_\alpha K(x - \mathbf{y}_\alpha),$$ (2.6a)

where:

$$F_\alpha = \frac{\rho \Delta S}{\Delta t \Delta x^2} (\mathbf{u}_B - \mathbf{u}_\alpha) \cdot \mathbf{n}_\alpha \mathbf{n}_\alpha.$$ (2.6b)

Here $\Delta S$ is the surface area corresponding to a single control point $\Delta S = 4\pi R^2 / N$. We used $\Delta S = 0.5 \Delta x^2$, since smaller values did not improve the accuracy of the method. The kernel given by Eq. (2.5) is also used to perform the interpolation. For a field $u$, the value $u_\alpha$ on point $\mathbf{y}_\alpha$ is interpolated from the values at the grid points $u_{i,j,k}$ as follows:

$$u_\alpha = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} K(x_{i,j,k} - \mathbf{y}_\alpha) u_{i,j,k}.$$ (2.7)

2.2.2. Zero Tangential Stress

In addition to no-penetration, free-slip on the bubble surface also requires zero tangential stress (Eq. 2.3). This condition is satisfied by adding the stress field $\sigma$ to Eq. (2.4a). Similar as the no-penetration force field $\mathbf{F}$, this tangential stress field $\sigma$ is non-zero only in the vicinity of the bubble surfaces. The stress field counterbalances the shear components of $2\mu S$ on the bubble surfaces. The procedure of computing $\sigma$ is analogous to that
Drag force in bubble swarms

of $F$, involving the control points on the bubble surface. On each control point the shear components of $2\mu S$ with respect to the bubble tangent plane are computed, using the interpolation scheme given in Eq. (2.7). The stress tensor $\sigma'$ that counterbalances these components is distributed to the neighboring grid nodes using the weights given in Eq. (2.5). In this procedure we identify at each control point the shear components of $2\mu S$ with respect to the bubble tangent plane. For this purpose we define at each control point an orthogonal coordinate system spanned by the unit normal $n$ and two unit tangents $t_1$ and $t_2$. A sketch of this system is provided in Fig. 2. Using spherical coordinates $(\theta, \phi)$ the transformation matrix between the $(e_x, e_y, e_z)$-frame and the $(n, t_1, t_2)$-frame reads:

$$M = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \end{pmatrix}.$$  

(2.8)

We construct a tensor $\sigma'$ that when added to $2\mu S$ produces zero viscous shear stress with respect to the tangent plane of the sphere, without affecting the normal stress components. In the $(n, t_1, t_2)$-frame the 12, 21, 13 and 31-components correspond to this shear stress. Therefore in the $(n, t_1, t_2)$-frame the 12, 21, 13 and 31-components of $\sigma'$ are given opposite values as those of $2\mu S$, while all others are set to be zero. In the $(e_x, e_y, e_z)$-frame this tensor is written as:

$$\sigma'_{ij} = -2\mu_{ijkl}S_{kl},$$  

(2.9a)

where the fourth-order viscosity tensor equals:

$$\mu_{ijkl} = \mu (M_{i1}M_{j2}M_{k1}M_{l2} + M_{i2}M_{j1}M_{k2}M_{l1} + M_{i1}M_{j3}M_{k1}M_{l3} + M_{i3}M_{j1}M_{k3}M_{l1}).$$  

(2.9b)

Adding $\sigma'$ to $2\mu S$ satisfies zero tangential stress at a single point. Taking all control points on the surface into account, the tangential stress field is written as:

$$\sigma(x) = \frac{\Delta S}{\Delta x^2} \sum_{\alpha=1}^{N} \sigma'_\alpha K(x - y_\alpha),$$  

(2.10)

where $\sigma'_\alpha$ correspond to $\sigma'$ (Eq. 2.9) at control point $\alpha$.

2.2.3. Bubble Motion

The numerical integration of Newton’s second law for a bubble, having a small mass, requires a small time step. In order to maintain the computational efficiency of the present numerical method we circumvent this problem by assuming a zero bubble mass, which eliminates the acceleration term from the equation of bubble motion. Without the acceleration term, the equation of bubble motion (Eq. A-11) states that the hydrodynamic forces integrate to zero over the bubble surface. In addition to the hydrodynamic forces, the simulated gas bubble also experiences a contribution from the force field $F(x)$ (Eq. 2.6), which is introduced to satisfy the no-penetration condition. The contribution of the artificial force field $F$ can be minimized by equating its total sum to zero:

$$\sum_{\alpha=1}^{N} F_\alpha = 0.$$  

(2.11)
where $F_{\alpha}$ is defined in Eq. (2.6b). Combining Eqs. (2.6b) and (2.11) gives the following equation of bubble motion:

$$
\sum_{\alpha=1}^{N} (u_B - u_{\alpha}) \cdot n_{\alpha} n_{\alpha} = 0.
$$

(2.12)

By approximating $\sum_{\alpha=1}^{N} n_{\alpha} n_{\alpha} = N\delta/3$, we arrive at the following expression for the bubble velocity:

$$
u_B = \frac{3}{N} \sum_{\alpha=1}^{N} u_{\alpha} \cdot n_{\alpha} n_{\alpha}.
$$

(2.13)

It is noted that the model can easily be extended to account for non-zero bubble mass by incorporating an inertia term into Eq. (2.13).

2.3. Lattice Boltzmann Method

To approximate the solution to Eq. (2.4) we use our in-house lattice Boltzmann (LB) code (Ten Cate et al. 2004). The LB method is based on discretizing the Boltzmann equation, which governs the distribution function $f$ over the space of the molecular velocity $v$ and position $x$ (Cercignani 1988).

$$
\frac{\partial f}{\partial t} + v \cdot \nabla f + a \cdot \nabla_v f = \frac{f^{(0)} - f}{\tau}.
$$

(2.14a)

Here $\nabla_v$ is the nabla operator in velocity space, and $a$ is the acceleration due to gravity, mean pressure gradient and the no-penetration force field:

$$
a = ge_x \{-\phi\Psi(x) + (1 - \phi)[1 - \Psi(x)]\} + F/\rho.
$$

(2.14b)

The right hand side of Eq. (2.14a) is the BGK approximation for the redistribution of probability due to molecular collisions, which is modeled as a relaxation process towards the Maxwell Boltzmann distribution $f^{(0)}$ corresponding to maximum entropy (Bhatnagar et al. 1954).

The key of the LB method is to discretize the velocity space into a minimum set of velocities $v_{\alpha}$, that is still large enough to represent the essential features of $f$ that play a role in the Navier-Stokes limit. Therefore the set $v_{\alpha}$ is chosen such that it facilitates a spectral representation of $f$ in Hermitian basis functions up to second order by using the Gauss-Hermite quadrature (He & Luo 1997; Philippi et al. 2006). The Hermite polynomials play a special role, since it can be shown that the corresponding expansion coefficients $g_{\alpha}$ are identical to the microscopic velocity moments of the distribution function, which represent the macroscopic flow quantities. Since the velocity directions $v_{\alpha}$ are such that $\Delta t v_{\alpha}$ equal the distances between neighboring lattice points, the Lagrangian derivative in Eq. (2.14a) is numerically integrated over one time step $\Delta t$ by simply shifting $f_{\alpha}$ between neighboring lattice points, where $f_{\alpha}$ equals $f$ evaluated at $v_{\alpha}$, multiplied with the corresponding weight of the Gauss-Hermite quadrature.

In the present LB method the acceleration $a \cdot \nabla_v f$ and the collision operator $(f^{(0)} - f)/\tau$ in Eq. (2.14) are applied in the space spanned by the hydrodynamical moments $g_{\alpha}$ which are linear combinations of $f_{\alpha}$. These moments represent the mass density $\rho$, the components of the momentum density $\rho u$ and the components of the momentum flux $\rho uu - 2\mu S - \sigma$. With the present set of 18 discrete velocities, only moments up to second order can be accurately calculated and therefore the third- and higher-order moments have no physical significance in the present numerical method. It is noted that these higher order moments can be incorporated by extending the set of discrete velocities,
Table 1. Parameters and numerical results for steady, individual bubble rise. The domain dimensions are $L_x = 2L_y = 2L_z$, the time step is $\Delta t_{\text{slip}}/\Delta x = 0.05$, the $C_{D,0}$ corresponds to a correlation from the literature (Eq. 1.4), $\delta_V = R\sqrt{2/Re}$ is the boundary layer thickness.

<table>
<thead>
<tr>
<th>$R/\Delta x$</th>
<th>$L_x/\Delta x$</th>
<th>$Re$</th>
<th>$C_D$</th>
<th>$\frac{C_D - C_{D,0}}{C_{D,0}}$</th>
<th>$\Delta/\delta_V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>48</td>
<td>4.92</td>
<td>4.48</td>
<td>0.02</td>
<td>0.35</td>
</tr>
<tr>
<td>4.5</td>
<td>48</td>
<td>16.9</td>
<td>1.42</td>
<td>-0.13</td>
<td>0.64</td>
</tr>
<tr>
<td>4.5</td>
<td>48</td>
<td>56.3</td>
<td>0.53</td>
<td>-0.13</td>
<td>1.2</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>53.6</td>
<td>0.59</td>
<td>-0.08</td>
<td>0.86</td>
</tr>
<tr>
<td>9</td>
<td>96</td>
<td>50.8</td>
<td>0.64</td>
<td>-0.04</td>
<td>0.56</td>
</tr>
<tr>
<td>12</td>
<td>128</td>
<td>50.4</td>
<td>0.67</td>
<td>-0.008</td>
<td>0.42</td>
</tr>
<tr>
<td>9</td>
<td>96</td>
<td>140</td>
<td>0.300</td>
<td>0.08</td>
<td>1.4</td>
</tr>
<tr>
<td>12</td>
<td>128</td>
<td>436</td>
<td>0.114</td>
<td>0.16</td>
<td>1.2</td>
</tr>
</tbody>
</table>

which then lead to higher-order approximations of the Boltzmann equation describing fluid mechanics beyond the Navier-Stokes equations (Shan et al. 2006; Colosqui 2010). In the present method, the acceleration and the collision operator are applied in $g_\alpha$-space, by adding $\Delta t\rho a$ to the momentum density and adding $\frac{4}{3}\rho c_s^2\Delta t S$ to the momentum flux, where $c_s$ is the speed of sound which in the present LB method equals $c_s = \Delta x/(\sqrt{2}\Delta t)$. The spurious higher-order components are relaxed towards zero. Further details of this LB method can be found in Somers (1993).

The combination of the LB and the immersed boundary methods provide several advantages over conventional Navier-Stokes solvers. Firstly, the method provides a direct control over the hydrodynamic stresses. This property allows the use of a very efficient algorithm to drive the shear stresses on the bubble to zero. Secondly, all the hydrodynamic quantities are defined on the same grid, which provides more accuracy when performing the velocity and stress interpolations. Thirdly, since the LB method permits a small compressibility to the fluid there is no need for a Poisson solver. In incompressible flow simulations the Poisson solver alters the immersed boundary force field. The costly operations needed to negate the corresponding errors in the boundary conditions (Kim et al. 2001) do not have to be applied when using the LB method.

2.4. Single Bubble

In this section we verify the accuracy of this method for the case of a single, steadily rising, spherical gas bubble by comparing the drag coefficient to a correlation from the literature (Eq. 1.4). Results of these test cases are presented in Table 1, showing the computed $C_D$ and the corresponding difference to Eq. (1.4) for different grid resolutions and Reynolds numbers.

When simulating single bubble rise in a periodic domain, the size of the domain should be chosen to be sufficiently large, so that the hydrodynamic interaction between the bubble and its periodic images can be ignored. The dominant interaction is due the wake produced by its upstream image, whose strength increases with increasing Reynolds number. Reducing these effects requires a rather large domain size in the $x$-direction corresponding to a large number of computational grid cells. In the single bubble simulations we have alleviated these computational loads by changing the boundary conditions in the $x$-direction from periodic to inflow and outflow, thereby excluding the wake of the periodic upstream image. The inflow and the outflow conditions correspond to a uniform velocity on the $(x = x_B + L_x/2)$-plane, and zero derivatives in $x$ of all hydrodynamic quantities on the $(x = x_B - L_x/2)$-plane, where $x_B$ is the $x$-coordinate of the bubble.
Figure 3. Penetration velocity (a) and shear stress (b) on the bubble surface for steady isolated bubble rise using $Re = 50$, $R/\Delta x = 4.5$ (solid line), $R/\Delta x = 6.0$ (dashed line), $R/\Delta x = 9.0$ (dotted line), $R/\Delta x = 12.0$ (dash-dotted line). Additional numerical parameters are given in Table 1.

Figure 4. Simulation results of a single rising gas bubble. The grayscale indicates the velocity difference magnitude, in $U_{slip}$-units, between our method and the method of Mei & Klausner (1992). The result from our method are obtained using $Re = 10$, $L_x/R = 422^2$, $L_y/L_x = L_z/L_x = 1/2$, $R/\Delta x = 6$, $L_x/\Delta x = 256$ (a) and $R/\Delta x = 12$, $L_x/\Delta x = 512$ (b). The vectors in the left figure correspond to the velocity field predicted by our method We have set the vectors in the gas bubble to zero.

Being determined by $x_B$, the inflow and outflow planes move periodically in the $x$-direction. It is noted that the inflow and outflow conditions are only applied in the single bubble simulations. They are not applied in the bubble swarm simulations of Sec. 3. Using these conditions in the $x$-direction and using periodic boundary conditions in the $y$- and $z$-directions the model predicts a $C_D$ within 10% of the literature value, using domain sizes of $L_x/R \approx 10$ and $L_y/R = L_z/R \approx 5$. For smaller domain sizes the drag coefficient is generally predicted to be larger.

In addition to the domain size the second parameter that determines the accuracy of the numerical result is the grid resolution. Obtaining an accurate numerical solution requires that the grid spacing $\Delta x$ is smaller than the viscous boundary layer thickness $\delta_V = R \sqrt{2/Re}$ (Blanco & Magnaudet 1995). In the present work we study bubble swarms at large Reynolds numbers up to $Re \approx 2000$. In order to keep the computational loads within feasible limits we have used $\Delta x \approx \delta_V$, which corresponds to $R/\Delta x = 12$ for
Drag force in bubble swarms

11

Re = 500 and R/Δx = 24 for Re = 2000. Although this resolution is somewhat too coarse to fully resolve the velocity profile inside the viscous boundary layer, the C_D results for single bubble rise are reasonable. As shown in Table 1 this resolution yields a drag coefficient within 15% from the literature value for Re ≈ 56, 140 and 440. For smaller Reynolds numbers Re ≤ 20 however a resolution of Δx = 0.5% yields a too small value for R/Δx, such that the spherical object can no longer be represented accurately in the grid. We found that for Re ≤ 20 a minimum of R/Δx = 4.5 must be used to obtain a C_D within 15% of the literature value. This minimum of R/Δx = 4.5 is related to the immersed boundary method, which effectively smears out the gas-liquid interface over three grid nodes. Apparently the bubble diameter must be at least a few times as large as this thickness in order to obtain reasonable results.

In Table 1 we illustrate the grid dependence of the method for Re ≈ 50. For this purpose we show results for four simulations with increasing grid resolution, corresponding to Δx/δ_V = 1.2, 0.9, 0.6 and 0.4. Table 1 shows that with increasing grid resolution C_D approaches the literature and reaches a value within 1% of the literature for Δx/δ_V = 0.4.

In Fig. 3 we examine for these four cases the accuracy of the immersed boundary method to satisfy the free-slip boundary condition. In that figure we show the penetration velocity u_s, scaled with U_slip and the shear stress on the bubble 2μS_θ + σ_θ scaled with C_D1/2ρU^2 slip. These variables are shown as functions of the spherical coordinate θ, which is defined in Fig. 2. From Fig. 3 we conclude that the no-penetration condition is satisfied within 2% of the slip velocity and the zero tangential stress condition is satisfied within 0.5% of the average stress on the bubble. These results are roughly independent of grid resolution. With diminishing grid size, the shear stress is seen to increase slightly.

In Fig. 4 we examine the accuracy of the method to produce the correct velocity field around the bubble. For this purpose we compare our result for Re = 10 to the finite difference result of Mei & Klausner (1992), who used a finite difference method on a spherical grid with a fine grid spacing in the radial direction close to the bubble surface. From Fig. 4 it is seen that the difference between our result and the result of Mei & Klausner (1992) is ≤ 0.1U_slip for R/Δx = 6 and ≤ 0.04U_slip for R/Δx = 12. The maximum difference occurs within one Δx to the bubble surface, due to the presence of the unphysical immersed boundary force and stress fields.

Finally it is noted that the numerical results are insensitive to the time step, provided that ΔU_slip/Δx ≤ 0.1.

3. Bubble Swarms

3.1. Numerical Parameters

The principal aim of the present work is to study the relation between the drag coefficient C_D, the gas volume fraction φ and the bubble Reynolds number Re in buoyancy-driven suspensions of spherical gas bubbles. For this purpose we have conducted ten simulations using three values of the gas volume fraction φ = 0.028, 0.11 and 0.44 and four values of the Reynolds number Re = 5, 50, 500 and 2000. All parameters are listed in Table 2. For each simulation gravity is tuned iteratively such that Re reaches the desired value. In all simulations ΔtU_slip/Δx = 0.05 and the domain is cubical with linear dimensions L_x/R = 24, except for the Re = 2000-case, where the domain size is L_x/R = 104. For Re = 5, 50, 500 and 2000 we have used R/Δx = 4.5, 6, 12 and 24, respectively. The corresponding grid spacing Δx per viscous boundary layer thickness δ_V are Δx/δ_V = 0.4, 0.9, 1.2 and 1.2, respectively. As discussed in Sec. 2.4 a grid resolution of Δx/δ_V = 1 is somewhat too coarse to fully resolve the velocity profiles within the viscous boundary
Table 2. Parameters and numerical results of buoyancy-driven bubble suspensions. The time step is $\Delta t_{U_{slip}}/\Delta x = 0.05$.

<table>
<thead>
<tr>
<th>$N_B$</th>
<th>$R/\Delta x$</th>
<th>$L_x/\Delta x$</th>
<th>$\phi$</th>
<th>$Re$</th>
<th>$C_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>4.5</td>
<td>96</td>
<td>0.027</td>
<td>5.1</td>
<td>4.9</td>
</tr>
<tr>
<td>256</td>
<td>4.5</td>
<td>96</td>
<td>0.110</td>
<td>5.0</td>
<td>5.9</td>
</tr>
<tr>
<td>1024</td>
<td>4.5</td>
<td>96</td>
<td>0.442</td>
<td>5.0</td>
<td>8.2</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
<td>128</td>
<td>0.027</td>
<td>50</td>
<td>0.87</td>
</tr>
<tr>
<td>256</td>
<td>6</td>
<td>128</td>
<td>0.110</td>
<td>49</td>
<td>1.2</td>
</tr>
<tr>
<td>1024</td>
<td>6</td>
<td>128</td>
<td>0.442</td>
<td>49</td>
<td>2.0</td>
</tr>
<tr>
<td>64</td>
<td>12</td>
<td>256</td>
<td>0.027</td>
<td>532</td>
<td>0.26</td>
</tr>
<tr>
<td>256</td>
<td>12</td>
<td>256</td>
<td>0.110</td>
<td>508</td>
<td>0.51</td>
</tr>
<tr>
<td>1024</td>
<td>12</td>
<td>256</td>
<td>0.442</td>
<td>479</td>
<td>1.60</td>
</tr>
<tr>
<td>32</td>
<td>24</td>
<td>256</td>
<td>0.110</td>
<td>2084</td>
<td>0.64</td>
</tr>
</tbody>
</table>

layer. However for single bubble rise the associated errors were shown to be modest, with a drag coefficient within 15% of the literature value. We have chosen to use this resolution, since it allows simulations of bubble swarms at large Reynolds numbers using reasonable amounts of computational resources. In this context it is further noted that the error of 15% is of minor concern compared to more fundamental inconsistencies between the simulations and real bubble swarms, such as the neglect of bubble deformation, bubble coalescence, bubble polydispersity, walls and impurities at the gas-liquid interface. These effects are expected to cause larger discrepancies between the simulations and experiments as compared to the 15% error due to the slightly under-resolved viscous boundary layer.

After starting the simulation from a random distribution of bubbles, a transition period of $\sim 100R/U_{slip}$ time units is required before the flow reaches a statistically steady state, after which statistics are collected during another $\sim 100R/U_{slip}$ time units.

3.2. Microstructure

Fig. 1 shows the $y$-component of the liquid velocity vector in the $xz$-plane at $\phi \approx 0.11$ for the four different Reynolds numbers considered $Re \approx 5, 50, 500, 2000$. The circles in the figure indicate the cross-sections of the bubbles with the plane. As can be seen from these plots, the structure of the flow is markedly different for $Re \approx 5$ and $Re \approx 50$ as compared to $Re \approx 500$ and $Re \approx 2000$. The larger $Re$ cases exhibit fluid velocity fluctuations on scales smaller than the bubble radius, which are absent for the smaller $Re$ cases. We will therefore refer to flow as ‘high agitation’ for $Re \gtrsim 100$, and we use ‘low agitation’ to refer to the flow for $Re \lesssim 100$.

By inspecting the bubble positions from plots such as those in Fig. 1, we verified that the bubbles were dispersed homogeneously for all cases. Although bubbles tend to form pairs, there was no sign of large-scale structuring into horizontal planes, such as observed in the potential flow simulations of Sangani & Didwania (1993), nor into vertical drafts, such as observed in the DNS of flexible bubbles of Bunner & Tryggvason (2003). To study the relative positioning of the bubbles we have computed the bubble pair probability density function $g(r)$. This function is defined such that $g(r)dr$ is proportional to the probability of finding a bubble pair whose separation vector lies within a volume $dr$ around $r$. The function $g(r)$ is normalized such that $g = 1$ corresponds to a random distribution of bubbles. The separation vector $r$ is parameterized using spherical coordi-
Drag force in bubble swarms

Figure 5. Radial pair distribution \( g_r(r) \) and polar pair distribution \( g_\theta(\theta) \) for \( Re \approx 5 \) (grey lines), \( Re \approx 50 \) (dashed black lines) and \( Re \approx 500 \) (solid black lines).

For the small volume fraction \( \phi = 0.028 \) there is a clear \( Re \)-dependence of the microstructure. For \( Re \approx 5 \) bubbles do not come close together, reflected by small \( g_r(r/R < 4) \). The opposite holds for \( Re \approx 50 \) and 500 where a large peak is seen at \( r/R = 2 \) corresponding to direct bubble contacts. For the intermediate volume fraction \( \phi = 0.11 \) the trends are similar as for \( \phi = 0.028 \), but the density variations are confined to smaller radial distances. For the large volume fraction \( \phi = 0.44 \), there is no apparent Reynolds dependency in \( g(r) \). The curves for all three \( Re \)-cases collapse showing a large peak at \( r/R = 2 \) and a secondary peak at \( r/R = 4 \).

Fig. 5 also shows the polar pair distribution which is defined as:

\[
g_\theta(\theta) = \frac{\int_0^{\pi} \int_{\theta-\Delta\theta}^{\theta+\Delta\theta} \sin \theta \sin \hat{r} \, d\hat{r} \, g(\hat{r}, \hat{\theta}) \, d\theta}{3R^3 \sin \theta \Delta \theta},
\]

where we only consider bubbles which are close together \( 2 < r/R < 5 \). Preferred horizontal alignment corresponds to large \( g_\theta \) around \( \theta = \pi/2 \), while vertical alignment corresponds to large \( g_\theta \) around \( \theta = 0 \). For \( \phi = 0.028 \) we find a strong \( Re \)-dependence in \( g_\theta \). For \( Re \approx 5 \) and 50 there is a modest and a strong preferred horizontal alignment, while for \( Re \approx 500 \) there is a preferred vertical alignment. For \( \phi \approx 0.11 \) all three \( Re \)-cases.
show preferred horizontal alignment, which is strongest for the $Re \approx 50$. For $\phi = 0.44$ the curves for all three $Re$-cases collapse showing no significant preferred alignment angle.

Another interesting property of the microstructure is presented in Fig. 6(a), where we plot the scaled kinetic energy contained in the liquid velocity fluctuations: $k = \frac{1}{2} \mathbf{u} \cdot \mathbf{u}' / U_{\text{slip}}^2$. In agreement with previous experimental observations (Garnier et al. 2002; Riboux et al. 2010), we find that this quantity scales linearly with $\phi$ while being roughly independent of $Re$. This similarity for different $Re$ is striking, given the marked flow changes, as shown in Fig. 1. In the next section we will exploit this similarity when formulating a model for the drag force in terms of an effective viscosity based on the stress carried by the velocity fluctuations in the liquid.

### 3.3. Drag Coefficient

In Fig. 6(b) we plot the computed $C_D$ as a function of $Re$. For clarity we have included lines of constant $\phi = 0, 0.028, 0.11$ and 0.44 as predicted by our model (Eq. 3.7) using $We = 0$. The model is derived below. To verify our numerical results we also plot the results of Yin & Koch (2008), who used a lattice Boltzmann method to simulate buoyancy-driven rise of spherical bubble swarms, for $5 \lesssim Re \lesssim 20$ and $0 < \phi < 0.25$. To facilitate a proper comparison, we have interpolated the data of Yin & Koch (2008) to the same $\phi$-values as those used in the present simulations, i.e. $\phi = 0.028$ and 0.11. As can be seen in Fig. 6(b) there is a good match between the data of Yin & Koch (2008) and our model (Eq. 3.7).

Fig. 6(b) shows a transition. In the ‘low agitation’ regime ($Re \lesssim 100$) the lines of constant $\phi$ are shifted parallel w.r.t. the line for an individual bubble (Eq. 1.4), while in the ‘high agitation’ regime ($Re \gtrsim 100$) the lines reach plateau-values which are independent of $Re$. The observations made in Sec. 3.2 that for $\phi = 0.44$ the $g(r, \theta)$ is $Re$-independent suggest that the differences in $C_D$ between the ‘low agitation’ and the ‘high agitation’ regimes are not linked to differences in the microstructure. Conversely we will argue that the different regimes are due to the occurrence of small scale velocity fluctuations in the liquid phase, while changes in bubble cluster configurations are of minor importance.
In the following we will develop a relation for $C_D$ by assuming a similarity between the $C_D(Re_{eff})$-relation in the suspension and the $C_D(Re)$-relation of an individual bubble (Eq. 1.4). Here the effective Reynolds number $Re_{eff}$ is based on the effective viscosity $\nu_{eff}$ which is allowed to depend on the properties of the suspension. The concept of an effective viscosity has previously been used to capture the effect of hydrodynamic interactions on the drag force in suspensions (see for instance: Barnea & Mizrahi (1973); Ishii & Zuber (1979)). To our knowledge previous models always assumed that $\nu_{eff}/\nu$ depends on $\phi$ only and is independent of $Re$. Here we will argue that for bubble swarms this assumption holds only for $Re \lesssim 100$. For $Re \gtrsim 100$ on the other hand, the occurrence of small scale velocity fluctuations introduces a $Re$-dependence in $\nu_{eff}/\nu$. To demonstrate this, we have computed $\nu_{eff}/\nu = Re/Re_{eff}$ by inserting the simulated $C_D$-values into Eq. (1.4), substituting $Re_{eff}$ for $Re$ and subsequently solving for $Re_{eff}$. The results are plotted in Fig. 7(a).

As expected the data for $Re \approx 5$ and 50, which correspond to the ‘low agitation’ regime, are relatively close to each other. We parametrize the ‘low agitation’ regime by a curve-fit to the $Re \approx 5$-data:

$$\frac{\nu_{eff}}{\nu} = \frac{1}{1 - 0.6\phi^2}, \quad (3.3)$$

The form of this relation is similar to the analytical solution $\nu_{eff}/\nu = (1 - 1.1964\phi^2 + 0.3508\phi^2)^{-1}$ for a fixed array of spherical gas bubbles in the creeping flow limit (Sangani & Acrivos 1983). For small $\phi$ Eq. (3.3) predicts: $\nu_{eff}/\nu = 1 + K\phi^2$, with $K$ a numerical constant. The term $\phi^2 \sim R/D$ can be understood by considering its proportionality to the magnitude of a velocity disturbance carried over the distance $D$ between two neighboring bubbles (Barnea & Mizrahi 1973).

As opposed to the ‘low agitation’ regime Fig. 7(a) shows a clear $Re$-dependence of $\nu_{eff}/\nu$ in the ‘high agitation’ regime. To construct a model that captures the physics in both regimes, we assume that the effective viscosity $\nu_{eff} = \nu_L + \nu_T$ is composed of a ‘laminar’ contribution $\nu_L$ as given by Eq. (3.3) and a ‘turbulent’ contribution $\nu_T$. We define $\nu_T$ as the ratio of the stress carried by the velocity fluctuations: $k = \frac{1}{2} \overrightarrow{u} \cdot \overrightarrow{u}$, and
the typical shear rate $U_{\text{slip}}/R$.

\[ \nu_{\text{eff}} = \nu_L + \nu_T = \frac{\nu}{1 - 0.6\phi^2} + \frac{CkR}{U_{\text{slip}}}, \]  

(3.4)

where $C$ is a constant of order unity. As shown in Fig. 6(a) and discussed in Sec. 3.2, we use that: $k \approx U_{\text{slip}}^2 \phi$. Inserting this into Eq. (3.4) gives the following relation for the effective viscosity ratio:

\[ \frac{\nu_{\text{eff}}}{\nu} = \frac{1}{1 - 0.6\phi^2} + C\phi Re. \]  

(3.5)

A value of 0.13 for $C$ provides the best correlation between Eq. (3.5) and the simulation data. The difference between our relation (3.5) and earlier proposed relations is the $Re$-dependence of $\nu_{\text{eff}}/\nu$. We argue that this dependence is essential to capture the effects of the small scale velocity fluctuations in the interstitial liquid, due to the randomly moving bubbles at large $Re$.

Fig. 7(b) shows the simulation data on the $(Re_{\text{eff}}, CD)$-plane where the effective Reynolds number $Re_{\text{eff}} = 2RU_{\text{slip}}/\nu_{\text{eff}}$ is obtained from Eq. (3.5):

\[ Re_{\text{eff}} = \frac{1}{Re(1 - 0.6\phi^2)} + 0.13\phi. \]  

(3.6)

The proposed model which is plotted as the solid line, assumes that $CD$ is described by Eq. (1.4) where $Re$ is replaced by $Re_{\text{eff}}$, which is given by Eq. (3.6). The validity of this approach is demonstrated by the collapse of the simulation data within 20% of the solid line.

3.4. Comparison to Experimental Data

To summarize, we have derived a $CD(Re, \phi)$-relation for spherical bubble swarms by assuming a similarity between the $CD(Re_{\text{eff}})$-relation for the swarm and the $CD(Re)$-relation for a single bubble (Eq. 1.4). In the model the effective Reynolds number $Re_{\text{eff}}$ (Eq. 3.6) is based on an effective viscosity which accounts for hindrance effects and turbulence effects. In order to extend our model to account for bubble deformation we hypothesize a similarity between the $CD(Re_{\text{eff}}, We)$-relation in the suspension and the $CD(Re, We)$-relation for a single bubble (Eq. 1.6). We therefore propose the following model for rising swarms of deformable bubbles:

\[ \frac{C_D}{G} = \frac{16}{Re_{\text{eff}}} \left\{ 1 + \left[ \frac{8}{Re_{\text{eff}}} + \frac{1}{2} \left( 1 + 3.315Re_{\text{eff}}^{-\frac{4}{5}} \right) \right]^{-1} \right\}, \]  

(3.7)

where the distortion factor $G$ and the effective Reynolds number $Re_{\text{eff}}$ are defined in Eqs. (1.5) and (3.6).

In order to determine the accuracy of this model we compare it to experimental data from the literature. For this comparison we use experimental $(\phi, U_{\text{slip}}, R)$-data, that have been obtained under homogeneous conditions, which means that no large scale circulation is present and that bubble coalescence and break-up do not play important roles. These conditions can be satisfied when bubbles are released in the column using carefully designed gas injection systems, that ensure a homogeneous distribution over the column cross-section and the bubbles are nearly monodisperse. We found nine experimental data sets from five different research groups, that were obtained under sufficiently well-controlled conditions to be suitable for the comparison. In Fig. 8 we plot these data on the $(Re_{\text{eff}}, C_D/G)$-plane. The experimental conditions are summarized in the caption.
4. Conclusions

The complexity of bubbly flow at large Re lies in the combination of pseudo-turbulence and bubble deformation. To gain insight we have reduced this complexity by ignoring bubble deformation and focusing on numerical simulations of spherical gas bubbles. These simulations were performed using a novel scheme based on lattice Boltzmann and immersed boundary methods. This strategy has provided a clear picture of the effect of pseudo-turbulence on the drag force in bubble swarms. We found that these effects can be captured using an effective viscosity $\nu_{\text{eff}} \approx kR/U_{\text{slip}}$, where the liquid velocity fluctuations are observed to behave as $k \approx \phi U_{\text{slip}}^2$. To account for bubble deformation we propose a modification of the model by using the distortion theory for a single gas bubble due to Moore (1965).

The principle assumption of our approach is a decoupling of the effects of the pseudo-turbulence from the effects of bubble deformation. Although the validity of this assumption is questionable, our approach resulted in a model that agrees reasonably well with experimental data from the literature. Furthermore it has provided insights that are dif-
ficult to obtain from experiments where effects of turbulence and bubble deformation are interrelated in a complicated and yet unknown way. Therefore we conclude that spherical bubble swarms provide a useful, limiting case in the analysis of deformable bubble swarms.

Appendix. Derivation of the hydrostatic pressure gradient, the bubble equation of motion and the drag coefficient

4.1. Hydrostatic Pressure Gradient

The flow in the liquid phase is described by the incompressible Navier-Stokes equation:

$$
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = - \left( \frac{dP}{dx} + \rho g \right) e_x + \nabla \cdot (-p\delta + 2\mu S), \quad \forall x \in V_L. \quad (A-1)
$$

Pressure is partitioned into two terms. The hydrostatic pressure $P$ varies linearly in the $x$-direction and the fluctuating pressure $p$ varies periodically. The hydrostatic pressure gradient $dP/dx$ counteracts gravity such that conservation of overall momentum is guaranteed. An expression for $dP/dx$ can be obtained by averaging the $x$-component of Eq. (A-1) over the liquid phase. The averaging operator $\overline{\cdot}$ acts on a variable $u$ in the following way:

$$
\overline{u} = \frac{1}{V} (1 - \phi) \int_V \Psi(x) u(x) dV. \quad (A-2)
$$

Here $\Psi$ is a marker function being 0 inside the gas phase and 1 inside the liquid phase.

The flow is assumed statistically steady and $V$ is assumed large enough that volume averages of hydrodynamic quantities are time independent. If we also use that all flow variables are periodic we arrive at the following expression for the $x$-component of the volume-averaged Navier-Stokes equation:

$$
0 = - (1 - \phi) \left( \frac{dP}{dx} + \rho g \right) + I_x. \quad (A-3)
$$

Here $I_x$ is the $x$-component of the momentum transfer per unit volume from the gas phase to the liquid phase due to viscous and fluctuating pressure forces:

$$
I_x = \frac{1}{V} \sum_{j=1}^{N_B} \int_{S_j} \left( p\delta - 2\mu S \right) : nnn \cdot e_x dS. \quad (A-4)
$$

Here $n$ is the outward-pointing normal on the bubble surface. Only normal stress components contribute to $I_x$, since tangential components are identically zero according to the free-slip boundary condition. The integral in Eq. (A-4) is taken over all bubble surfaces $S_j$. Assuming zero bubble mass, the total force acting on the bubbles equals zero:

$$
0 = - \frac{1}{V} \sum_{j=1}^{N_B} \int_{S_j} \left( P\delta + p\delta - 2\mu S \right) : nnn dS. \quad (A-5)
$$

Applying Gauss’ theorem to the $P$-term and using Eq. (A-4) gives the following force balance for the gas phase:

$$
0 = -\phi \frac{dP}{dx} - I_x. \quad (A-6)
$$

Using Eq. (A-6) to eliminate the interaction force $I_x$ in Eq. (A-3) gives the following
expression for the mean pressure gradient:

$$\frac{dP}{dx} = -(1 - \phi) \rho g. \quad (A-7)$$

Combining Eqs. (A-6) and (A-7) gives the following expression for the interaction force:

$$I_x = \phi (1 - \phi) \rho g. \quad (A-8)$$

Inserting Eq. (A-7) into Eq. (A-1b) gives the following equation for the liquid momentum:

$$\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\phi \rho g e_x + \nabla \cdot \left( -p \delta + 2 \mu S \right), \quad \forall x \in V_L. \quad (A-9)$$

### 4.2. Bubble Motion

Assuming zero bubble mass, the bubbles move such that the hydrodynamic forces integrate to zero over the bubble surface:

$$0 = -\int_S (p \delta + p \delta - 2 \mu S) : nnn dS. \quad (A-10)$$

Applying Gauss' theorem to the \(P\)-term and using Eq. (A-7) gives the following equation of bubble motion:

$$0 = (1 - \phi) \rho g V_B e_x - \int_S (p \delta - 2 \mu S) : nnn dS, \quad (A-11)$$

where \(V_B\) is the bubble volume.

### 4.3. Drag Coefficient

The drag coefficient is defined as the interaction force per bubble:

$$C_D = \frac{I_x V}{N_B \pi R^2 \frac{1}{2} \rho U_{slip}^2}. \quad (A-12)$$

Inserting the expression (A-8) for \(I_x\) into Eq. (A-12) yields the following expression for \(C_D\):

$$C_D = \frac{\frac{4}{3} R (1 - \phi) g}{U_{slip}^2}. \quad (A-13)$$

### REFERENCES


