“Bubbles, Ambiguity and the Lack of Impatience”

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Preliminary version

September 13, 2009

Abstract

We show that when infinite lived agents are not impatient, the transversality condition (necessary for optimality) becomes weaker, allowing for agents to be lenders at infinity and for efficient bubbles in the prices of assets in positive net supply. We provide results on the occurence of bubbles and give examples when lack of impatience arises from caring specifically about the worst possible lifetime outcome. This precautionary attitude can be reinterpreted in a context of ambiguous discounting, where each consumption plan is being valued using the most penalizing deflator. Pessimism over consumption creates a precautionary asset demand that induces an optimistic appraisal of the asset, as it becomes priced above the fundamental value. We address also the case of two-dates economies with countably many states of nature, where the lack of myopia or a certain kind of ambiguous beliefs lead, analogously, to an overpricing of assets.

1 Introduction

Occurrence and bursting of bubbles in the prices of assets in positive net supply are important phenomena in real world economies, but the theory of general equilibrium with infinite lived agents has not managed to accommodate these events satisfactorily. We focus on deterministic economies[1], which seemed to be, for standard preferences and debt constraints, the most unfriendly ones to this kind of speculation, since in this case the unique non-arbitrage deflator had to yield an infinite present value of wealth. In this paper, we depart from the impatience assumption on preferences and this led us to a more general transversality condition, necessary for individual optimality, that reduces to the usual one when agents are impatient but allows now for asymptotic deflated long positions. Agents can now be creditors at infinity: it all depends on what is the portfolio needed to accommodate the optimal consumption plan. Transversality conditions are now compatible with bubbles in the prices of assets in positive net supply, for deflators yielding finite present value of wealth. The portfolio constraints that avoid Ponzi schemes can be chosen to mimic these transversality consitions, becoming an extension of the usual ones.

We start by looking at preferences that put a particular weight on the worst outcome. We give examples of utility functions that deviate from the standard time-separable utility by adding a term concerned with the infimum of the utilities at all dates. We can reinterpret this as a situation where agents might be using several discount factors and tend to pick, for each consumption plan, the factor that discounts the future in the most severe way. Such framework had already been addressed by Gilboa (1989). The multiplicity of the possible discount factor may not be just a result of a difficulty in choosing one factor. It can be due to uncertainty, if an agent is unsure about his own preferences in a distant future. Hence, this imprecise discounting feature of the consumer problem with precautionary preferences can be seen a form of ambiguity (see Gilboa and Schmeidler (1989) and Schmeidler (1989)).

Actually, in a two-dates economy with infinitely many states of nature, analogous preferences including the infimum of utilities at all states, could also be reinterpreted in terms of aversion to the ambiguity in beliefs. Both in the infinite dates and the infinite states cases, the resulting consumer's optimization problem is a particular case of a maxmin problem where the functional being maximized is the minimum expected utility over all probabilities (discount factors) bounded from below by a set function that puts on any strict subset of events (or dates) a weight that is uniformly lower than one. Epstein and Wang (1992) showed that this discontinuity is actually equivalent to the (lower semi) *Mackey discountinuity* of the utility function (on the space of bounded sequences). Mackey discontinuity characterizes non-impatience in the infinite horizon problem and characterizes non-myopia in the infinitely many states problem. Moreover, the discontinuity of the set function at the full set is as if some state of nature or date were missing. Intuitively, that missing subjective event might not be captured in the fundamental value of assets and could give rise to a bubble. This is what occurs under lack of impatience (see section 3). Similarly, under lack of myopia, asset prices exceed the series of returns weighted by state prices (see section 6). In either case, pessimistic agents end up overvaluing the asset: pessimism on consumption creates a precautionary demand for the asset and this makes agents have an optimistic view on the asset price.

Mackey discontinuous preferences have been studied before in the general equilibrium

[1] As we are mostly interested in efficient bubbles, we could have dealt also with the stochastic case of complete markets. For simplicity of the presentation, we look at the deterministic case.

literature and, initially, in the context of a contingent claims economy with a single budget constraint. Bewley's (1972) stronger result, on the existence of Arrow-Debreu equilibrium with summable price sequences, does not hold [2]. However, the weaker version tells us that equilibrium prices are bounded finitely additive set functions. The first examples of such Arrow-Debreu equilibrium prices driven by the presence of the infimum term in the utility were given by Sawyer (1997), Barrios (1996) and **Werner (1997)**. Bounded (finitely) additive functionals can be decomposed into a countably additive functional and a pure charge (by the Yosida-Hewitt theorem). We identify sufficient conditions that actually prevent Arrow-Debreu prices from being countably additive.

The above non-countably additive Arrow-Debreu prices had been referred to as bubbles by Gilles and LeRoy (1992), although it was not clear what was the relation with the usual concept of bubbles of long-lived assets. The latter occur in robust cases for zero net supply assets (see Kocherlakota (1992) and Magill and Quinzii (1996)) but are ruled out under standard borrowing constraints when the net supply is positive, unless the present value of wealth is infinite (see the example by Bewley (1980) and the result in Santos and Woodford (1996)). The relation between pure charges in Arrow-Debreu prices and bubbles in asset prices was addressed by Huang and Werner (2000, 2004) in two papers. The first one, contains an example of sequential implementation, with asset price bubble, under a constraint requiring portfolios to be constant after some date. The second paper, shows that, under the usual borrowing constraints, any sequential equilibrium, with or without asset price bubbles, can be put in correspondence with Arrow-Debreu equilibria, with or without transfers, respectively, but nevertheless with countably additive prices.

Having found Arrow-Debreu equilibria whose prices are not countably additive, we implement the allocations in a sequential economy, where agents face one bugdet constraint at each date and there is an asset allowing for transfers of wealth across dates. The asset prices are determined using as deflator the countably additive part of the Arrow-Debreu price. This is the only possible choice (at least under interiority and some differentiability conditions), since the deflator ratios must be the marginal rates of intertemporal substitution of any agent, which were known to coincide with the ratios of elements in the Arrow-Debreu countably additive component. It follows that the present value of bounded endowments will always be finite. In this implementation we impose portfolio constraints that mimic the transversality condition (as it is usual in the literature) and related ones that bound debt at each date (but in a more general way than was done before).

Two important points should be noticed. First, we show that, once we allow for non-impatient agents, there are Arrow-Debreu allocations that could not be implemented sequentially using standard borrowing constraints. Secondly, there are Arrow-Debreu equilibria whose sequential implementation using a positive net supply asset requires a price bubble, as long as the constraint satisfies some weak minimal requirement[3] This allows us to say that it is not the form of the portfolio constraint that drives our speculation result. It is rather the weaker form of the transversality condition, which, as shown, must hold at any optimal plan chosen by a non-impatient agent. Other portfolio constraints, like the standard ones, might be introducing a friction preventing the implementation of efficient allocations.

A different approach was followed recently by Kocherlakota (2008) where new portfolio

[2]The earlier examples by Radner(1967) and Majumdar(1972) illustrated this.

[3]When the constraint allows for the admissibility of a scalar multiple of the equilibrium portfolio plan, for a scalar close enough to one, or implies the inclusion of the sequential choice set in the Arrow-Debreu budget set.

constraints are discussed, independently of what are agents' preferences. There the lower bound on asset positions is no longer non positive and may prevent agents from selling the bubble on the initial holdings. Our constraints mimic or imply transversality conditions that follow from agents' preferences. Whether there is an asymptotic deflated long position or not depends on what is the net trade in commodities that the agent intends to do. Namely, when the constraints take the form of restrictions on the value of debt at each date, the bound may now depend on the consumption plan: there is no longer an exogenous bound or a bound given by the present value of future endowments.

In our examples, when assets pay dividends, we pick a portfolio constraint that mimics the transversality condition computed using the supporting Arrow-Debreu price. In this case, the sequential choice set becomes a subset of the Arrow-Debreu budget set and the bubble is equal to the pure charge of the Arrow-Debreu price evaluated at the sequence of asset returns. The constraint could have imitated instead the transversality condition computed using another super-gradient, namely the one taking the highest value at the Arrow-Debreu net trade, with a resulting higher price bubble. When endowments do not converge but the optimal consumption plan does, these two choices differ and the latter allows for sequential implementation using fiat money. We give also an example where one agent is impatient but the other one is not. The optimal consumption of the impatient agent is not uniformly bounded away from zero (as the agent chooses to sell gradually the endowments to the counter-party, who places a higher value on distant consumption) and this allows for Arrow-Debreu prices with pure charges, implementable with asset price bubbles.

The paper is organized as follows. Section 2 outlines the guiding example. Section 3 relates ambiguity with the precautionary attitute prevailing in the example and addresses, in general, Arrow-Debreu equilibrium under ambiguity. Section 4 discusses sequential equilibria under lack of impatience, in a general framework. The last section discusses the occurrence of price bubbles, for assets in positive net supply, in sequential equilibria (that may or not implement sequentially Arrow-Debreu equilibria).

2 Guiding Example

Consider a deterministic infinite horizon economy with a single commodity and two agents (indexed by $i = 1, 2$) whose preferences depart from the standard time separable utilities since agents are particularly worried about the worst possible outcome of each consumption plan $x = (x_1, ..., x_t, ...) \in \ell^\infty_+$ [4]:

$$U^i(x) = \sum_{t \geq 1} \delta^{t-1} u^i(x_t) + \beta \inf_{t \geq 1} u^i(x_t)$$

with $\delta \in (0, 1)$, $\beta \in [0, \infty)$ and $u^i(r) = \sqrt{r}$.

This precautionary behavior is actually equivalent to a maxmin attitude, in the sense that the consumer looks for a consumption plan that maximizes the worst discounted time separable utility, within a class of discount factors (not necessarily of the exponential form) having δ^{t-1} as lower bound at each date t. It is as if the consumer were unsure about the discount factor that should be used and, therefore, uses the severest one. More precisely

[4] ℓ^∞ is the set of all bounded and nonnegative real sequences. See some properties of this set in section 3

(as shown in subsection 3.2.1 and appendix A),

$$\sum_{t\geq 1} \delta^{t-1} u^i\left(x_t\right) + \beta \inf_{t\geq 1} u^i(x_t) = \inf_{(\sigma_t)_{t\geq 1}\in \mathcal{A}} \sum_t \sigma_t u^i\left(x_t\right)$$

where $\mathcal{A}$ is the set of all real sequences $(\sigma_t)_{t\geq 1}$ such that $\sum_{t\geq 1}\sigma_t = \frac{1}{1-\delta}+\beta$ [5] and $\sigma_t \geq \delta^{t-1}$ for every date t. Hence, by analogy with the Ambiguity literature, we reinterpret the above precautionary behavior in terms of imprecise impatience (or flexible discount factor) and refer to it as ambiguous discounting.

There are endowment shocks that agents try to get rid of. Let $W^1 = (\frac{t+1}{t}+\varphi_t)_{t\geq 1}$ and $W^2 = (\frac{t+1}{t}-\varphi_t)_{t\geq 1}$, where φ_t is $1/2$ when t is even and $-1/4$ when t is odd.

Let us start by finding an Arrow-Debreu equilibrium. We will show (proposition 2 in section 4) that the consumption plan $x^a = a(W^1+W^2) = 2a(\frac{t+1}{t})_{t\geq 1}$ is agent i optimal for the single budget constraint $\pi^a x \leq \pi^\alpha W^i$ when the prices π^a are given by

$$\pi^a x \equiv \sum_t \delta^{t-1} u'(x_t^a)\, x_t + \beta u'(\inf x^a) B(x) = \frac{1}{2^{3/2}a^{1/2}}\left[\sum_t \delta^{t-1}\sqrt{\frac{t}{t+1}}\, x_t + \beta B(x)\right],$$

if it is true that $\pi^a x^a = \pi^a W^i$. The functional B denotes a Banach limit[6].

Now, if we define the prices π by

$$\pi x \equiv \sum_t \delta^{t-1}\sqrt{\frac{t}{t+1}}\, x_t + \beta B(x)$$

and, picking out $a(i) = \pi(W^i)/\pi(W^1+W^2)$, for the plans $\widetilde{x}^i = x^{a(i)}$, the condition $\pi\widetilde{x}^i = \pi W^i$ holds for $i = 1,2$ [7]. Since $\widetilde{x}^1 + \widetilde{x}^2 = W^1 + W^2$, we have that the prices π together with the plans $(\widetilde{x}^1, \widetilde{x}^2)$ are an Arrow-Debreu equilibrium for this economy.

As explained in detail in subsection 5.3, this Arrow-Debreu equilibrium can be implemented as a sequential equilibrium for the following economy: we introduce an infinite-lived real asset that pays as dividends 1 unit of the consumption good at every date[8] and is in positive net supply. At each date, agents trade an amount $z_t \in I\!R$ of the asset at a price $q_t \geq 0$. Moreover, we replace the Arrow-Debreu economy endowments W^i by $\omega^i = W^i - z_0^i$ that are the endowments adjusted to the asset initial holding $z_0^i > 0$. Notice that the choice of z_0^i must be such that $W_t^i - z_0^i > 0$ at each date. Since $W_t^i > 1/2$ for every t, one possible choice is $z_0^i \in (0, 1/2)$.. Notice that the endowments ω^i of the sequential economy accrued by the returns from asset initial holdings are the resources that each agent has when he does not trade in the asset market and, therefore, these resources should coincide with the Arrow-Debreu endowments W^i.

Taking the good to serve as numeraire at each date, the budget constraints that replace the Arrow-Debreu constraint are

$$x_t \,+\, q_t z_t \,\leq\, \omega_t^i \,+\, (q_t+1)z_{t-1} \quad \forall\, t \geq 1.$$

[5]So the sum of discount factors σ_t must be equal to the sum of the bounds δ^{t-1} and the extra weight β.

[6]That is, B is a linear and norm-continuous functional defined on the space of bounded real sequences such that $B((x_1, ..., x_t, ...)) = \lim_n \sum_{t=1}^n \frac{x_t}{n}$ when this limit exists. See appendix B for more details.

[7]Notice that $\widetilde{x}^i$ remains optimal under π once the budget constraint is homogeneous of degree zero.

[8]In fact, we will show in section 5.3 that the returns paid by the asset can be of a more general form.

It is clear that some borrowing constraints have to be added to avoid Ponzi schemes. If $\widetilde{x}^i$ is a solution for which those constraints are not binding, then (see subsection 5.1.1) the usual Euler equations must hold:

$$q_t u'(\widetilde{x}^i_t) = \delta(q_{t+1} + 1)u'(\widetilde{x}^i_{t+1})$$

However, the transversality condition is now quite different (see subsection 5.1.2), requiring only

$$\lim_t \delta^{t-1} u'(\widetilde{x}^i_t) q_t z^i_t \in [\beta \liminf_t (q_t(z^i_{t-1} - z^i_t) + z^i_{t-1}), \beta \limsup_t (q_t(z^i_{t-1} - z^i_t) + z^i_{t-1})]$$

which becomes the usual one when $\beta = 0$. That is, contrary to the standard case, for $\beta > 0$, agents can be (present value) lenders or borrowers at infinity. It is the precautionary behavior, trying to avoid a bad outcome at distant dates, that leads agents not to vanish the present value long position as time goes to infinity.

Following the usual transversality approach, we impose the above transversality condition (which must hold at an suitable optimal solution) on every admissible portfolio plan and look for borrowing constraints that imply those conditions.

For a suitable choice of those borrowing constraints, the above Arrow-Debreu consumption plans can be implemented in an equilibrium of the sequential economy with a bubble in the price of the asset (see subsection 5.3). More specifically, asset prices can be obtained from the Euler equations $p_t q_t = p_{t+1}(q_{t+1} + 1)$, where the deflator process is given by $p_t = \delta^{t-1}\sqrt{\frac{t}{t+1}}$ (i.e., the summable component of Arrow-Debreu equilibrium price). Asset holdings are given by $z^i_t = \sum_{s=1}^{t}(\omega^i_s - x_s)\vartheta_{s,t}/q_s + z^i_0 \vartheta_{1,t}$, where $\vartheta_{s,t} = \Pi_{s<\tau\leq t}(1 + 1/q_\tau)$, if $s < t$, and $\vartheta_{t,t} = 1$.

3 On non-impatient Preferences

Agents with ambiguous discount that end up picking the worst discount sequence for each consumption plan, put a special emphasis on the worst possible outcome and, therefore, exhibit a particular form of lack of impatience (or more precisely, Mackey discontinuity of preferences). In general, the lack of impatience implies more flexible transversality conditions that no longer prevent and, in many situations, actually determine bubbles in the prices of assets in positive net supply

3.1 Preliminary Concepts and Notation

We will denote by ℓ^∞ the space of real bounded sequences. With the usual norm $\|\cdot\| \to I\!R_+$ defined by $\|x\| = \sup_t |x_t|$, the set ℓ^∞ becomes a Banach space.

The space ℓ^1 is the set of all absolutely convergent real sequences. This space also becomes a Banach space with the norm $\|\cdot\|_1 \to I\!R_+$ such that $x \mapsto \sum_{t=1}^{\infty} |x_t|$. It is clear that $\ell^1 \subsetneqq \ell^\infty$.

The sequence whose terms are all equal to 1 will be denoted by $1\!\!1$. Given a set $A \subset \mathbb{N}$, the sequence $1\!\!1_A$ is defined as the one whose t index term is equal to 1, if it belongs to A, and zero otherwise. Given $n \in \mathbb{N}$, the set E_n stands for the set of natural numbers that are bigger than n. Therefore, $1\!\!1_{E_n}$ is the sequence whose t index term is equal to zero, if $t \leq n$, and equal to zero, if $t > n$.

Let x be a sequence in ℓ^∞. We say that x is *non negative* - which will be denoted by $x \geq 0$ - when $x_t \geq 0 \ \forall t \geq 0$ and we say that x is *bounded away from 0* - which will be denoted by $x \gg 0$ - when $\exists h > 0$ such that $x_s \geq h \ \forall s$.

Given two distinct elements x and x' in ℓ^∞, we say that $x > x'$ if $x_n \geq x'_n$ for each n.

Let ba is the space of bounded finitely-additive set functions, also known as charges, denoted by ba. A set function η is said to be finitely-additive set if $\eta(\emptyset) = 0, \eta(\cup_{i=1}^n A_i) = \sum_{i=1}^n \eta(A_i), \forall \{A_i\}$ where $A_i \cap A_j = \emptyset \ \forall i \neq j\}$.

We define generalized limit LIM as a continuous linear mapping from ℓ^∞ into $I\!R$ such that $\text{LIM}(x) = \lim_n x_n$ when this limit exists. Clearly, this functional is not uniquely defined and $\text{LIM}(x) \in [\liminf(x), \limsup(x)]$ (See Dunford and Schwartz (1958)).

A charge $\rho \geq 0$ is a *pure charge* when $[\lambda \in ca_+, \ \rho \geq \lambda \Rightarrow \lambda \equiv 0]$.

Denote by $\text{int}_{\|\cdot\|}\ell^\infty_+$ the interior of the positive orthant of ℓ^∞ in the norm topology.

3.2 Preferences

Throughout the paper we assume that the utility function $U : \ell^\infty \to I\!R \cup \{-\infty\}$ of an agent is a concave extended real valued function, in the usual sense that, on its effective domain M it has finite values and outside it takes the value $-\infty$. We assume that M contains $\{x : x >> 0\}$. The set M may contain, even strictly, the whole positive orthant. Notice that U will be finite and norm continuous at a point x if and only there is a norm neighborhood of x where the function is bounded from below. We assume that U is norm continuous on the norm interior of its effective domain. At some of the following statements we will also impose that U be a be *monotonous* function, i.e., that $x > x'$ implies $U(x) > U(x')$.

A coarser topology, for which the utility function may be assumed to be continuous in some results is the Mackey topology, which is the finest topology on ℓ^∞ for which the dual is ℓ^1. A sequence (x_n) converges to x in this topology if and only if for any weakly compact subset A of ℓ^1 we have $\langle x_n, y\rangle \to \langle x, y\rangle$ uniformly on $y \in A$.

To characterize the solutions of optimization problems that will be defined below, we use the concept of super-gradient of a concave function. A supergradient of $U : \ell^\infty_+ \to I\!R$ at x is an element T in the dual space such that $U(x+h) - U(x) \leq Th$, for any $h \in \ell^\infty$. The set of all supergradients of U at x is called the superdifferential of U at x and is denoted by $\partial U(x)$.

Given $x \in M$ and $v \in \ell^\infty$, when exists, the limit $\lim_{h\to 0} \frac{U(x+hv)-U(x)}{h}$ is called the *directional derivative* of U at the point x along (the direction) v and it is denoted by $\delta U(x; v)$. When the limit is evaluted only $h > 0$ (i.e., $\lim_{h\downarrow 0}$) it is called the *right-directional limit*, with notation $\delta^+ U(x; v)$. In the analogous way one defines the *left-directional derivative* $\delta^- U(x; v)$. Since U is concave, the limits $\delta^+ U(x; v)$ and $\delta^+ U(x; v)$ exist for all $x \in M$.

3.2.1 Ambiguous Discount with Lack of Impatience

Now we will deepen the outlined analysis (section 2) about ambiguous discounting preferences.

Suppose that some agents care specifically about the worst future outcome. More precisely, the utility function allows for precautionary behavior as follows:

$$U(x) = \sum_{t=1}^{\infty} \zeta_t u(x_t) + \beta \inf_{t \geq 1} u(x_t) \tag{1}$$

with $\zeta \in \ell^1_+$, $\beta \in [0,\infty)$, where the function $u : [0,\infty) \to I\!R \cup \{-\infty\}$ is: i) increasing, ii) concave and iii) of class C^1 at $(0,\infty)$.

We can assume, without loss of generality, that $\|\zeta\|_\infty + \beta < 1$ (and, so, $\zeta_t + \beta < 1 \;\forall t$). Thus, we can say that the agent has several ways of discount the future. For instance, for a date t_0, let $y = (y_t)_{t\in\mathbb{N}}$ be a consumption plan such that $u(y_{t_0}) = \inf_t u(y_t)$. Then $U(y) = \sum_{t\neq t_0} \zeta_t u(y_t) + (\zeta_{t_0} + \beta)u(y_{t_0})$, i.e., the date t_0 has been discounted by the rate $(\zeta_{t_0}+\beta)^{1/(t_0-1)}$ (the one within the possible that less depreciates the consumption welfare at this date), whereas all the other dates by t the rate $\zeta_t^{1/(t-1)}$ (the most severe one). Thus the degree of impatience for an arbitrary date can change according to his relative level of consumption.

Let us make two important remarks about the above utility function. **First**, it fails the usual impatience requirement of Mackey continuity in the space of bounded sequences. In fact, it is Mackey-u.s.c., but it is not Mackey-l.s.c. since for $z_n = c\mathbb{1}_{\mathbb{N}} - \frac{c}{2}\mathbb{1}_{E_n}$, with $c > 0$, we have $z_n \to c\mathbb{1}_{\mathbb{N}}$ in the Mackey topology[9], but it is not true that $U(z_n) \to U(c\mathbb{1}_{\mathbb{N}})$ (as $u(c/2) = \inf u(z_n)$ does not converge to $\inf u(c\mathbb{1}_{\mathbb{N}}) = u(c)$).

Besides, a sufficient condition to get $\partial U(x) \not\subset \ell^1$ is that the infimum $\underline{x}$ be a cluster point of sequence x. In fact, as we have seen in subsection 3.1, for every $T \in \mathrm{ba}_+$ there exists $\alpha \geq 0$ such that, at every $x \in \ell^\infty$, $T(x) = \sum_{t+1}^\infty T(e_t)x_t + \alpha\,\mathrm{LIM}(x)$. Let us denote by $\widehat{\partial U}(x) \subset \mathrm{ba}$ the set of all linear and norm continuous operators T such that $T(e_t) = u'(x_t)(\zeta_t + \gamma_t\beta)$ and $\alpha = \sigma\beta u'(\underline{x})$ where i) $\gamma_t \geq 0 \;\forall\, t \geq 1$, ii) $\gamma_t = 0$, if $x_t > \underline{x}$, iii) $\sigma \geq 0$ is zero when $\underline{x}$ is not a cluster point of the sequence x and iv) $\sum_{t=1}^\infty \gamma_t + \sigma = 1$. We state that:

Proposition 1: *If $x \in \mathrm{int}_{\|.\|}\ell^\infty_+$, then the superdifferential $\partial U(x)$ is equal to $\widehat{\partial U}(x)$.*

Proof: See appendix B. ■

Remark 1: Under the assumptions and notation of proposition 1, it is true that:

a) If $\underline{x}$ is not a cluster point of x, $\partial U(x) \subset \ell^1$;

b) As we have claim, if $\underline{x}$ is attained for infinite indices t, $\partial U(x) \cap \ell^1 \neq \emptyset$ but $\partial U(x)$ is not contained in ℓ^1;

c) If $\underline{x}$ is not attained, $\partial U(x) \cap \ell^1 = \emptyset$.

Secondly, the above utility function can be reinterpreted as the minimal separable utility when the discount factors have a certain lower bound at each date. In this reinterpretation consumers are not sure how to discount future events and they end up maximizing the worst discounted utility, over a certain set of possible discount factors. Let us be more precise.

The preferences represented by (1) are a particular case of

$$(1-\epsilon)\int_{\mathbb{N}} u \circ x\, d\mu + \epsilon \inf u \circ x \quad \text{with} \quad \epsilon \in [0,1) \quad \text{and} \quad \|\mu\|_1 = 1 \tag{2}$$

In fact, we can define $\mu = (1/\|\zeta\|_1)\,\zeta$ and multiply (1) by $1/(\|\zeta\|_1 + \beta)$ to obtain (2) by setting $\epsilon = \beta/(\|\zeta\|_1 + \beta)$. In this form, preferences can be easily related to work on the Choquet integral and non-additive expected utility present in Gilboa and Schmeidler (1989) and Schmeidler (1989).

[9]See the proposition **??** in the appendix **??**.

Given an arbitrary set Ω and $\mathcal{F}$, a σ algebra of subsets of Ω, we say that a set-function $\nu : \mathcal{F} \to I\!R$ is a *capacity* if $\nu(\emptyset) = 0$, and $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$. A capacity ν is said to be *convex* when $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)\ \forall A, B \in \mathcal{F}$.

Now, the utility representation that we considered above can be shown to be the infimum over all integrals of $u \circ x$ with respect to all probability measures that dominate the convex capacity ν obtained by making a linear distortion with coefficient $(1-\epsilon) \in (0,1]$ of μ, i.e., taking[10]

$$\nu(A) = \begin{cases} (1-\epsilon)\,\mu(A)\,, & \text{for } A \subsetneqq \Omega \\ \quad 1 \quad\quad , & \text{for } A = \Omega \end{cases} \tag{3}$$

In fact, let $M(\Omega, \mathcal{F})$ be the set of all probability measures on $\mathcal{F}$. Then[11]

$$(1-\epsilon) \int_\Omega u \circ x \, d\mu \, + \, \epsilon \inf_\Omega u \circ x = \inf_{\substack{\eta \in M(\Omega,\mathcal{F}) \\ \eta \geq \nu}} \int_\Omega u \circ x \, d\eta \tag{4}$$

for every $\mu \in M(\Omega, \mathcal{F})$, $\epsilon \in [0,1)$ and ν given by (3).

The representation of preferences as a minimal integral over a set of beliefs was suggested first by Schmeidler (1989) in his pioneering work on Knightian uncertainty. The right-hand side of (4) is actually a particular case of the Choquet integral, proposed by Schmeidler, where the minimum is taken over charges[12] η that dominate the convex capacity ν and such that $\eta(S) = 1$. When ν is given by (3), the minimum over dominating charges coincides with the infimum over dominating probability measures[13] (see lemma 1 of appendix A).

Notice that the minimal integral over the beliefs set tries to represent a precautionary or pessimistic behavior. The minimization solution η^* is selected so that it puts more weight on sets where the utility reaches its lowest values. When each agent has a "collection" of beliefs about nature states, this approach tries to capture the concept of ambiguity aversion. In a deterministic setting, each agent may be unsure about the discount factor and, therefore, by analogy, we say that preferences given by (1) or (2) represent agents' ambiguous discounting. Gilboa (1989) had already remarked that a possible explanation for the use of the Choquet integral as a representation of deterministic preferences may be that the agent dislikes great wobbles in his consumption level along time and is actually concerned with the worst future outcome.

The imprecise discount factors may be a particular case of a more general situation where the agent may not know what he will be later on - he may have what is usually called a *divided self*) or where there is some random element, besides those that may affect endowments and prices, determines future preferences, namely how to discount them. Notice that, even in a context where each agent had standard discounted utilities, if a policy-maker ignores the specific discount factor of each agent but is particularly worried about not leaving some agent extremely unhappy, then he might want to use a representative consumer(s) model with the above imprecise discounting feature (and pick the worst deflator as some agent might be using that deflator).

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[10]This is the called ϵ-contamination capacity, that appears in statistical works since the fifties - see, for instance, Hodges and Lehmann (1952) -. Some applications to Finance literature can be found in **?**) and in **?**)

[11]A proof of (4) is given in the appendix section A.

[12]See Dunford and Schwartz (1958) for the definition of an integral with respect to a charge η.

[13]That this Choquet integral coincides with the left-hand side of (4) is shown, for instance, in Dow and Werlang (1992).

The preferences described in this section are not necessarily dynamically consistent. Under consistency, if the decisions would be made only tomorrow, the choices for next dates would be the same as those taken today. Some authors look at this property as a desirable feature. In the example, this consistency holds when the infimum of the utilities in not attained in finite time (as it is the case for the endowments that we consider). However, our approach and the results presented below hold for more general Mackey discontinuous preferences.

4 Arrow-Debreu Equilibrium

In this section we address the Arrow-Debreu equilibria of the infinite-dimensional economy with preferences defined in subsection 3.2. Consumption plans are nonnegative bounded sequences, that is, elements in ℓ^∞_+. For an endowment sequence $W \in \ell^\infty_+$ and prices π given by a positive linear functional on ℓ^∞, we define the Arrow-Debreu budget set as the set

$$B_{AD}(\pi, W) = \{x \in \ell^\infty_+ : \pi(x - W) \leq 0\}$$

Consumer's problem is defined as

$$\begin{array}{rl} \text{maximize} & U(x) \\ \text{subject to} & x \in B_{AD}(\pi, \omega) \end{array} \tag{5}$$

Suppose that there are a number finite I of consumers each one characterized by (U^i, W^i). A couple $(\pi, (\overline{x}^i)_{i=1}^I)$ is said to be an *Arrow-Debreu equilibrium* (A-D equilibrium, for short) when:

- $\overline{x}^i$ is a solution of the problem of consumer i for (π, ω^i);
- Markets clear: $\sum_{i=1}^I (\overline{x}^i - \omega^i) = 0$.

Let us recall Bewley's (1972) result on existence of Arrow-Debreu equilibrium. Endowments are such that $\omega^i \gg 0 \; \forall i$ and preferences are assumed to be monotonous, $\|.\|_\infty$-lower-semi-continuous and such that have convex upper-contour sets. There exist equilibrium prices in ℓ^1_{++} if preferences are Mackey continuous, but if only the Mackey upper-semi-continuity holds, the equilibrium prices are in the dual of ℓ^∞, i.e., the Banach space ba.

If the utility functions U^i are not all Mackey lower semi-continuous, sometimes we have $\pi \notin \ell^1_{++}$. Indeed, we already gave one example in that A-D equilibrium prices are not countable-additive (section 2), but postpone some details until this moment. Now, we justify them with next proposition.

Proposition 2: *Let x be a consumption plan in ℓ^∞_+ such that $\underline{x} > 0$. If $\underline{x}$ is a cluster point never attained of the sequence x, then x is maximal for U in $B_{AD}(\pi, W)$ when*

(i) $\pi \in ba$ is given by

$$\pi y = \sum_{t \geq 1} \zeta_t \, u'(x_t) \, y_t \; + \; \beta \, u'(\underline{x}) \, \mathrm{LIM}(y)$$

and (ii) W is such that $\pi W = \pi x$.

Remark 2: By the Yosida-Hewitt decomposition theorem (see subsection 3.1) every $\pi \in ba_+$ can be written as $\pi = p + \xi$ where p is a non-negative countably additive set function and ξ is a non-negative pure charge. In the proposition 2, $\beta u'(\underline{y})$ LIM is the pure charge component. Gilles and LeRoy (1992) suggested that the pure charge component of an A-D equilibrium price system could be interpreted as a speculative bubble (we will this discussion in subsection **??**). However, these authors did not develop a sequential general equilibrium model and failed to relate this pure charge component to price bubbles in the assets that serve to complete the markets. Our goal will be to find sufficient conditions for the A-D equilibrium consumption plans can be implemented as an equilibrium of a deterministic sequential economy with an asset completing the markets in such a way that the countably additive component of the A-D equilibrium price induces the deflator process whereas the pure charge component induces the asset price bubble.

We close this section with two examples of Arrow-Debreu equilibria with non-summable prices.

Example 1 Consider the example outlined in section 2. Let $u^1(x) = \log x, \quad \omega^1 = (\frac{s+1}{s} + \varphi_s)_{s\geq 1}$ and $u^2(x) = \log x, \quad \omega^2 = (\frac{s+1}{s} - \varphi_s)_{s\geq 1}$ where φ_s is 1/2 when s is even and $-1/4$ when s is odd.

The candidate to Arrow-Debreu equilibrium consumption is $\overline{x}^1 = x^a = a(W^1 + W^2) = 2a(\frac{t+1}{t})_{t\geq 1}$. Let us check what is the supporting price.

$\overline{x}^i$ is maximal (by Proposition 2) for price system π and income $\pi\overline{x}^i$ when π is the functional defined by

$$\pi x = \sum_s \delta^{s-1} \frac{s}{s+1} x_s + \beta b(x)$$

where b is a Banach limit (it is such that $b((x_1, .., x_s)) = \lim_n \sum_{s=1}^n \frac{x_s}{n}$ when this limit exists[14]).

It remains to show that $\pi\omega^1 = \pi\overline{x}^i$. Here we use the fact that $b(\varphi)$ is uiquely defined.

Let $S_n(y) = \frac{\sum_{j=1}^n y_n}{n}$. We will show that $\lim_n S_n(\varphi) = 1/8$. In fact, $S_{2n} = \frac{n(-1/4+1/2)}{2n} = 1/8$ and that $S_{2n+1} = \frac{n(-1/4+1/2)-1/4}{2n+1} = \frac{-1/4+1/2}{2+1/n} + \frac{-1/4}{2n+1}$ converges to 1/8 when $n \to \infty$.

This is an example of an Arrow-Debreu equilibrium whose prices are not in ℓ^1, as contemplated by Bewley's existence theorem dispensing Mackey lower semicontinuity of preferences.

Example 2 The economy has two consumers whose preferences and endowments are given by

$u^1(x) = x^{1/2}, \quad \omega^1 = \frac{n+8}{n}$ and $\delta = 1/2$,

$u^2(x) = \log x, \quad \omega^2 = \frac{1}{4}(\frac{n+8}{n})^2$ and $\delta = 1/2$.

Let the consumption bundles be $y^1 = \omega^2$ and $y^2 = \omega^1$. We have market clearing. Let us construct the price functionals induced by marginal utilities (given by Proposition 2), which are the same for the two agents. In fact, $\frac{du^1}{dz}(y^1_n) = \frac{du^2}{dz}(y^2_n) = \frac{n}{n+8}$ and $\frac{du^1}{dz}(\underline{y}^1) = \frac{du^2}{dz}(\underline{y}^2) = 1$. Hence, for any bundle x, we have

[14]See Bhaskaro Rao, K. and Bhaskara Rao, M. (1983).

$$\pi x = \alpha \sum_{n\geq 1} \frac{n}{n+8} \frac{1}{2^{n-1}}\, x_n + (1-\alpha)\, b(x)$$

i.e., π does not depend on i.

Finally, for a suitable choice of α we have that $\pi\omega^1 = \pi y^1$ (and, therefore, $\pi\omega^2 = \pi y^2$). In fact, $\pi\omega^1 = 1$ and $\pi y^1 = \frac{1}{4}[\alpha \sum_{n\geq 1} \frac{n+8}{n} \frac{1}{2^{n-1}} + (1-\alpha)]$. So $\alpha = \frac{3}{\sum_{n\geq 1} \frac{n+8}{n} \frac{1}{2^{n-1}} - 1}$ and we want show that $\sum_{n\geq 1} \frac{n+8}{n} \frac{1}{2^{n-1}} > 4$. If suffices to notice that the first term in the series is already greater than 4. Then, (π, y^1, y^2) is an Arrow-Debreu Equilibrium.

5 Sequential Equilibrium

Let us define now a sequential economy whose equilibria will implement the above Arrow-Debreu equilibria, under certain portfolio constraints. In this sequential economy agents can transfer income across different dates through a financial structure. More specifically, it will be enough to introduce an infinite-lived real asset that pays $R_t > 0$ unit of consumption good as dividend at each date $t > 1$ and may be in positive net supply. We will take the single consumption good to serve as numeraire at each date and denote by $q_t > 0$ the asset price at date t, The sequential optimization problem of an agent i consists in choosing a consumption plan $x = (x_t)_{t\geq 1} \in \ell^\infty_+$ and a portfolio plan $z = (z_t)_{t\geq 1} \in I\!R^\infty$, in order to maximize his utility U^i subject to the sequential budget constraints

$$x_t \,+\, q_t z_t \,\leq\, \omega^i_t \,+\, (q_t + R_t) z_{t-1} \quad \forall\, t \geq 1 \tag{6}$$

given the sequence of endowments $\omega^i \in \ell^\infty_+$ and the asset initial holdings $z^i_0 \geq 0$, for the asset prices sequence q.

This optimization problem does not have a solution, under monotonicity of preferences, as the agent could always improve upon by doing a Ponzi scheme. To prevent this, portfolio constraints should be added to (6). Examples of such constraints, in the previous literature, were of the form $q_t z_t \geq -M_t$, where sequence M is positive valued and may be required to be bounded. We refer to the problem with a portfolio constraint P (to be specified) as the *Sequential P-Constrained Optimization Problem*:

$$\begin{array}{ll} \text{maximize} & U^i(x) \\ \text{subject to} & x \,\in\, \ell^\infty_+ \\ & \exists\, z \,\in\, I\!R^\infty \,:\, (6) \quad \text{and} \quad P \quad \text{are} \quad \text{valid} \end{array} \tag{7}$$

Since there is only one good in each period t, the decision of portfolio z determines the consumption plan $x^i(z) := (\omega^i_t + q_t\,(z_{t-1} - z_t) + R_t z_{t-1})_{t\geq 1}$, i.e., the variable of choice becomes the portfolio z. So, problem (7) can be restated in the form of a typical problem of dynamic programming:

$$\begin{array}{ll} \text{maximize} & U(x(z)) \\ \text{subject to} & \omega_t + q_t(z_{t-1} - z_t) + R_{t+1} z_{t-1} \geq 0 \quad ,\forall\, t \\ & x(z) \in \ell^\infty \\ & P \text{ is valid for } z \end{array} \tag{8}$$

For short, we will say that the real sequence z is a *P-feasible portfolio* for (q, ω^i, z_0) when $x^i(z) \in \ell^\infty_+$ and P is valid for z. Denoting by $B_P(q, \omega^i, z_0)$ the set of all consumption plans $x \geq 0$ for which there is a P-feasible portfolio z such that $x \leq x^i(z)$, we can restate the problem (7) as maximize $U(x)$ subject to $x \in B_P(q, \omega, z_0)$.

An *equilibrium* for the sequential economy with (P) portfolio constraints consists in an allocation $(x^i, z^i)_i$ and an asset prices sequence (q) such that (x^i, z^i) solves the sequential (P)-constrained optimization problem of agent i at prices (q).

5.1 Necessary Conditions for Optimality

In order to study the optimal solutions for (8), we need to define the concept of admissibility. Fixed a portfolio constraint P, the real sequence v is said to be a *right-admissible direction* at $x \in B_P(q, \omega^i, z_0)$ when $\exists\ \varepsilon' > 0$ such that $x + \tau v \in B_P(q, \omega^i, z_0) \quad \forall \tau \in [0, \varepsilon')$. It is said to be *left-admissible* when $\exists\ \varepsilon'' > 0$ such that $x + \tau v \in B_P(q, \omega^i, z_0) \quad \forall \tau \in (-\varepsilon'', 0]$ and is said to be *admissible* if it is right and left-admissible. Notice that if z^* is optimal on (8) and v is an admissible direction (at $x(z^*)$), then the directional derivative $\delta U(x(z^*); v)$, when exists, is equal to zero.

5.1.1 Euler Conditions

Let us start by examining Euler conditions, which will always hold under very mild assumptions. Denote by e_m an element in ℓ^∞ such that the m-th coordinate is 1 and all the other coordinates are zero.

Proposition 3: *Let x be an optimal solution to the Sequential (P)-Constrained Optimization Problem and suppose that U^i is norm continuous at x. If the direction $v(t) = -q_t e_t + (q_{t+1} + R_{t+1}) e_{t+1}$ is , right-admissible, then*

$$\exists \quad T \in \partial U^i(x) \ : \ q_t T e_t \geq (q_{t+1} + R_{t+1}) T e_{t+1}. \tag{9}$$

Proof: See appendix section C. ∎

Remark 3: (a) Notice that for different directions v_t, the supergradient given by the above Proposition might not be the same[15].

(b) The right-adimissibility of the above direction holds whenever portfolio constraints do not restrict increases in asset positions at a particular date (namely, for the classical constraints $q_t z_t \geq -M_t$ or for the new constraints that we will consider in this paper).

(c) Notice that, under the additional assumption that v_t is admissible for $h < 0$ (what we call left-admissible), then the opposite inequality would hold for some $T' \in \partial U(x)$ and, therefore, *Euler equation* holds for some supergradient of U at x since the superdifferential is a convex set.

(d) When, in addition to the condition in (c), the directional derivative exists for every canonical direction, we have conditions 9 with equality at every date and with Te_t given by the marginal utility with respect to consumption at t. In particular, for the preferences in examples 1 and 2, when the infimum is not attained at the optimal solution x, we have $q_t \delta^{t-1} u‘(x_t) = (q_{t+1} + R_{t+1}) \delta^t u‘(x_{t+1})$.

[15]This occurs, in general, even in the case of time separable preferences (see Araujo, Pascoa and Torres-Martinez (2005) and Pascoa, Petrassi and Torres-Martinez (2008))

5.1.2 Transversality Conditions

Transversality conditions, as necessary conditions for individual optimality, should not be confused with portfolio constraints that have been imposed by several authors on portfolio plans in order to guarantee the existence of a finite solution. The former are properties that the optimal solution always exhibits. The latter restrict the choice set by requiring all eligible portfolio plans to mimic somehow that property.

We claim that the transversality conditions that are necessary for individual optimality do not prevent speculation in positive net supply assets, irrespective of the portfolio restriction that is imposed in order to guarantee existence of a constrained optimal solution. We present now our most general result on transversality conditions. To do this we need to study some particular directions for changes in the portfolio plan. should not

For a given $n \in \mathbb{N}$, consider the direction $y(n)$ defined by $y_t(n) = 0$ if $t < n$, $y_n(n) = -q_n z_n$ and $y_t(n) = q_t(z_{t-1} - z_t) + R_t z_{t-1}$ if $t > n$. Notice that $y_t(n)$ is equal to $x_t - \omega_t$, for $t > n$. This direction will be left-admissible when the portfolio constraint (P) allows the agent to reduce, for any $t > n$, the absolute difference between x_t and ω_t, by replacing his portfolio plan $(z_t)_{t \geq n}$ by $((1+h)z_t)_{t \geq n}$ with $h < 0$. This left-admissibility holds for constraints of the form $q_t z_t \geq -M_t((q_s, z_s)_{s>t})$, where $M_t(.) = K_t + A_t(.)$, for some $K_t \geq 0$ and some homogeneous (possibly independent of t) function A_t of $(z_s)_{s>t})$ (as illustrated below). The right-admissibility requires the portfolio $((1+h)z_t)_{t \geq n}$ to be admissible for $h > 0$, which would hold for constraints of the form $q_t z_t \geq -M_t$ if the original portfolio were uniformly bounded away from $-M_t$.

We assume throughout this section that portfolio constraints (P) are such that, when $x >> 0$, (i) the direction $y(n)$ is always left-admissible , for every n, and that (ii) if $y(n)$ is right-admissible, then $y(n+1)$ is also right-admissible, for the plan z that is budget-consistent with x. These properties hold for the constraints $q_t z_t \geq -M_t((q_s, z_s)_{s>t})$ discussed above and also for constraints that mimic transversality conditions (classical ones or the ones will we present below).

Proposition 4: *Let $x >> 0$ be an optimal solution to the Sequential (P)-Constrained Optimization Problem and assume that the utility function U^i is norm continuous at x. Then,*

(i)

$$\lim_n \nu^n(x - \omega) \geq \limsup \mu_n^n q_n z_n$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components for some weak converging sequence $T^n \in \partial U^i(x)$;*

(ii) If the direction $y(n)$ is right-admissible, for some n, then the following transversality condition holds:

$$\lim_n \nu^n(x - \omega) \leq \liminf \mu_n^n q_n z_n$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components for some weak converging sequence $T^n \in \partial U^i(x)$;*

(iii) If the directional derivative $\delta U(x; y(n))$ exists and $y(n)$ is right-admissible, for every n large enough[16], then the following transversality condition holds

$$\nu(x - \omega) = \lim \mu_n q_n z_n$$

for every $\widetilde{T} \in \partial U^i(x)$ such that $\widetilde{T} = \mu + \nu$, where $\mu \in \ell^1$ and ν is a pure charge.

[16] or for just $y(t_0)$ when (9) hold as equalities beyond a certain date t_0 for a same operator $T \in \partial U(x)$

PROOF: See appendix section C. ■

If, as has been done in the previous literature, the portfolio constraint (P) mimics the transversality condition that must hold at the optimal solution (say in (i) or in (ii) above), then both the left and the right admissibility would hold (as both sides of the inequality, or the equality, would be multiplied by the positive term $1 + h$ for h sufficiently small).

For other portfolio constraints, such as those of the form $q_t z_t \geq -M_t$, where M_t may be a function of (x, ω, z), the right-admissibility of the direction $y(n)$ may fail (when the portfolio is not uniformly bounded away from $-M_t$).

When the utility function is differentiable along any canonical direction (as in examples 1 and 2 when the optimal plan is such that infimum consumption is never attained), then the right-hand side of items (i) and (ii) can be replaced by $\limsup \mu_n q_n z_n$ and $\liminf \mu_n q_n z_n$, where μ is the countably additive part common to $T \in \partial U^i(x)$. Otherwise, it is possible to rewrite these terms using the lower and upper bounds on marginal rates of inter-temporal substitution (see remark(5) in the appendix).

In the context of examples 1 and 2, under the portfolio costraints that were considered, as the infimum of the optimal consumption plan is not attained in finite time, we have $\lim_n \mu_n q_n z_n \in \beta \left[\liminf(x^i - \omega^i), \limsup(x^i - \omega^i)\right]$

5.2 Sufficient Conditions for Optimality

We already know that Euler equation is true at date t at every optimal solution for which the direction $v(t)$ is admissible. Now we investigate what are the other conditions that should be added in order to ensure that a feasible portfolio will indeed be a solution.

Proposition 5: *(i) Let z^* be a feasible portfolio for which U is norm continuous at the consumption plan associated $x^* = x(z^*)$. (ii) Suppose that exists $T \in \partial U(x^*)$ with decomposition $T = \mu + \nu$, $\mu \in \ell^1_+$ and $\nu \in \mathrm{pch}_+$ such that, for every t,*

$$\mu_t q_t = \mu_{t+1}(q_{t+1} + R_{t+1}) \tag{10}$$

and

$$\lim \mu_t q_t z^*_t = \nu(x^* - \omega). \tag{11}$$

(iii) Suppose also that every feasible portfolio z satisfies the condition

$$\lim_t \mu_t q_t z_t \geq \nu(x(z) - \omega), \tag{12}$$

Then z^* is a optimal solution for the problem (8).

PROOF: Given a feasible portfolio z, it is true that $U(x(z)) - U(x^*) \leq T(x(z) - x^*) = T(x(z) - \omega) + (\omega - x^*)$. Moreover, $\mu(x(z) - \omega) = \sum_{t=1}^{\infty}[\mu_t(R_t + q_t)z_{t-1} - \mu_t q_t z_t)]$. By (10), $\mu(x(z) - \omega) = \mu_1 q_1 z_0 - \mu_1 q_1 z_1 + \sum_{t=2}^{\infty}[\mu_{t-1} q_{t-1} z_{t-1} - \mu_t q_t z_t)] = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z_t$ (at particular, this limit always exists under proposition assumptions). By the same argument, $\mu(x^* - \omega) = \mu_1 q_1 z_0 - \lim_t \mu_t q_t z^*_t$. Now applying (11), results $U(x(z)) - U(x^*) \leq \nu(x(z) - \omega) - \lim_t \mu_t q_t z_t$. Finally, by condition, (12) we get $U(x(z)) - U(x^*) \leq 0$ and, so, z^* is optimal. ■

Corollary 1: *Under assumption (i) of proposition 5, suppose that, for each i, U^i is differentiable along any canonical direction e_t at x^i and denote by μ^i the countably additive component common to all suoer-gradients.*

Assume that for every i, $\delta^- U^i(x^i)(x^i, x^i - \omega^i) = \mu^i(x^i - \omega^i) + \alpha \limsup(x^i - \omega^i)$, for some $\alpha > 0$, and that assumption (ii) of the same proposition holds for ν such that $\mu^i(x^i - \omega^i) + \nu(x^i - \omega^i) = \delta^- U^i(x^i)(x^i, x^i - \omega^i)$. Then z^* is an optimal solution for problem (8) when requirement (12) is replaced by the following stronger one

$$\lim_t \mu_t q_t z_t \geq \alpha \limsup(x(z) - \omega),$$

5.3 Sequential Implementation and Room for Efficient Bubbles

Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium for the endowments $(W^i)_i$. We say that $(x^i)_i$ can be implemented sequentially with an asset with returns $R \in \ell^\infty_+$, if we can find asset initial holdings $z^i_0 > 0$ for which there are non-negative endowments of the sequential economy satisfying $\sum_i \omega^i + Rz^i_0 = \sum_i W^i$, asset prices q and portfolios $(z^i)_i$ such that $((x^i, z^i)_i, q$ is an equilibrium of the sequential economy, for given portfolio constraints.

The implementation depends on the choice of a deflator, since asset prices and, therefore, portfolios, will be determined by this choice.

Definition: Given asset returns R and asset prices q, we say that a sequence $(\lambda_t)_t >> 0$ is a *non-arbitrage deflator* if $\lambda_t q_t = \lambda_{t+1}(q_{t+1} + R_{t+1})$ for every $t \geq 1$.

It is immediate to see that, for given asset prices and returns, this sequence is uniquely determined up to the choice of the some term, say the initial term λ_1. Conversely, given λ and the asset returns, (q_t) is uniquely determined up to the choice of one term (and we will discuss this degree of freedom below).

Notice that, at date t, $\lambda_t q_t = \sum_{s>1} \lambda_s R_s + \lim_s \lambda_s q_s$, where both the series and the limit are finite, since asset returns are positive and q_t is finite. The series is called the *fundamental value* of the asset and the limit is the *bubble*.

Once the deflator λ is chosen, asset prices (q_t) become determined and (z^i_t) will also be determined so that x^i satisfies the sequential budget constraints $x^i - \omega^i = q_t(z^i_{t-1} - z^i_t) + R_t z^i_{t-1})$. When, for every agent i, $x^i \in \mathrm{int}_{\|.\|}\ell^\infty_+$ and marginal rates of substitution are uniquely defined, we can use the countably additive component p of the Arrow-Debreu price as a deflator [17].

Can we use a deflator which is not, up to a scalar multiple, equal to the countably additive part of the Arrow-Debreu price? Could we have distributed the pure charge across all dates to get a different deflator ? We show that this could not be done whenever the Arrow-Debreu allocation is an interior point and utility of each agent is differentiable along the canonical directions at the optimal solution (namely, when the infimum is not attained for the preferences in examples 1 and 2).

Proposition 6: *Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium such that, for each i, x^i is an interior point of ℓ^∞_+ and U^i is differentiable along any canonical direction e_t at x^i, then the countably additive component p of π is, up to a scalar multiple, the only possible choice of a deflator to implement sequentially $((x^i)_i$.*

[17] as the first order condition of the Arrow-Debreu problem requires $\rho^i p \in \partial U^i(x^i)$, for some $\rho^i > 0$, see Zeidler (1985), p. 391, Theorem 47.C).

Proof: We can not implememt sequentially $((x^i)_i$ with frictions, that is, with positive shadow prices for the portfolio constraints, otherwise, as utility is differentiable along the canonical directions, the uniquely defined marginal rates of intertemporal substitution would become different across agents (as some agent, the one purchasing the positive net supply asset, will not have the constraint binding), contradicting efficiency. Now, any non-arbitrage deflator λ must satisfy $\frac{\lambda_t}{\lambda_{t+1}} = \frac{q_{t+1}+R_{t+1}}{q_t}$. On the other hand, in the absence of frictions, ,we have $\frac{\lambda_t}{\lambda_{t+1}} = \frac{\delta U(x;e_t)}{\delta U(x;e_{t+1})}$ but $\frac{\delta U(x;e_t)}{\delta U(x;e_{t+1})} = \frac{p_t}{p_{t+1}}$. Then, in the absence of frictions, the deflator λ would be proportional to p. Hence, the asset prices induced by p are the same that would be induced by λ and are uniquely determined up to the choice of q_1. ∎

As x^i satisfies the Arrow-Debreu budget constraint at π it follows that $\sum_{t=1}^{\infty}(q_t(z^i_{t-1} - z^i_t) + R_t z^i_{t-1}) + \omega^i) - z^i_0\pi(R) \leq 0$ where the series is equal to $p_1 q_1 z^i_o -$. That is, at any sequential implementation using p as deflator we have (under local non-satiation)

$$\nu(x^i - \omega^i) - \lim_t p_t q_t z^i_t = z^i_0(\nu(R) - \lim p_t q_t) \tag{13}$$

which implies that $\lim_t p_t q_t z^i_t$ exists.

For the above $(z``i)_i, q$ to succeed implementing $(x^i)_i$, $(z``i)_i$ must be optimal on the sequential choice set $B^i_P(\overline{q}, \omega^i, z^i_0)$ for some portfolio constraint (P).

Could we use standard portfolio constraints to implement sequentially efficient allocations? Take three types of borrowing constraints that have been extensively used in the literature to avoid Ponzi schemes under impatience assumptions on consumers preferences.

1. $\lim p_t q_t z^i_t = 0$ (transversality constraint)
2. $q_t z^i_t \geq -M_t$ (bounded debt)
3. $p_t q_t z^i_t \geq -\sum_{s>t} p_s \omega_s$ (debt dependent on future ability to repay)

We know that any of these three types of constraints rules out bubbles for positive net supply assets, in the deterministic case (actually in the complete markets case), when the present value of wealth is finite. Actually, item 2 when M is bounded or item 3 when $p \in \ell^1$ (which is sufficient for the present value of wealth to be finite, for bounded endowments, and also necessary when endowments are uniformly bounded away from zero) imply $\lim p_t q_t z^i_t \geq 0$ and, therefore, in equilibrium, item 1 must hold, for a positive net supply asset (as the sum across agents of item 1 constraints is equal to $(\sum_i z^i_0) \lim p_t q_t$, where the bubble, $\lim p_t q_t$, is zero).

Proposition 7: *For any asset, with non-negative returns, in positive or zero net supply, under portfolio constraints of type 1, 2 or 3, it is impossible to implement sequentially an Arrow-Debreu equilibrium whose price has a pure charge that is non-zero valued at the net trades of some agent and, for each i, x^i is an interior point of ℓ^∞_+ and U^i is differentiable along any canonical direction e_t at x^i.*

Proof: In equilibrium, under any of these constraints, $\lim p_t q_t z^i_t = 0.$ and as $\nu(x^i - W^i) = \nu(x^i - \omega^i) - z^i_0 \nu(R)$, the Arrow-Debreu budget constraint (see equation (13)) implies that $\nu(x^i - W^i) = -z^i_0 \lim p_t q_t$. If the asset is in positive net supply, $\lim p_t q_t = 0$, since the asset has no price bubble for deflators yielding finite present value of wealth, such as p), under the constraints that were assumed. Otherwise, $z^i_0 = 0$. In either case, $\nu(x^i - W^i) = 0$. ∎

In examples 1 and 2 $\nu(x^i - W^i) = \beta LIM(x^i - W^i)$ which is positive for one agent and negative for the other, in the Arrow-Debreu equilibrium allocation.

We have shown that it is impossible to implement sequentially a large class of Arrow-Debreu equilibria using standard borrowing constraints, which are known to rule out bubbles (for positive net supply assets, in the deterministic case and for finite present values of wealth). Do positive net supply assets have price bubbles when other portfolio constraints are used instead?

The first important observation is that, under portfolio constraints that implement sequentially an Arrow-Debreu equilibrium, *transversality conditions do not prevent bubbles.* In fact, take the simplest case when, for each agent i, the utility function is differentiable along canonical directions at the optimal consumption plan x^i and $x^i \in \text{int}_{\|.\|}\ell^\infty_+$.

The transversality condition that must always hold, given by item (i) in Proposition(4), implies (see footnote(5.3)) by adding across agents that $\sum_i \lim p_t q_t z^i_t \leq \sum_i \alpha^i(\limsup(x^i - W^i) + z^i_0 \limsup R)$, where α^i is a positive scalar (see also lemma ??? in the appendix).

Even when $x^i - W^i$ converges, by normalization of the utility functions to get $\alpha^i = 1$ (adjusting accordingly the multiplier ρ^i) we just get $\lim_t p_t q_t \leq \limsup R$, which does not allow us to rule out bubbles of assets that pay dividends. Moreover, these bubbles would occur for the non-arbitrage kernel deflator p which yields a *finite present value of wealth* (as $p \in \ell^1$ and $W^i \in \ell^\infty$).

If, for all agents, optimal consumption plans were norm-interior points of the positive orthant and utility functions were Mackey continuous at these points, then the super-differentials of the utility functions had to be contained in ℓ^1 (see Lemma(6)) and, therefore, the transversality conditions became $\lim p_t q_t z^i_t = 0$, for every i, implyong $(\sum_i z^i_0) \lim p_t q_t = 0$, that is, bubbles of assets in positive net supply would be ruled out.

Intuitively, if agents were impatient, they would like to sell the bubble (exploring an infinite horizon arbitrage), at least when equilibrium bundles are interior points, and this rules out the ocurrence of bubbles whenever they can actually do it. However, in some cases (in the absence of asset initial holdings and under appropriate short-sales constraints) prevented from doing it. Let us be more precise.

Suppose there is a bubble in the price of the asset. Given an agent i and a consumption plan $x \in \ell^\infty_+$, we say that the agent is *willing to sell the bubble* at date t if there is some $h > 0$ such that $U(x + h(q_t e_t - \sum_{\tau > t} R_\tau e_\tau)) > U(x)$.

Proposition 8: *Suppose the utility function U of agent i is monotonous and Mackey continuous at point $x \in \text{int}_{\|.\|}\ell^\infty_+$. Suppose condition (D). If there is a bubble in the asset price and x satisfies, for each t, (9) with equality, then agent i is willing to sell the bubble at some date.*

PROOF: See appendix section D. ■

5.4 When bubbles occur

When do bubbles actually occur? We will resume Examples 1 and 2, illustrating how the Arrow-Debreu equilibria computed in Section(4) can be implemented sequentially and will provide also some more general results. Clearly, the equilibrium of the sequential economy must satisfy the Arrow-Debreu budget equation, which, in the context of Proposition(6), requires $p_1 q_1 z^i_0 - \lim p_t q_t z^i_t + p(\omega^i - W^i) = \nu(W^i - \omega^i) + \nu(\omega^i - x^i)$, that is, $\nu(x^i - \omega^i) - \lim p_t q_t z^i_t = z^i_0(\nu(R) - \lim p_t q_t)$. It follows immediately that $\lim p_t q_t z^i_t$ must exist.

Additionally, x^i must be optimal in the sequential choice set $B^i_P(q, \omega^i, z^i_0)$. We will try to accomplish this using Proposition(5) or by showing that $B^i_P(q, \omega^i, z^i_0)$ is contained in the Arrow-Debreu budget set. Let us start with the latter.

We will look at portfolio constraints that mimic transversality conditions. Recall that when, at the optimal plan $z^i, x(z^i)$, the utility function is differentiable along every canonical direction (as in Examples 1 and 2, since the infimum is not attained in finite time) and the portfolio constraint is such that λz^i is still admissible, for $\lambda > 0$ sufficiently close to one, the necessary transversality condition requires $\lim \mu^i_t q_t z^i_t = \nu^i(x(z^i) - \omega^i)$, for the common countably additive component μ^i of all super-gradients of U^i at $x(z^i)$ and the pure charge component ν^i of one of these super-gradients. We will consider portfolio constraints that require $\lim \mu^i_t q_t z_t \geq \nu^i(x(z) - \omega^i)$ for every admissible portfolio plan z and the equality at the optimal one, for one of those pure charges. One possible choice is to pick precisely the pure charge of the super-gradient that satisfies the first-order condition of the Arrow-Debreu equilibrium (which requires, at a norm-interior optimal bundle, $\rho^i(p+\nu) \in \partial U^i(x^i)$, for some $\rho^i > 0$). This is equivalent (dividing both sides by the Lagrange multilpler ρ^i)to require, at q in the set $Q(p)$ of asset price sequences for which p is a non-arbitrage deflator, we have

$$\lim p_t q_t z_t \geq \nu(x(z) - \omega^i) \tag{14}$$

, for any (P)-admissible plan z of agent i, where $x_t(z) = \omega^i + q_t(z_{t-1} - z_t) + R_t z_{t-1}$.

The constraint $p_t q_t z^i_t \geq -\sum_{s>t} p_s \omega^i_s + \nu(x^i - \omega^i)$ implies (14). The next two propositions relates (14) to the inclusion of $B^i_P(q, \omega^i, z^i_0)$ in the Arrow-Debreu budget set.

Proposition 9: *Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium such that π has a positive pure charge ν and, for each i, x^i is an interior point of ℓ^∞_+ and U^i is differentiable along any canonical direction e_t at x^i. Suppose (14) holds for any i and any $q \in Q(p)$. Then, the sequential implementation of $(x^i)_i$*

(i) using a zero net supply asset can be done for any $q \in Q(p)$.

(ii) using a positive net supply asset paying dividends requires a bubble always and can be done by finding $q \in Q(p)$ such that $B^i_P(\overline{q}, \omega^i, z^i_0) \subseteq B^i_{AD}(\pi, W^i)$ for every i.

Proof: Under the assumption, the sequential equilibrium satisfies the Arrow-Debreu budget equation if and only if $z^i_0(\nu(R) - \lim p_t q_t) \leq 0$, which implies immediately that an asset in positive net supply paying dividends must have a price bubble. Moreover, $B^i_P(\overline{q}, \omega^i, z^i_0) \subseteq B^i_{AD}(\pi, W^i)$ for any $q \in Q(p)$ when $z^i_0 = 0$ and for some $q \in Q(p)$ when $z^i_0 > 0$ and $R > 0$, namely for q such that $\lim p_t q_t = \nu(R)$.

The converse holds under an additional assumption. Consider the direction $u(1) = q_1 e_1 - \sum_{t>1} R_t e_t$. This direction is sequential left-admissible if, for $h < 0$ close enough to zero, the agent is able to increase his asset position by $|h|$ in every date, thus decresing x^i_1 by $q_1 |h|$ and increasing x^i_t by $R_t |h|$ at $t > 1$. This left-admissibility means that the agent can purchase some arbitrarily small amount of the asset and hold this additional position for ever.

Proposition 10: *Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium such that π has a positive pure charge ν and, for each i, x^i is an interior point of ℓ^∞_+ and U^i is differentiable along any canonical direction e_t at x^i. Suppose that, at an equilibrium asset price q, implementing the above Arrow-Debreu equilibrium, $B^i_P(\overline{q}, \omega^i, z^i_0) \subseteq B^i_{AD}(\pi, W^i)$, for some agent i. Then,* *(i) (14) holds at q for agent i if $z_0^i = 0$ or if $z^i > 0$ and the direction $u(1)$ is left-admissible.*

(ii) when q is such that $u(1)$ is left-admissible, the asset must have a price bubble when it pays dividends.

(iii) if the asset is fiat money and $x \neq W$, the portfolio constraint had to prevent holding for ever positions that are lower than the initial holdings.

PROOF: When $z_0^i = 0$, (i) is immediate . Otherwise, it holds when $\lim p_t q_t \geq \nu(R)$, that is, when $\pi(u(1)) = p1q1 - \pi(R) \geq 0$, which is the condition for the left-adimissibility of $u(1)$ with respect to the Arrow-Debreu budget set. Now, the sequential choice set is contained in the Arrow-Debreu budget set and, therefore, the sequential left-admissibility of $u(1)$ implies the Arrow-Debreu left-admissibility of $u(1)$. We have also shown (ii). To see (iii), let $z_t = c$ for any $t \geq 1$. Recall that the Arrow-Debreu budget constraint can be written as $\lim p_t q_t z_t \geq \nu(x(z) - W^i) + z_0^i \lim p_t q_t$, which would hold only if $c \geq z_0^i$. ∎

Notice that when (14) holds, the sequential left-admissibility of $u(1)$ requires $\lim p_t q_t \geq \nu(R)$, that is, when the asset is in zero-net supply and pays dividends, the left-admissibility of $u(1)$ requires a bubble.

Item (iii) shows that in the case of fiat money an implementation where $B_P^i(\overline{q}, \omega^i, z_0^i) \subseteq B_{AD}^i(\pi, W^i)$ would require a very awkward portfolio constraint preventing agents from choosing a portfolio that consists in keeping for ever part of the initial holdings.

Finally, we address the ocurrence of bubbles in the prices of positive net supply assets, for non-specified portfolio constraints, when Arrow-Debreu net trades converge. We will assume only that the constraint allows us to replace the equilibrium portfolio by a scalar multiple λ of it, for λ close enough to one (so that the direction $y^i(n)$ considered in the derivation of the transversality condition becomes right-admissible). This admissibility propoerty holds always for homogenous constraints (namely under (14) or, in general, for another super-gradient as in (12)) and for constraints of the form $p_t q_t z_t \geq M_t(x^i, W^i, \pi, R, z_0^i)$ when the implementation is done outside of the norm-boundary of the constraint set.

Proposition 11: *Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium such that $\pi = p + \nu$ where ν is a positive pure charge ν. Suppose that for each i, x^i is an interior point of ℓ_+^∞ and U^i is differentiable along any canonical direction e_t at x^i. Suppose $(x^i)_i$ is implemented sequentially using a positive net supply asset with asset prices q and that the direction $y^i(n)$ is right-admissible for some agent i for whom $x^i - \omega^i$ converges. Then, $\lim p_t q_t = \nu(R)$ and, therefore, a bubble occurs if and only if the asset pays dividends, irrespective of the choice of initial asset holdings (z_0^i)*

PROOF: The Arrow-Debreu budget constraint $p(x^i - W^i) = \nu(W^i - x^i)$ can be rewritten as $p_1 q_1 z_0^i - \lim p_t q_t z_t^i + p(\omega^i - W^i) = \nu(W^i - \omega^i) + \nu(\omega^i - x^i)$, that is, $\nu(x^i - \omega^i) - \lim p_t q_t z_t^i = pRz_0^i + z_0^i \nu(R) - p_1 q_1 z_0^i$. It follows immediately that $\lim p_t q_t z_t^i$ exists. Now, $\nu(x^i - \omega^i) - \lim p_t q_t z_t^i = z_0^i(\nu(R) - \lim p_t q_t)$. Now, as both x^i and ω^i converge, $\nu(x^i - \omega^i) = \alpha \lim(x^i - \omega^i)$ (for some positive α, by lemma... in the appendix) and should coincide with $\lim p_t q_t z_t^i$, by the transversality condition (as the direction in the statement of Proposition(4) item (ii) is admissible). Hence, $z_0^i(\nu(R) - \lim p_t q_t)$ should be zero, for every i and, by assumption, $z_0^i > 0$ for some agent. ∎

The assumption that, at the optimal solution, utility is differentiable along the canonical directions can be weakened by requiring these non-differentiabilities to fade away as time evolves, as discussed in the appendix.

5.5 Does the coexistence with an impatient agent kill the bubble?

If some agents are impatient and other agents are not, is it still possible to obtain efficient bubbles in the prices of assets in positive net supply ? Let us look at an economy with two agents, one with Mackey continuous preferences and another with ambiguous discounting. Since we are looking for efficient bubbles, it makes sense to search for Arrow-Debreu prices with pure charges and the try to implement sequentially the associated equilibrium allocations.

Huang and Werner (2000) already gave an example where the impatient agent, with time separable linear preferences, consumed zero after the initial date and the Arrow-Debreu price had a pure charge induced by the preferences of the other agent. Sequential implementation was obtained for portfolio constraints that require asset positions to be eventually constant. In this interesting example, the impatient agent had a peculiar behavior that was compatible with the pure charge in prices. One may wonder what are, in general, the features of the impatient agent's problem that allow for speculation under other portfolio constraints.

If an agent has Mackey continuous concave utility U and the optimal consumption plan x were an interior point of the positive orthant of ℓ^∞_+, then the superdifferential $\partial U(x)$ would be contained in ℓ^1 (see Lemma (6) in the appendix). In this case, x could not be the optimal choice for an Arrow-Debreu price $\pi = p + \nu$ with a nonzero pure charge ν. In fact, the necessary and sufficient (together with the budget constraint) optimality condition (see Zeidler (1985), p. 391, Theorem 47.C, adapted from the convex function to the concave function case) is that, for some $\lambda > 0$, we have

$$\lambda \pi \in \partial U(x) + \partial \chi_{\ell^\infty_+}(x) \tag{15}$$

where $\chi_{\ell^\infty_+}$ is the extended real valued functional that takes value 0 on ℓ^∞_+ and $-\infty$ elsewhere. Now, as shown in Lemma (4) in the appendix, the superdifferential $\partial \chi_{\ell^\infty_+}(x)$ is the set $\{T \in \text{ba} : T(y) \geq 0 \forall y \in \ell^\infty_+ \text{ and } T(x) = 0\}$[18]. Hence, if $x \in \text{int}_{\|\cdot\|} \ell^\infty_+$, then $\partial \chi_{\ell^\infty_+}(x) = \{0\}$ and, therefore, as $\partial U(x) \subset \ell^1$, π could not have a pure charge[19].

That is, when the Arrow-Debreu price has a pure charge, impatient agents must have consumption bundles on the boundary of ℓ^∞_+. In other words, the bundle should have a subsequence converging to zero. This property holds trivially in the example by Huang and Werner (2000), but it is compatible with a much larger class of preferences, namely when utility is stricly concave and, therefore, the agent may not want be consuming only at the first date, in spite of being impatient. Inada conditions hold, ruling out zero consumption at every date.

When consumption is positive at each date, $\partial \chi_{\ell^\infty_+}(x)$ only contains pure charges. Indeed, for $T \in \partial \chi_{\ell^\infty_+}(x)$, with countably additive nonnegative component η and pure charge nonnegative component ρ, $T(x) = 0$ implies $\eta(x) = \rho(-x)$. Hence, $\eta(x)$ is equal to zero [20]

[18] Loosely speaking, the element belonging to $\partial \chi_{\ell^\infty_+}(x)$ entering (15) plays the role of a multiplier for the constraint defining the consumption set.

[19] An immediate implication is that it is enough to have one agent with Mackey continuous preferences with an equilibrium interior plan to guarantee that the Arrow-Debreu price is countably additive.

[20] Note that $\eta(x) = 0$ implies $x_t \eta(e_t) = 0$ at each t. This complementary slackness relation together with $\lambda \pi(e_t) - \eta(e_t) \in \partial U(x)$ (implied by (15))is a generalization of the finite dimensional Kuhn-Tucker condition.

and, since $\eta(e_t) \geq 0$, it follows that η is a null operator.

For this reason, an agent with Mackey continuous utility can satisfy the above optimality condition at a boundary point $x >> 0$ of ℓ^∞_+ for an Arrow-Debreu price with a pure charge ν. In fact, the pure charge ρ of the multiplier of the consumption set constraint cancels out the price pure charge ($\lambda\nu = \rho$)[21]. Let us illustrate with a two agents example.

Example 4: Agent 1 has a time separable utility given by $U^1(x) = \sum_{t=1}^{\infty}(\frac{1}{4})^{t-1} \ln x_t$ and agent 2 has preferences described by $U^2(x) = \sum_{t=1}^{\infty}(\frac{1}{2})^{t-1} \ln x_t + 2 \inf_t \ln x_t$. The endowments of both agents are equal and given at each date t by $W^i_t = (1 + 3\left(\frac{1}{2}\right)^{t-1})\frac{t+1}{2t}$. We will show that an Arrow-Debreu equilibrium is given by the feasible consumption plans $\overline{x}^1_t = 3\left(\frac{1}{2}\right)^{t-1}\frac{t+1}{t}$, $\overline{x}^2_t = \frac{t+1}{t}$ and the price functional π such that, for $x \in \ell^\infty_+$, $\pi(x) = \sum_{t=1}^{\infty}(1/2)^{t-1}\frac{t}{t+1}x_t + 2\text{LIM}(x)$.

Let us start by checking that the budget constraint holds for agent 1 (which implies also that the budget constraint of agent 2 is satisfied). This reduces to show that $\sum_{t=1}^{\infty}(1/2)^{t-1}\frac{t}{t+1}(\overline{x}^1_t - W^1_t) + 2\text{LIM}(\overline{x}^1 - W^1) = 0$. This holds since $\sum_{t=1}^{\infty}(1/2)^t(3(1/2)^{t-1} - 1) - 1 = 0$.

The first order optimality conditions for agents 1 and 2 are given by (15). The latter holds (by making $\lambda^2 = 1$ and noticing that $\overline{x}^2$ is an interior point of the positive orthant) since the π is a supergradient for the utility of agent 2, at $\overline{x}^2$, . To check the former, recall that we can make the price pure charge cancels out with the pure charge on the multiplier of the positive orthant constraint, so that it suffices to find $\lambda \geq 0$ verifying $\lambda p_t = \frac{(1/4)^{t-1}}{\overline{x}^1_t}$. That is, we must have $\lambda(1/2)^{t-1}\frac{t}{t+1} = (1/4)^{t-1}(1/3)(2)^{t-1}\frac{t}{t+1}$, which holds for $\lambda = 1/3$.

Although agent 1 has endowments that are uniformly bounded away from zero, in equilibrium, consumption by agent 1 tends to zero. This happens since the pessimistic agent 2 is placing a high value on consumption at arbitrarily large dates, inducing the impatient agent 1 to sell the endowments at distant dates.

The above Arrow-Debreu equilibrium can be implemented sequentially as an equilibrium of an economy with one asset in positive net supply. Let asset initial holdings be $z^i_0 > 0$ and denote asset returns by $R_t \geq 0$. Commodity endowments net of returns from asset endowments are given by $\omega^i_t = W^i_t - R_t z^i_0$. We need to find portfolio constraints such that the sequential constraint set is contained in the Arrow-Debreu constraint set. As before, non-arbitrage deflators are given by the countably additive component p of the Arrow-Debreu price and, therefore, asset prices q_t satisfy $q_t = (1/p_t)(q_{t+1} + R_{t+1})p_{t+1}$. Multiplying the budget constraint at each date by the respective state price p_t and adding across dates we get the following condition for the Arrow-Debreu constraint to hold:

$$\sum_{t=1}^{\infty} p_t(q_t(z^i_{t-1} - z^i_t) + R_t z^i_{t-1})) + \nu(x^i - \omega^i) + \pi(\omega^i - W^i) \leq 0$$

[21]To see this, notice that, by Machey continuity, the countably additive component of any supergradient of U at x is still a supergradient (see Lemma (8) in the appendix) and it can replace the supergradient satisfying condition (2) (by moving the associated pure charge to the supergradient of $\chi_{\ell^\infty_+}$)

The above series is equal to $p_1 q_1 z_0^i - \lim_t p_t q_t z_t^i$ whereas the third term is equal to $-z_0^i \pi(R)$. Now $p_1 q_1 = \pi(R)$ (where the latter can be decomposed as the fundamental value computed using p plus the bubble). Therefore, the condition becomes $\lim_t p_t q_t z_t^i \geq \nu(x^i - \omega^i)$.

Hence, for the above condition to hold, we have two possibilities. We impose the transversality condition of the pessimistic agent (agent 2) on the portfolio plans of both agents or, under an appropriate choice of asset initial holdings, we require the portfolio plans of each agent to satisfy a constraint that mimics the respective transversality condition. To simplify assume that $R_t = 1$ for every $t > 1$. In the first case, the portfolio constraint would be $\lim_t p_t q_t z_t^i \geq (1/2) \limsup (x_t - \omega_t^i)$. [22] In the second case, this constraint will be imposed on agent 2, but agent 1 faces the usual constraint $\lim_t p_t q_t z_t^i \geq 0$ and sequential implementation can be achieved provided that $z_0^1 = 1/2$.[23]

Finally, notice that when both agents have Mackey continous preferences and the aggregate ewdowment is an interior point of the positive orthant, it is not possible to find sequential equilibria with asset price bubbles that implement Arrow-Debreu equilibria (as the latter would have a countably additive price functional, by Bewley (1972), Theorem 2). Actually, the same conclusion holds when aggregate endowments tend to zero, even though Arrow-Debreu equilibrium exist with a pure charge ν in prices. We would need an asset with asymptotically zero returns (otherwise the endowments of the sequential economy would become negative) and, therefore, a portfolio constraint of the form $\lim_t p_t q_t z_t = \nu(x - \omega^1)$, when added across agents, would necessarilly kill the bubble.

5.6 More on fiat money

The results in subsection 5.4 suggest that for fiat money to implement sequentially an Arrow-Debreu allocation $(x^i)_i$ the net trades $x^i - W^i$ can not converge for all agents and the implementation can not be achieved by forcing the sequential choice set $B_P(q, \omega^i, z_0^i)$ to be contained in the Arrow-Debreu budget set (as was the case when constraints (14) were used).

At a sequential implementation, the Arrow-Debreu budget set does not have to contain the sequential choice set $B_P(q, \omega, z_0)$, even when the latter is convex. This is a consequence of the non-differentiability of the utility function at the optimal point (more precisely, under differentiability along canonical directions, due to the non-uniqueness of the pure-charge components of the super-gradients). In fact, if a consumption bundle b belongs to $B_P(q, \omega, z_0)$ but $\pi(b) > \pi(W^i)$, then, under convexity of $B_P(q, \omega, z_0)$, we should have $0 \geq \delta^+ U(x^i, b - x^i)$ and this right-directional derivative is equal to $\widehat{T}(b - x^i)$ for some $\widehat{T} = \rho^i p + \widehat{\nu} \in \partial U(x^i)$, where ρ^i is the multiplier of the Arrow-Debreu constraint. Then, $\pi(x^i - W^i) = 0$ implies $\widehat{\nu}(b - x^i) < \rho^i \nu(b - x^i)$, which may occur when $b - x^i$ does not have a limit.

We could have considered other portfolio constraints that allow us to use Proposition(5) to guarantee optimality. Namely, when $x^i - W^i$ does not converge, for some agent, as in Example 2, there are other super-gradients of the utility whose pure charges could have

[22]Given a choice for the portfolio constraint, in equilibrium, agent 2 ends up demanding all the asset supply, as time goes to infinity. For this reason, if one is to impose the same portfolio constraint on both agents, it is reasonable to choose one that mimics agent 2 transversality condition.

[23]Conversely, for an arbitrary z_0^1, there are endowments W^1 for the Arrow-Debreu economy such that $\omega^1 \to 0$ and, therefore, an Arrow-Debreu with similar features could be implemented with the given asset endowments.

been considered in meeting the assumptions of the Proposition (instead of the super-gradient that is, up to a scalar multiple equal to the Arrow-Debreu price). This may allow us to find a positive price for *fiat money*. Let us focus on this asset. Notice that the absence of returns implies that $W = \omega$. We will use for each agent i a super-gradient whose pure charge $\widetilde{\nu}^i$ takes the highest value on the direction of the net trade. That is, $\widetilde{\nu}^i(x^i - W^i) = \max\left\{T(x^i - W^1) - \mu^i(x^i - W^i) : T \in \partial U^i(x^i)\right\}$, where μ^i is the countably additive component, common to all super-gradients (assuming as usual that the utility is differentiable along canonical directions at the optimum). Now, $\widetilde{\nu}^i = \alpha^i \mathrm{LIM}^i$ for some $\alpha^i > 0$ and some generalized limit LIM^i (see Lemma..., in the appendix).

Proposition 12: *Let $((x^i)_i, \pi)$ be an Arrow-Debreu equilibrium such that π has a positive pure charge ν. Suppose that for each i, x^i is an interior point of ℓ^∞_+ and U^i is differentiable along any canonical direction e_t at x^i; denote by μ^i the countably additive component common to all suoer-gradients.*

If, for some i, $x^i - W^i$ does not converge and $\delta^- U^i(x^i)(x^i, x^i - \omega^i) = \mu^i(x^i - \omega^i) + \alpha^i \limsup(x^i - \omega^i)$, for some $\alpha > 0$, then there exist initial holdings that allow for the sequential implementation of $(x^i)_i$ using fiat money under the following constraint

$$\lim \mu^i_t q_t z_t \geq \alpha^i \limsup(x(z) - W^i) \tag{16}$$

where $x_t(z) = W^i_t + q_t(z_{t-1} - z_t)$.

Notice that when $q = 0$ (16) does not restrict portfolios.

Proof: The hypothesis of Proposition(5) are satisfied if there is an asset price sequence q such that at the portfolio plan z^i implementing x^i we have $\lim \mu^i_t q_t z^i_t = \widetilde{\nu}^i(x^i - W^i)$. In addition, the equilibrium of the sequential economy should satisfy the Arrow-Debreu budget equation for some initial holdings. Both requirements are met if for any i

$$\widetilde{\nu}^i(x^i - W^i) - \nu^i(x^i - W^i) = z^i_0 \lim \mu^i_t q_t \tag{17}$$

To see that this suffices, notice first that, by construction of $\widetilde{\nu}^i$ the left-hand sides in all equations are non-negative. Now, $\lim \mu^i_t q_t = \rho^i \lim p_t q_t$. Suppose for the moment that we can find $b > 0$ as a choice for $\lim p_t q_t$ so that these equations hold, for some $(z^i_0) > 0$. Using $q_t = b(p_t)^{-1}$ we find a portfolio z^i for each agent so that the sequential budget equations hold, given z^i_0 and $x^i - W^i$. Now, as the Arrow-Debreu budget equation holds at x^i , with prices $p^i + \nu^i$, $p^i(x^i - W^i) = p^i_1 q_1 z^i_0 - \lim p^i_t q_t$ must be equal to $\nu^i(x^i - W^i)$. Hence, equation(17) implies $\lim \mu^i_t q_t z^i_t = \widetilde{\nu}^i(x^i - W^i)$, as desired, and the Arrow-Debreu budget equation holds also.

It remains to show that there exists a solution $b > 0$. This would not happen if $\widetilde{\nu}^i(x^i - W^i) = \nu^i(x^i - W^i)$ for all i, which, by definition of $\widetilde{\nu}^i$ implies that $\widetilde{\nu}^i = \nu^i$. Now, $\nu^i = \rho^i \nu$, $\nu = \alpha \mathrm{LIM}$ and $\widetilde{\nu}^i = \alpha^i \mathrm{LIM}^i$. Hence, $\mathrm{LIM}^i = \mathrm{LIM}$ for any i and, therefore, $\alpha^i = \alpha \rho^i$. Arrow-Debreu prices can be normalized so that $\alpha = 1$ and we get $\alpha^i = \rho^i$. By definition of $\widetilde{\nu}^i$ we have $\widetilde{\nu}^i(x^i - W^i) = \alpha^i \limsup(x^i - W^i)$, implying that $\nu(x^i - W^i) = \frac{\alpha^i}{\rho^i} \limsup(x^i - W^i) = \limsup(x^i - W^i)$, for any i. Adding across agents, $0 = \sum_i \limsup(x^i - W^i) \geq \limsup \sum_i (x^i - W^i) = 0$, where the strict inequatity would hold when for some i $x^i - W^i$ does not converge. That is, sequential implementation can be done using (16), for suitable initial holdings, except when $x^i - W^i$ converges for any i (actually, in this case, we were forced to use the pure charge of the super-gradient verifying the first-order condition of the Arrow-Debreu problem and we fall in the case of Proposition(9), which rules out monetary implementation for converging net trades). ∎

Notice that constraint (16) is stronger than constraint (**??**), which in the case of fiat money does not imply $B^i_P(\overline{q}, \omega^i, z^i_0) \subseteq B^i_{AD}(\pi, W^i)$[24]. The result is this proposition also holds under the weaker constraint $\lim \mu^i_t q_t z_t \geq \widetilde{\nu}^i(x(z) - W^i)$.

Example: Let us go back to example 2 and implement an Arrow-Debreu equilibrium using fiat money. Recall that the indeterminacy in the generalized limit considered in the Arrow-Debreu price leads to a real indeterminacy in Arrow-Debreu equilibrium allocations. Take the equilibrium allocation that results from using the Banach limit B. The supporting price π^i for agent i is equal to $\rho^i\pi$ where $\rho^i = \frac{1}{2^{3/2}a(i)^{1/2}}$.

Now $\widetilde{\nu}^i(x^i - W^i) = \alpha^i \limsup(x^i - W^i)$ where $\alpha^i = \frac{\beta}{2\sqrt{\inf_t x^i_t}} = \frac{\beta}{2^{3/2}a(i)^{1/2}}$. So $\alpha^i/\rho^i = \beta$ and constraint(16) becomes $\lim \delta^{t-1}\sqrt{\frac{t}{t+1}} q_t z_t \geq \beta \limsup q_t(z_{t-1} - z_t)$ for both agents. Let $\overline{W} = (W^1 + W^2)/2$ and notice that, as $x^i - \overline{W}$ converges, $\widetilde{\nu}^i(x^i - W^i) - \nu^i(x^i - W^i) = \widetilde{\nu}^i(\overline{W} - W^i) - \nu^i(\overline{W} - W^i)$.

Equations(17) become: $\beta(\frac{1}{4} + \frac{1}{8}) = z^1_0 \lim p_t q_t$ and $\beta(\frac{1}{2} - \frac{1}{8}) = z^2_0 \lim p_t q_t$. Initial holdings must be the same for the two agents but can take any positive value z_0. Take $z_0 = \frac{3}{8}\beta$ so that $p_t q_t = 1$, implying $q_t = \delta^{-(t-1)}\sqrt{\frac{t+1}{t}}$.

Equilibrium portfolios are determined by solving the sequential budget equations, which, for agent 1, are equivalent to $z^1_{t-1} - z^1_t = \delta^{t-1}\sqrt{\frac{t}{t+1}}\left[(2a(1) - 1)\frac{t+1}{t} - \varphi_t\right]$. Namely, when δ and β are such that $a(i) = 1/2$ we get $z^1_t = z_0 + \sum_{s=1}^t \delta^{s-1}\sqrt{\frac{s}{s+1}}\varphi_s$.

6 Two-dates economies under ambiguity and overpriced assets

The Arrow-Debreu equilibria discussed so far can be reinterpreted as equilibria of a two-dates economy with countably many states of nature at the second date and a complete set of assets, traded at the initial date and paying returns at the next date. We will show that, when the Arrow-Debreu price is not in ℓ^1 and the complete set of assets consists in Arrow securities, then the price of an asset whose returns are uniformly bounded away from zero will exceeed the series of returns weighted by state prices (given by the Arrow securities prices). If we think of this series as a fundamental value, the asset will be overpriced and the excess is precisely equal to the pure charge component in the Arrow-Debreu price evaluated at the returns stream. had already suggested that a pure charge in the Arrow-Debreu price could be interpreted as a bubble. Here we provide, in a two-dates economy, a precise relation between the Arrow-Debreu pure charge and the excess of asset prices over the series of returns weighted by state prices. This relation is analogous to the one with found in infinite horizon economies between that pure charge and the bubble in the long-lived asset. The interpretation of pure charges as bubbles, suggested by Gilles and LeRoy (1992), for which we had already given a precise meaning in the infinite horizon context (see Section ...), can also be shown to have interesting insights for the price of one-period assets.

We recall that ambiguity refers to a situation where agents have a collection of beliefs

[24] $B^i_P(\overline{q}, \omega^i, z^i_0) \setminus B^i_{AD}(\pi, W^i)$ consists of bundles for which the difference $\alpha^i \limsup(x - \omega^i) - \nu(x - \omega^i)$ is smaller than the value that this difference takes at x^i

(that is, probability distributions). Gilboa and Schmeidler (..) and Schmeidler (..) modeled ambiguity aversion by considering an utility functional that is the minimum of all expected utilities computed using the different beliefs. It follows from Theorem 2.1 in Epstein and Wang (1995) that, when the collection of beliefs is determined by the core of a capacity η (that is, by the finitely additive probabilities that dominated the capacity, see section...), utilities are Mackey lower semi-continuous if and only if the capacity is continuous at certainty [25]. The discontinuity of the capacity at certainty opens some room for the occurence of Arrow-Debreu equilibrium whose prices have a pure charge. The discontinuity at certainty can be interpreted as if there were a missing state, in the sense that there is a sequence of events converging to the set of all states for which the respective sequence of capacity values does not tend to one.

This discontinuity at certainty occurs clearly in the preferences of examples 1 and 2, where the capacity is the epsilon-contamination (see appendix ...). in this case, the capacity gives to any strict subset of states a value that is bounded from above by $\epsilon < 1$.

Let us see, more precisely, for any Mackey lower semi-discontinuous preference, how can the Arrow-Debreu equilibria be put in one-to-one correspondence with equilibria of two-dates economies with a complete set of one-period assets. In general, Mackey discontinuity can be interpreted, in the context of many states, in terms of lack of myopia (see Brown (1981) and Raut (...)). A myopic agents tends to focus on some finite set of states. This interpretation is similar to the interpretation of Mackey discontinuity in infinite-horizon (finitely many states) problems in terms of lack of impatience. Both phenomena can occur in double infinity models.

Given an Arrow-Debreu economy with endowments W^i we can always associate it with a two-dates economy with financial assets. Suppose the asset structure is described by the returns linear operator $R : \ell^\infty \to \ell^\infty$ that maps any portfolio $z \in \ell^\infty$ of countably many assets into a bounded sequence of payoffs over the countably many states. Asset j is identified by its returns $R(e_j)$. We can allow for initial asset holdings $z_0^i \geq 0$ by defining the endowments in the two-dates economy as $\omega^i = W^i - R(z_0^i)$ (assuming that z_0^i is such that $\omega^i \geq 0$).

For asset prices q in the dual of ℓ^∞, the budget constraints of the two-dates economy are

$$q \cdot z \leq q \cdot z_0^i \tag{18}$$

$$x_s - \omega_s^i \leq R(z)_s \ \forall\, s \in \mathbb{N} \tag{19}$$

$R(z)_s$ denotes the s^{th} component of $R(z)$.

We say that there are arbitrage opportunities if there exists $z \in \ell^\infty$ for which $(-q \cdot z, R(z)) > 0$. In the absence of arbitrage opportunities, there exists π in the dual of ℓ^∞, called the non-arbitrage functional, such that $q \cdot z = \pi \circ R(z)$.

Proposition 13: *If (x,z) satisfies constraints 18 and 19, with equality in the latter, at asset prices q in the dual of ℓ^∞, given ω^i and z_0^i, then $x \in B(\pi, W^i)$, for the non-arbitrage functional π.*

Proof: Let (p, ν) be the Yosida-Hewitt decomposition of π, with $p \in \ell^1$ and $\nu \in$ pch. Then, 18 is equivalent to $\sum_{s=1}^{\infty} p_s R(z - z_0^i)_s + \nu(R(z - z_0^i)) \leq 0$. Using 19, we have $\sum_{s=1}^{\infty} p_s (x_s - W_s^i) + \nu(x - W^i) \leq 0$, as claimed. ∎

[25] A capacity η is continuous at certainty if, for any sequence (A_n) of sets such that each $a_n \subset A_{n+1} \subset \mathbb{N}$ and $\cup_n A_n = \mathbb{N}$, we have $\lim \eta(A_n) = 1$.

The converse of the above proposition holds when markets are complete, that is, when $R(\ell^\infty) = \ell^\infty$.

Proposition 14: *Suppose markets are complete. Given the asset initial holdings z_0^i, if $x \in B(\pi, W^i)$, then there exists $z \in \ell^\infty$ such that (x, z) satisfies constraints 18 and 19, at asset prices q given by $\pi \circ R$.*

PROOF: Let z be such that $R(z) = x - \omega^i$. Then 19 holds. To see that 18 is satisfied we use the Arrow-Debreu constraint and the definition of z to get $\pi(R(z - z_0^i)) \leq 0$. Now, by definition of q, we obtain 18. ■

There is a one-to-one correspondence between Arrow-Debreu equilibria and equilibria of two-dates economies with complete markets. An equilibrium of the two-dates economy, with endowments $(\omega^i, z_0^i)_i$ and asset returns R, is a vector $((x^i, z^i)_i, q)$ such that (i) for each i, (x^i, z^i) is optimal for consumer i under 18 and 19, (ii) $\sum_i x^i = \sum_i(\omega^i + R(z_0^i))$ and (iii) $\sum_i z^i = \sum_i z_0^i$.

We assume that the linear operator R is not only surjective but also injective [26]. That is, different portfolios must have different return streams. Under monotonicity, an optimal solution under constraints 18 and 19 is also optimal in $B(\pi, W^i)$ and vice-versa. Commodity market clearing is immediate and it remains to show that an Arrow-Debreu equilibrium induces asset market clearing. Here we use injectivity of R, since $R(\sum_i(z^i - z_0^i)) = \sum_i(x^i - W^i) = 0$ implies $\sum_i(z^i - z_0^i) = 0$.

When the Arrow-Debreu price has a pure charge, asset prices will inherit that pure charge, that is, for any portfolio z,

$$q \cdot z = \sum_s p_s R(z)_s + \nu(R(z)). \tag{20}$$

In the simplest case, when $R(e_j) = e_j$, that is, when assets are Arrow securities, then $q \cdot e_j = p_j$. Consider an asset whose returns are uniformly bounded away from zero (which can be obtained from the primitive Arrow securities by choosing an appropriate portfolio). The price of this asset can be found using 20, as this is the non-arbitrage condition. Interpreting the series as a fundamental value, we can look at the non-zero value of the pure charge $\nu(R(z))$ as a bubble.

What happens when markets are incomplete? For a given subspace $R(\ell^\infty)$, the consumption set X^i of agent i, in the two-dates economy, will be the intersection of $R(\ell^\infty) + W^i$ with the positive orthant of ℓ^∞. Let us define an Arrow-Debreu economy where each agent has X^i as the respective consumption set. Utility functions will be restricted to these sets. Arrow-Debreu equilibrium prices will be bounded linear functionals on the set of net trades $R(\ell^\infty)$ [27]. Existence of Arrow-Debreu equilibrium with these consumption sets follows from Theorem 1 in Bewley (1972), provided that $R(\ell^\infty)$ is Mackey closed (namely when it is a finite dimensional subspace). Whenever the Arrow-Debreu equilibrium for these restricted consumption sets has a price functional with a pure charge we can implement it as an equilibrium for a two-dates economy whose assets are overpriced, in the sense that the price exceeds that series of returns weighted by state prices. However, as not all Arrow

[26] A surjective linear operator mapping a finite dimensional space into itself is always injective, but this not always true for infinite dimensional spaces

[27] Notice that the restriction of any element in the dual of ℓ^∞ to $R(\ell^\infty)$ will be a bounded linear functional on $R(\ell^\infty)$

securities are available, some states will have more than one state price (given by different non-arbitrage deflators). In particular, it is well-known that when $R(\ell^\infty)$ is finite dimensional, if state j is such that $e_j \notin R(\ell^\infty)$, but we were to add the respective Arrow security, its non-arbitrage price would belong to the interval $[\sup\{q \cdot z : z \in A\}, \inf\{q \cdot z : z \in B\}]$, where $A = \{z \in \ell^\infty : R(z) \leq e_j\}$ and $B = \{z \in \ell^\infty : R(z) \geq e_j\}$ are, respectively, the sets of superhedging and subhedging strategies (see Follmer and Schied).

Appendix

A

Let Ω be an arbitrary set and $\mathcal{F}$ a σ-field of Ω subsets. First, we will define the *core* of a capacity ν at $(\Omega, \mathcal{F})$:

$$\text{core}(\nu) = \{\eta \in ba(\Omega, \mathcal{F}) : \eta \geq \nu,\ \eta(S) = \nu(S)\}$$

and, for a given $r \geq 0$, the set $M(r) = \{\eta \in ca_+(\Omega, \mathcal{F}) : \eta(\Omega) = r\}$. Suppose that u is an utility index function with the properties listed in the subsection 3.2.1. Now we can state:

Lemma 1: *If $\mu \in ca_+(\Omega, \mathcal{F})$, $\beta \in [0, +\infty)$ and ν is the capacity defined by equation (3) of subsection 3.2.1, then*

$$\min_{\eta \in \text{core}(\nu)} \int_\Omega u \circ x \, d\eta = \inf_{\substack{\eta \in M(\mu(\Omega)+\beta) \\ \eta \geq \nu}} \int_\Omega u \circ x \, d\eta = \int_\Omega u \circ x \, d\mu + \beta \inf_\Omega u \circ x$$

is true that for every function $x \in B(\Omega, \mathcal{F})$ such that $u \circ x \in B(\Omega, \mathcal{F})$.

Remark 4: Even if $u \circ x \notin B(\Omega, \mathcal{F})$.

Proof: Let us denote $I(\nu) = \min_{\eta \in \text{core}(\nu)} \int_\Omega u \circ x \, d\eta$ and $F(\nu) = \inf_{\substack{\eta \in M(\mu(\Omega)+\beta) \\ \eta \geq \nu}} \int_\Omega u \circ x \, d\eta$. It is clear that $F(\nu) \geq I(\nu)$.

Let $\eta \in ba(S, \mathcal{S})$ such that $\eta \geq \nu$ and $\eta(S) = \nu(S)$. Thus $(\eta - \mu) \in ba_+$. So we get

$$\begin{aligned} \int u \circ x d\eta &= \int u \circ x d(\eta - \mu) + \int u \circ x d\mu \\ &\geq (\eta(S) - \mu(S)) \inf u \circ x + \int u \circ x d\mu \\ &= \beta \inf u \circ x + \int u \circ x d\mu \end{aligned}$$

hence $I(\nu) \geq \int_S u \circ x \, d\mu + \beta \inf_S u \circ x$. On other hand, let (x_n) a sequence in $x(S)$ such that $x_n \to \inf x$.

Let us define $\varsigma_n = \mu + \beta\delta_{x_n}$, where δ_{x_n} is the Dirac delta probability measure with mass at x_n. So $\varsigma_n \in M^{\mu(A)+\beta}$, $\varsigma_n \geq \nu$ and

$$I_n := \int u \circ x d\varsigma_n = \int_S u \circ x \, d\mu + \beta u(x_n) \geq F(\nu)$$

since $I_n \to \int u \circ x d\mu + \beta \inf u \circ x$ we are done because we get

$$\int u \circ x d\mu + \beta \inf u \circ x \geq F(\nu) \geq I(\nu) \geq \int u \circ x d\mu + \beta \inf u \circ x$$

∎

If $\phi \in ca_+^1$ and $\epsilon \in (0, 1)$, making $\mu = \epsilon\phi$ and $\beta = 1 - \epsilon$, we get the ϵ-contamination formula for the Choquet integral.

B

Lemma 2: *Let $\nu \in \mathrm{pch}_+$ such that $\nu(\mathbb{1}) = 1$. Then, $\nu(x) \in [\liminf x, \limsup x]$. In other words, ν is a Banach limit.*

PROOF OF PROPOSITION 1:

PROOF: Firt, we will show that $\partial U(x) \subset \widehat{\partial U}(x)$. Let us define $U^t : I\!R \to I\!R \cup \{-\infty\}$ by $U^t(z) = U(x_1, ..., x_{t-1}, z, x_{t-2}, ...)$. Given T an operator in $\partial U(x)$, we have, for each t, $T(e_t) \in \partial U^t(x_t) \equiv [\zeta_t u'(x_t), (\zeta_t + \beta)u'(x_t)]$ and, so, there is $\gamma_t \in [0,1] \ : \ T(e_t) = (\zeta_t + \gamma_t \beta)u'(x_t)$. When $x_t > \underline{x}$, the directional derivative $\delta U(x; e_t)$ there exists and it is equal to $\zeta_t u'(x_t)$, what implies $\gamma_t = 0$. We know (by lemma (2)) that the pure charge component of T can be written as $\alpha\,\mathrm{LIM}$ for some $\alpha \geq 0$. Define $\sigma = \alpha/\beta u'(\underline{x}) \geq 0$ and $\underline{N}$ be the ordered subset of $\mathbb{N}$ composed by all the indices $t \ : \ x_t = \underline{x}$. It is true that $T(\mathbb{1}) = \sum_{t\geq 1}(\zeta_t + \gamma_t)u'(x_t) + \sigma\beta u'(\underline{x}) = \sum_{t\geq 1}\zeta_t u'(x_t) + (\sum_{t\in \underline{N}}\gamma_t + \sigma)\beta u'(\underline{x})$. Since $\delta U(x; \mathbb{1}) = \sum_{t\geq 1}\zeta_t u'(x_t) + \beta u'(\underline{x})$, we get $T(\mathbb{1}) = \delta U(x;\mathbb{1})$ and, so, $\sum_{t=1}^{\infty}\gamma_t + \sigma = \sum_{t\in\underline{N}}\gamma_t + \sigma = 1$. It remains to show that $\sigma = 0$ when $\underline{x}$ is not a cluster point. Suppose $\sigma > 0$. There are $\varepsilon > 0$ and $t_0 \in \mathbb{N}$ such that $x_t > \underline{x} + \varepsilon$ for all $t > t_0$. Given $n \in \mathbb{N}$, let $\widetilde{x}^n$ be the sequence $(x_1, ..., x_n, \underline{x} + \frac{\varepsilon}{2}, \underline{x} + \frac{\varepsilon}{2}, ...)$. Then $\widetilde{x}^n_t - x_t < -\varepsilon/2$ for all $t > t_0$. As $T \in \partial U(x)$, the inequality $U(\widetilde{x}^n) - U(x) \leq T(\widetilde{x}^n - x)$ holds for each n. If $n > t_0$, it is true that $U(\widetilde{x}^n) - U(x) = \sum_{t>n}\zeta_t(u(\widetilde{x}^n_t) - u(x_t))$, and the right hand converges to zero when n goes to infinity. Then, $\liminf_n T(\widetilde{x}^n - x) \geq 0$. However, for all n large enough, it is true that $T(\widetilde{x}^n - x) \leq -\frac{\varepsilon}{2}\sum_{t>n}T(e_t) + \sigma\beta u'(\overline{x})\mathrm{LIM}(\widetilde{x}^n - x) < \sigma\beta u'(\overline{x})\mathrm{LIM}(\widetilde{x}^n - x)$. As $\mathrm{LIM}(\widetilde{x}^n - x) \leq \mathrm{LIM}(-\frac{\varepsilon}{2}\mathbb{1}) < 0$, we get a contradiction.

Now, take a operator $T \in \widehat{\partial U}(x)$. Let N_1 be the set of all indices $t \in \mathbb{N}$ such that $\gamma_t > 0$. Thus, $\inf u\circ y - \inf u \circ y \leq u(y_t) - u(x_t)$ for every $t \in N_1$. If $\sum_{t\in N_1}\gamma_t = 1$ (and, so, $\sigma = 0$), it is clear that, for every $y \in \ell^{\infty}_+$, we have $U(y) \leq \sum_{t\geq 1}(\zeta_t + \gamma_t\beta)u(y_t)$ and that $U(x) = \sum_{t\geq 1}(\zeta_t + \gamma_t\beta)u(x_t)$, which implies $U(y) - U(x) \leq \sum_{t\geq 1}(\zeta_t + \gamma_t\beta)(u(y_t) - u(x_t))$. Since $u(y_t) - u(x_t) \leq u'(x_t)(y_t - x_t)$ holds for each t, we get $U(y) - U(x) \leq \sum_{t\geq 1}(\zeta_t + \gamma_t\beta)u'(x_t)(y_t - x_t) = T(y - x)$. If $y \in \ell^{\infty} \cap (\ell^{\infty}_+)^c$ this inequality is obvious. If $\sum_{t\in N_1}\gamma_t < 1$ (and, so, $\sigma > 0$), $\underline{x}$ is a cluster point of x. We will show that, in this case, $\inf(u\circ y) - \inf(u\circ x) \leq u'(\underline{x})\mathrm{LIM}(x-y)$, what implies (using the same argument to dominate the series $\sum_{t\geq 1}(\zeta_t + \gamma_t\beta)u(y_t)$) the desired inequality $U(y) - U(x) \leq T(y - x)$. In fact, $\inf(u\circ y) - u(x_n) \leq u(y_m) - u(x_n)\ \forall\, n, m \in N$. Soon, $\inf(u \circ y) - u(x_n) \leq u'(x_n)(y_m - x_n)$ holds too. Making $n \in N_1$ goes to infinite and using that u is of class C^1 at $(0, +\infty)$, we get $\inf(u\circ y) - \inf(u \circ x) \leq u'(\overline{x})(y_m - \overline{x})$. Finally, making $m \to +\infty$, we obtain $\inf(u\circ y) - \inf(u\circ x) \leq u'(\overline{x})(\liminf y - \overline{x}) \leq u'(\overline{x})(\mathrm{LIM}(y) - \mathrm{LIM}(x))$. ∎

C Proofs of Section 5.1.1

PROOF OF PROPOSITION 3:

PROOF: For $h > 0$, $\frac{U(x+hv_t)-U(x)}{h} \leq 0$ so $\delta^+U(x; v_t) = \lim_{h\downarrow 0}\frac{U(x+hv_t)-U(x)}{h} \leq 0$. We know that there exists $T \in \partial U(x)$ such that $T(v_t) = \delta^+U(x; v_t)$, since $\delta^+U(x; v_t) = \inf\{L(v_t) : L \in \partial U(x)\}$, where the infimum can be replaced by the minimum, as U is norm continuous at x and therefore $\partial U(x)$ is weak* compact (see Zeidler (1984), theorem 47.A, p.387). ∎

PROOF OF PROPOSITION **??**

PROOF: (i) For every n satisfying the assumption we have

$0 \leq \lim_{h\uparrow 0} \frac{U(x+hyn)-U(x)}{h} = T^n(\sum_{t>n}^{\infty}(q_t(z_{t-1}-z_t)+R_t z_{t-1})e_t) - q_n z_n T^n e_n,$

where T^n is some supergradient of U at point x (since the limit is $\delta^- U(x; v_t)$, which is equal to $\max\{L(v_t) : L \in \partial U(x)\}$, as U is norm continuous at x. Now, $T^n = \mu^n + \nu^n$, where $\mu^n \in \ell^1$ and ν^n is a pure charge. Hence,

$$\sum_{t>n}^{\infty}(q_t(z_{t-1}-z_t)+R_t z_{t-1})\mu_t^n + \nu^n(\sum_{t>n}^{\infty}(q_t(z_{t-1}-z_t)+R_t z_{t-1}) - q_n z_n \mu_n^n \geq 0. \qquad (21)$$

Let $n \to \infty$. The sequence (T^n) lies in the weak* compact set $\partial U(x)$ and therefore has a subsequence converging, in the weak* topology, to some T in this set. Without loss of generality and to simplify the notation, we take this subsequence to be the initial sequence. Let us show that the associated pure charges sequence (ν^n) lies in a bounded set and, therefore, without loss of generality, in a weak* compact set. Now, as (T^n) lies in a weak* compact set, there is $N > 0$ such that $\|T^n\| \equiv \max\{|T^n(g)| : |g| \leq 1\} \leq N$. For every g in the unit ball of ℓ^∞, then $g^m = (0, ..., 0, g_{m+1}, g_{m+2}, ...)$ is also in the unit ball. So $\left|\sum_{k>m} \mu_k^n g_k^m + \nu^n(g^m)\right| = |T^n(g^m)| \leq N$, where $\mu^n = T^n - \nu^n$. Take the limit as m goes to ∞, $\nu(g) \leq N$ since $\nu(g) = \nu(g^m)$ for every m. Then we can take a subsequence of (ν^n) converging in the weak* topology to some ν and along this subsequence the associated countably additive components converges also to some μ. Again, without loss of generality, we stick to the same sequence. Then, (μ^n) converges in the weak topology of ℓ^1 to μ and, therefore, the union set of terms of the sequence and its limit constitutes a weakly compact set.

Let $\widetilde{y}(n) = y(n) + q_n z_n e_n$. By Lemma (5) in the appendix, the sequence $(\widetilde{y}(n))$ tends, as n goes to ∞, to zero in the Mackey topology (and, therefore, uniformly on weakly compact sets in ℓ^1). Hence, the series in inequality (21) tends to zero, as n goes to ∞.

Notice that, for every n, $\nu^n(y(n)) = \nu^n(y(1))$ and, as (ν^n) converges in the weak* topology, $\lim \nu^n(y(1))$ exists. This completes the proof of item (i). The proof of item (ii) is analogous.

(iii) Inequality (21) holds now as an equality for any $T \in \partial U(x)$ decomposable into a countably additive part μ and a pure charge ν. The series also tends to zero as n goes to ∞. By the argument at the end of the proof of item (i) we complete the proof. ∎

Remark 5: Given a point x where U is norm continuous, define $k_n = \dfrac{\min_{T\in\partial U(x)} T(e_n)}{\max_{T\in\partial U(x)} T(e_n)}$.

So, for any $T^1, T^2 \in \partial U(x)$, we have $k_n \leq \frac{T^1(e_n)}{T^2(e_n)} \leq \frac{1}{k_n}$.

Under the same assumptions of the previous proposition, the inequality in item (i) can be replaced by

$$\nu(x-\omega) \geq \limsup \gamma_n \mu_n q_n z_n$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components of some $T \in \partial U(x)$ and the inequality in item (ii) can be replaced by

$$\nu(x-\omega) \leq \liminf \frac{1}{\gamma_n} \mu_n q_n z_n$$

where μ^n and ν^n are, respectively, the countably additive and the pure charge components of some $T \in \partial U(x)$ for γ_n given by $1/k_n$ when $z_n \geq 0$ and by k_n when $z_n < 0$.

(in fact, in each item, we can use the limit points of the sequences μ^n and ν^n (see also the proof of the proposition))

Notice that, when k_n tends to one, we obtain in item (i) $\nu(x-\omega) \geq \limsup \mu_n q_n z_n$ and in item (ii) $\nu(x-\omega) \leq \liminf \mu_n q_n z_n$.

D

PROOF OF LEMMA 8

PROOF: Notice that for each t there exists $S^t \in \partial U(x)$ (by the same argument used in the proof of (3)) such that

$$\lim_{h \downarrow 0} U(x + h(q_t e_t - \sum_{\tau > 1} R_\tau e_\tau) - U(x) = S^t(q_t e_t - \sum_{\tau > 1} R_\tau e_\tau) \tag{22}$$

There are two cases to consider. If the directional derivatives exist, for the directions $-q_t e_t + (q_{t+1} + R_{t+1})e_{t+1}$, then, by the assumption of the Proposition, at every t, Euler equations will hold for every element in superdifferential, namely for S^1, $-q_t S^1(e_t) + (q_{t+1} + R_{t+1})S^1(e_{t+1})$. Choosing $\lambda_1 = S^1(e_1)$ we get $\lambda_t = S^1(e_t)$ as the non-arbitrage deflator. Since, by assumption, there is a bubble, we have $S^1(q_1 e_1 - \sum_{\tau>1} R_\tau e_\tau)) > 0$, which implies that $U(x + h(q_1 e_1 - \sum_{\tau>1} R_\tau e_\tau) > U(x)$ for $h > 0$ sufficiently small.

In the other case allowed by condition D, as the bubble never bursts, we have $q_t \lambda_t > \sum_{s>t} R_s S^t(e_s) \frac{\lambda_s}{S^t e_s}$. Now, $\frac{\lambda_s}{S^t(e_s)}$ can be written as $\left[\Pi_{t \leq \tau \leq s} \frac{\varphi_\tau^{\tau-1}/\varphi_{\tau-1}^{\tau-1}}{S^t(e_\tau)/S^t(e_{\tau-1})}\right] \times \frac{\lambda_t}{S^t(e_t)}$, where each term in the product inside square brackets is greater or equal to c_τ which is less than or equal to one. Then $\frac{\lambda_s}{S^t(e_s)} \geq \frac{\lambda_t}{S^t(e_t)} \times \Pi_{\tau \geq t}^{\infty} c_\tau$.

By condition D, $\Pi_{\tau \geq 1}^{\infty} c_\tau$ is a positive number k. Then $\Pi_{\tau \geq t}^{\infty} c_\tau = k / \Pi_{1 \leq \tau < t} c_\tau$. So, given $\varepsilon > 0$, there exists $t(\varepsilon)$ such that $\Pi_{\tau \geq t}^{\infty} c_\tau$ is greater than $1 - \varepsilon$, for $t > t(\varepsilon)$. Hence, $q_t \lambda_t > (1-\varepsilon) \frac{\lambda_t}{S^t(e_t)} \sum_{s>t} R_s S^t(e_s)$, for $t > t(\varepsilon)$. Now, we know that $\lim_t q_t \lambda_t > 0$ (since there is a bubble), $\frac{\lambda_t}{S^t(e_t)}$ is bounded (by lemma (3 in the appendix) and $(R_s S^t(e_s))_s \in \ell^1$. So the right-hand side tends to zero as t goes to infinity and, therefore, we can find t large enough so that $q_t > \frac{1}{S^t(e_t)} \sum_{s>t} R_s S^t(e_s)$, as desired. ■

E Other Results

We know by Proposition (3) and the subsequent remark that, if for every t the direction $v_t = -q_t e_t + (q_{t+1} + R_{t+1})e_{t+1}$ is admissible, then, for each t, there is a supergradient whose values at e_t and e_{t+1}, denoted by φ_t^t and φ_{t+1}^t, such that $\varphi_t^t q_t = \varphi_{t+1}^t (q_{t+1} + R_{t+1})$. Recall that we may have $\varphi_{t+1}^t \neq \varphi_{t+1}^{t+1}$ and therefore the non-arbitrage deflator may not be obtained directly from the Euler equations. However, fixing $\lambda_1 = \varphi_1^1$, we have for $t > 1$ the following relation:

$$\lambda_t = \frac{\varphi_t^{t-1}}{\varphi_{t-1}^{t-1}} \times \frac{\varphi_{t-1}^{t-2}}{\varphi_{t-2}^{t-2}} \times \ldots \times \frac{\varphi_2^1}{\varphi_1^1} \times \varphi_1^1 \tag{23}$$

Lemma 3: *Suppose that, at some point x, for every t condition (9) holds with equality and, for (c_t) evaluated at x, $\Pi_{t \geq 1} c_t > 0$, then:*

(i) for any $L \in \partial U(x)$ we have $\Pi_{s \geq 1} c_s \dfrac{L(e_t)}{L(e_1)} \leq \dfrac{\lambda_t}{\lambda_1} \leq \dfrac{1}{\Pi_{s \geq 1} c_s} \dfrac{L(e_t)}{L(e_1)}$

(ii) $\lambda \in \ell^1$.

PROOF: (i) Monotonicity of U guarantees that $L(e_t) > 0$ for every t. Then we can always write $L(e_t) = \frac{L(e_t)}{L(e_{t-1})} \times \frac{L(e_{t-1})}{L(e_{t-2})} \times ... \times \frac{L(e_2)}{L(e_1)} \times L(e_1)$. By definition of c_t, $\frac{L(e_t)}{L(e_{t-1})} \geq c_t \frac{\mu_t^{t-1}}{\mu_{t-1}^{t-1}}$. Then, using (23), $L(e_t) \geq \lambda_t \frac{L(e_1)}{\lambda_1} \Pi_{s=2}^t c_s$ and therefore $\lambda_t \leq \frac{1}{\Pi_{s=2}^\infty c_s} L(e_t) \frac{\lambda_1}{L(e_1)}$. We can replace $L(e_t)$ by λ_t and vice-versa to obtain the other inequality.

(ii) From (i), since $(L(e_t))_t \in \ell^1$ and $\lambda_t > 0$. ■

Lemma 4: *Let $\chi_{\ell_+^\infty}$ be the extended real valued functional that takes value 0 on ℓ_+^∞ and $-\infty$ elsewhere. Then, for $x \in \ell_+^\infty$, the superdifferential $\partial\chi_{\ell_+^\infty}(x)$ is the set* $\{T \in \text{ba} : T(y) \geq 0 \forall y \in \ell_+^\infty \text{ and } T(x) = 0\}$.

PROOF: Using example 47.9 in Zeidler (1985), p.385, we have, for $x \in \ell_+^\infty$, that $\partial\chi_{\ell_+^\infty}(x)$ is the set $\{T \in \text{ba} : T(y) \geq T(x) \forall y \in \ell_+^\infty\}$ (we have changed the sign of $\chi_{\ell_+^\infty}$ to make it concave and then we work with the superdifferential instead of using the subdifferential). Now, applying T to 0 and to $2x$, we get, using the above inequality, $T(x) = 0$. ■

Lemma 5: *If $x \in \ell^\infty$, let $x^n = (0, ..., 0, x_{n+1}, x_{n+2}, ...)$. Then, x^n converges to 0 in the Mackey topology.*

PROOF: Let E_n be the set $\{n+1, n+2, ...\}$. We know that $\mathbb{1}_{E_n}$ converges to 0 in the Mackey topology (see Bewley (1972), p.534, remark 24). We have to show that, for any weakly compact subset K of ℓ^1, $\langle y, x^n \rangle$ tends to zero, uniformly on $y \in K$. Let $|\overline{x}|$ and $|\underline{x}|$ be the supremum and the infimum of the sequence x. Then $\underline{x} \langle y, \mathbb{1}_{E_n} \rangle \leq \langle y, x^n \rangle \leq \overline{x} \langle y, \mathbb{1}_{E_n} \rangle$. Now $\langle y, \mathbb{1}_{E_n} \rangle$ converges to zero uniformly in $y \in K$ and the result follows. ■

Lemma 6: *Let U be a monotonous concave utility function on ℓ_+^∞. If U is Mackey continuous at $x \in \text{int}_{\|.\|}\ell_+^\infty$, then $\partial U(x) \subset \ell^1$.*

PROOF: By monotonicity we have $U(x + e_t) > U(x)$ for any t. Then, by the previuos lemma, $e_t + x - \alpha\mathbb{1}_{E_n}$ converges in the Mackey topology to $e_t + x$, for any $\alpha \in I\!R$. Let us pick a positive α such that $x - \alpha\mathbb{1}_{E_n} \in \ell_+^\infty$. Hence, for n sufficiently large, without loss of generality above t, it is true that $U(e_t + x - \alpha\mathbb{1}_{E_n}) > U(x)$. However, for any $T \in \partial U(x)$ and n large enough, we know that $0 < U(e_t + x - \alpha\mathbb{1}_{E_n}) - U(x) \leq T(e_t - \alpha\mathbb{1}_{E_n})$. Now by the Yosida-Hewitt theorem, $T = \mu + \nu$, where $\mu \in \ell^1$ and ν is a pure charge (**see footnote** ...). If $T \notin \ell^1$, then, by monotonicity, μ and ν are positively valued on $\text{int}_{\|.\|}\ell_+^\infty$. This implies that, for n large enough, $0 < \mu_t - (\sum_{s>n} \mu_s + \alpha\nu(\mathbb{1}_{E_n}))$. We claim that t can be chosen sufficiently large so that the right hand side becomes negative, which is a contradiction. In fact, since ν is a pure charge, we have $\nu(\mathbb{1}_{E_n}) = \nu(\mathbb{1}) > 0$ and, on the other hand, $\sum_{s>n} \mu_s$ tends to zero because n goes to infinity (since $t \to \infty$). Given α and $\nu(\mathbb{1})$ we can choose t sufficiently large so that the claim holds. ■

This lemma does not extend to boundary points. Actually, we have the following result (that does not depend on U being Mackey continuous):

Lemma 7: *If x is boundary point x of ℓ_+^∞ such that $\partial U(x) \neq \emptyset$, then $\partial U(x)$ is not contained in ℓ^1.*

PROOF: Given x not uniformly bounded away from zero and $T \in \partial U(x)$, it is always possible to find a nonnegative pure charge ν such that $\nu(x) = 0$, which implies that $T + \nu \in \partial U(x)$. In fact, take the subsequence x_{n_i} converging to zero and let N' the ordered set of natural numbers constituted by the respective indices. Define a continuous linear functional $\nu : \ell^\infty \to I\!R$ in the following way. Consider the collection of points $y \in \ell^\infty$ such that the subsequence y_{n_i} with $n_i \in N'$ converges. The function mapping each of these points y into $\lim_{n_i \in N'} y_{n_i}$ is linear, nonnegative and continuous in the norm topology, so it can be extended to a linear, nonnegative and continuous functional ν on the whole space (see Schaeffer (1966), p.227, corollary 2). Now ν is a pure charge if and only if for every $\lambda \in \ell^1$ and $\varepsilon > 0$ there is $B \subset \mathbb{N}$ such that $\nu(\mathbb{1}_B) = 0$ and $\lambda(\mathbb{1}_{B^c}) < \varepsilon$ (see Bhaskara Rao and Bhaskara Rao (1983), p.244, Theorem 10.3.2). This holds since we can find a large finite set B such that $\lambda(\mathbb{1}_{\mathbb{N}}) - \sum_{t \in B} \lambda_t < \epsilon$ and, moreover, by definition of ν, $\nu(\mathbb{1}_B) = 0$. It remains to show that $T + \nu \in \partial U(x)$. The inequality $U(x') - U(x) \leq T(x' - x)$, together with $\nu(x) = 0$ and $\nu(x') \geq 0$, implies that $U(x') - U(x) \leq (T + \nu)(x' - x)$, as desired. ■

We can netherless say the following about superdifferentials, even on the boundary points of ℓ^∞_+.

Lemma 8: *Suppose U is a monotonous, concave and Mackey continuous function on ℓ^∞_+. Let $x \in \ell^\infty_+$. For any $\mu + \nu \in \partial U(x)$, where $\mu \in \ell^1$ and $\nu \in \text{pch}$, we have $\mu \in \partial U(x)$. Moreover, $\nu(x) = 0$.*

PROOF: Let us show that μ is also a supergradient. Take any $y \in \ell^\infty_+$, we know, by Lemma (5), that $y^n = (0, ..., 0, x_{n+1} - y_{n+1}, x_{n+2} - y_{n+2}, ...)$ converges in the Mackey topology to 0. Then, given $m \in \mathbb{N}$, there exists n_0 such that, for $n > n_0$, $U(y) \leq U(y + y^n) + 1/m$. Hence, $U(y) - U(x) \leq \sum_{t=1}^n \mu_t(y_t - x_t) + 1/m$. Since $\mu \in \ell^1$, $\sum_{t=1}^\infty \mu_t$ is finite. So taking the limit as m goes to ∞, we see that $U(y) - U(x) \leq \sum_{t=1}^\infty \mu_t(y_t - x_t) < \infty$. We know that ν is a nonnegative operator. Suppose $\nu(x) > 0$. Let $x^n = (x_1, ..., x_n, x_{n+1}/2, x_{n+2}/2, ...)$, then $U(x^n) - U(x) \leq (\mu + \nu)(x^n - x)$. Now, x^n converges to x in the Mackey topology, so the left hand side tends to zero. However, the right hand side is equal to $-\sum_{t>n} \mu_t x_t/2 - \nu(x/2) < -\nu(x/2) < 0$. As the sum tends to zero as n goes to ∞, $\lim_n(\mu + \nu)(x^n - x)$ is negative, a contradiction. ■

Let us see an additional property for pure charge components of a supergradient. We have shown that each positive pure charge is a distortion of a generalized limit. Now, we will say more about the distortion coefficient.

Proposition 15: *Let $T \in \partial U(x)$ and $(\mu, \nu) \in \text{ca} \times \text{pch}$ its Yosida-Hewitt decomposition. There are a generalized limit* LIM *and a real constant $\alpha \in [\lim_n \delta^+ U(x; \mathbb{1}_{E_n}), \lim_n \delta^- U(x; \mathbb{1}_{E_n})]$ such that $\nu(x) = \alpha \, \text{LIM}(x) \;\; \forall \, x \in \ell^\infty$.*

PROOF: We just need to show that α belongs to the mentioned interval. Given $n \in \mathbb{N}$, it is true that $\delta^+ U(x; \mathbb{1}_{E_n}) \leq T(\mathbb{1}_{E_n}) \leq \delta^- U(x; \mathbb{1}_{E_n})$. Moreover, $T(\mathbb{1}_{E_n}) = \sum_{t>n} \mu_t + \nu(\mathbb{1}_{E_n})$. Since, $\forall n$, $\nu(\mathbb{1}_{E_n}) = \nu(\mathbb{1}) = \alpha$ and $\lim_n \sum_{t>n} \mu_t = 0$, we get $\lim_n \delta^+ U(x; \mathbb{1}_{E_n}) \leq \alpha \leq \lim_n \delta^- U(x; \mathbb{1}_{E_n})$. ■

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