Uniform Topologies on Types*

Yi-Chun Chen[†] Alfredo Di Tillio[‡] Eduardo Faingold[§] Siyang Xiong[¶]

April 16, 2009

Abstract

We study the continuity of the correspondence of interim correlated ε-rationalizable actions in incomplete information games. We introduce a topology on types, called *uniform-weak topology*, under which two types of a player are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. First, we show that the uniform-weak topology is finer than the uniform-strategic topology introduced by Dekel, Fudenberg, and Morris (2006). Second, we prove a partial converse: around finite types, the uniform-weak topology is equivalent to the strategic topology (and hence to the uniform-strategic topology). Finally, we show that the set of finite types is nowhere dense under the uniform strategic topology, which implies that the uniform-strategic topology is strictly finer than the strategic topology.

1 Introduction.

Incomplete information games are games in which some payoff-relevant states are not common knowledge among the players. Harsanyi (1967-68) observes that the Bayesian

*We thank Pierpaolo Battigalli, Martin Cripps, Eddie Dekel, Jeffrey C. Ely, Amanda Friedenberg, Drew Fudenberg, Qingmin Liu, George J. Mailath, Stephen Morris, Marcin Peski, Aaron Sojourner, Satoru Takahashi and Muhamet Yildiz for useful comments and interesting conversations.

[†]Department of Economics, Northwestern University, yichun.chen@northwestern.edu

[‡]IGIER and Department of Economics, Università Luigi Bocconi, alfredo.ditillio@unibocconi.it

[§]Department of Economics, Yale University, eduardo.faingold@yale.edu

[¶]Department of Economics, Northwestern University, siyang.xiong@northwestern.edu

analysis of incomplete information games requires a model in which each player is equipped with an infinite hierarchy of beliefs: a belief about the payoff-relevant states, a belief about his opponents' beliefs about the payoff-relevant states, and so on. Following this observation, Harsanyi (1967-68) introduces type spaces as a parsimonious model that encodes the belief hierarchies and is suitable for game theoretic analysis, in that interim best-reply sets can be appropriately defined. Mertens and Zamir (1985) provide a foundation for the use of type spaces showing that the space $\mathcal{T}$ of coherent belief hierarchies is a *universal type space*. That is, $\mathcal{T}$ is a type space itself and, moreover, every type space can be embedded in $\mathcal{T}$ via a belief-preserving morphism. Hence, the universal type space $\mathcal{T}$ captures the richness of any abstract type space, and not more.

The Mertens-Zamir universal type space comes with a natural topology: the product topology.[1] A distinctive feature of the product topology is that it is insensitive to the tails of belief hierarchies: two types are close in the product topology if, and only if, their k^{th}-order beliefs are close for some large *finite* $k \geq 1$. Strategic behavior, however, can be very sensitive to high order beliefs. This is true even for interim rationalizability (see Dekel, Fudenberg, and Morris (2007)), the most permissive solution concept consistent with common knowledge of rationality. In effect, in Rubinstein (1989)'s electronic mail game, an action – "attack" – is *strictly* rationalizable for a type t, but fails to be rationalizable for all types in a sequence that converges to t in the product topology. As shown in Weinstein and Yildiz (2007), this discontinuity is not peculiar to the electronic mail game but is rather pervasive: in any game satisfying a certain (generic) payoff-richness condition, any type with multiple rationalizable actions is the limit (in product topology) of a sequence of types with unique rationalizable actions. Hence, to the extent that strategic behavior is what one ultimately cares about, the product topology yields an inadequate notion of proximity of types.

From this point of view, the appropriateness of a topology on types depends on what is meant by strategic behavior. But given a solution concept, it is natural to consider the coarsest topology under which the correspondence that maps types into solutions is continuous in every game. For the solution concept of interim ε-rationalizability

[1] It is only when $\mathcal{T}$ is equipped with the measurable structure induced by the product topology that $\mathcal{T}$ can be shown to be a universal type space. This is the sense in which the product topology is natural.

this yields the *strategic topology* on types introduced by Dekel, Fudenberg, and Morris (2006), hereafter DFM. The strategic topology, while being strong enough to render ε-rationalizable behavior continuous, is remarkably weak: DFM show that finite types are dense.

Given the importance of the strategic topology,[2] and the fact that it is a topology on types that is independent of the strategic situation (i.e., action sets and payoffs), we find it conceptually important to give it an internal characterization, i.e. a characterization in terms of properties of the belief hierarchies, with no direct reference to rationalizability. We introduce a topology on types, called the *uniform-weak topology*, under which two types of a player are close if they have similar first-order beliefs, attach similar probabilities to other players having similar first-order beliefs, and so on, where the degree of similarity is uniform over the levels of the belief hierarchy. More rigorously, the uniform weak topology is the topology induced by the metric d^{uw}, defined as follows: for each order $k \geq 1$, let d^k be the Prokhorov metric on k-order beliefs; metric d^{uw} is defined as the supremum of d^k over all orders $k \geq 1$. In Theorem 1 we show that convergence in the uniform-weak topology implies convergence in the *uniform-strategic topology*. The latter is a variation of the strategic topology, also introduced in Dekel, Fudenberg, and Morris (2006), in which strategic behavior (as given by interim correlated rationalizability) is required to converge at a rate that is uniform over all games. In Theorem 2, we provide a partial converse: uniform-weak convergence is a necessary condition for strategic convergence to *finite types*. Thus, as far as convergence to finite types is concerned, the strategic, the uniform-strategic and the uniform-weak topologies are all equivalent. Finally, in Theorem 3 we show that the set of finite types is nowhere dense under the uniform-strategic topology. In light of this theorem and DFM's result that finite types are dense under the strategic topology, we conclude that the uniform-strategic topology is strictly finer than the strategic topology.

The connection between uniform topologies on types and the strategic topology was first suggested by Morris (2002), who studies a particular class of infinite-action games, called higher-order expectation games (HOE), and shows that a certain topology on types (different from ours) is equivalent to the weakest topology under which the

[2] One reason why the study of strategic convergence seems important is that it appears to be a useful step for the examination of robustness questions in mechanism design.

ε-rationalizability correspondence is continuous in every game of the HOE class. This uniform topology is too strong for our purposes: there exists a sequence of types, (t_n), which fails to converge (in this uniform topology) to a type finite type t, and yet in every *finite* game, every rationalizable action for t remains ε-rationalizable for t_n for all n large enough.

The connection between uniform and strategic convergence of types also underlies the main result in Monderer and Samet (1989). They show that a sufficient condition for the correspondence of Bayesian-Nash ε-equilibrium to be continuous at a *complete-information* type profile is that the sequence of approximating type profiles converges to its complete information limit in the *common p-belief sense.* (That is, for every $p > 0$, at every type profile sufficiently far in the tail of the sequence there is common p-belief of the state that is common certainty in the limit.) Moreover, they show that this notion of convergence of type profiles yields strategic continuity in every game. Kajii and Morris (1997) prove the converse: If a sequence of type profiles fails to converge to a complete information type in the common p-belief sense, then a finite game exists such that for some $\varepsilon > 0$, some equilibrium of the complete information game will fail to be an ε-equilibrium at every type profile in the tail of the sequence. It is interesting to note that a sequence of types converges to a complete information type in the uniform-weak topology if, and only if, it converges in the common p-belief sense. Hence, the topology of uniform-weak convergence extends the notion of common p-belief convergence to incomplete information limit types.

This paper is also closely related to contemporaneous work by Ely and Peski (2008). Following their terminology, a type is called *regular* if for every finite game the ε-rationalizability correspondence is continuous in the product topology. Ely and Peski (2008) provide an insightful characterization of regular types in terms of properties of the belief hierarchies and show that the set of regular types is generic (in a topological sense). They prove:

Theorem (Ely and Peski (2008))**.** *A type t is regular if, and only if, for every $p > 0$ and every closed (in the product topology), proper subset W of the universal type space, W is not common p-belief at t. Furthermore, the set of regular types is residual, that is, it contains a countable intersection of open and dense sets.*

Thus, around all types in a generic set (in product topology), the strategic topology coincides with the product topology. While topological genericity is interesting, we think it should not be the end of the story. We find it conceptually important to characterize the strategic topology also around *critical types*, namely, those types which are not regular. In fact, given Ely and Peski (2008)'s result, it appears to us that every type space ever considered in applications consists entirely of critical types. We take a first step towards such characterization by proving the equivalence between the strategic topology and the uniform-weak topology around *finite types*. All finite types are critical, but not conversely.

2 Preliminaries

Throughout the paper, we fix a two-player set I and a finite set Θ of payoff-relevant states. Given a player $i \in I$, we write $-i$ to designate the other player in I.[3] All topological spaces, when viewed as measurable spaces, are endowed with their Borel σ-algebra. For a topological space S we write $\Delta(S)$ to designate the space of probability measures over S endowed with the weak* topology. Unless explicitly noted, all product spaces will be endowed with the product topology and subspaces with the relative topology.

2.1 Belief hierarchies and types.

Our formulation of incomplete information follows Mertens and Zamir (1985).[4] Define $X^0 = \Theta$, $X^1 = X^0 \times \Delta(X^0)$, and for each $k \geq 2$ define recursively

$$X^k = \left\{(\theta, \mu^1, \dots, \mu^k) \in X^0 \times \bigtimes_{\ell=1}^{k} \Delta(X^{\ell-1}) \ : \ \mathsf{marg}_{X^{\ell-2}} \mu^\ell = \mu^{\ell-1} \ \forall \ell = 2, \dots, k\right\}.$$

By virtue of the above coherency condition on marginal distributions, each element of X^k is determined by its first and last coordinates, so we can identify X^k with $\Theta \times \Delta(X^{k-1})$. For each $i \in I$ and $k \geq 1$ we let $\mathcal{T}_i^k = \Delta(X^{k-1})$ designate the space of *k-order beliefs* of player i, so that $\mathcal{T}_i^k = \Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$. The space $\mathcal{T}_i$ of *hierarchies of beliefs* of

[3] We restrict attention to two-player games with finitely many payoff-relevant states for ease of notation. All our results remain valid when there is an arbitrary finite number of players and Θ is a compact metric space.

[4] An alternative, equivalent formulation is found in **?**.

player i is defined as

$$\mathcal{T}_i = \Big\{(\mu^k)_{k\geq 1} \in \bigtimes_{k\geq 1} \Delta(X^k) \ : \ \mathrm{marg}_{X^{k-2}}\, \mu^k = \mu^{k-1} \ \ \forall k \geq 2\Big\}.$$

Since Θ is finite, $\mathcal{T}_i$ is a compact metrizable space. Moreover, there is a unique mapping $\mu_i : \mathcal{T}_i \to \Delta(\Theta \times \mathcal{T}_{-i})$ that is *belief-preserving*, i.e. for all $t_i = (t_i^1, t_i^2, \ldots) \in \mathcal{T}_i$ and $k \geq 1$,

$$\mu_i(t_i)[\theta \times (\pi_{-i}^k)^{-1}(E)] = t_i^{k+1}[\theta \times E] \quad \text{for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^k,$$

where π_i^k is the natural projection of $\mathcal{T}_i$ onto $\mathcal{T}_i^k$. Furthermore, the mapping μ_i is a homeomorphism, and so, to save on notation, we will identify each hierarchy of belief $t_i \in \mathcal{T}_i$ with its corresponding belief $\mu_i(t_i)$ over $\Theta \times \mathcal{T}_{-i}$. Similarly, for each $t_i \in \mathcal{T}_i$ we will write $t_i^k \in \mathcal{T}_i^k$ instead of the more cumbersome $\pi_i^k(t_i)$.

Hierarchies of beliefs can be implicitly described using a *type space*, i.e. a tuple $(T_i, \phi_i)_{i\in I}$ where each T_i is a Polish space of *types* and each $\phi_i : T_i \to \Delta(\Theta \times T_{-i})$ is a measurable function. Indeed, every type $t_i \in T_i$ is mapped into a belief hierarchy $\nu_i(t_i) = (\nu_i^k(t_i))_{k\geq 1}$ in a natural way: $\nu_i^1(t_i) = \mathrm{marg}_\Theta\, \phi_i(t_i)$ and for $k \geq 2$,

$$\nu_i^k(t_i)[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i}^{k-1})^{-1}(E)] \ \text{ for all } \theta \in \Theta \text{ and measurable } E \subseteq \mathcal{T}_{-i}^{k-1}.$$

The type space $(\mathcal{T}_i, \mu_i)_{i\in I}$ is called the *universal type space*, since for every type space $(T_i, \phi_i)_{i\in I}$ there is a unique belief-preserving mapping from T_i into $\mathcal{T}_i$, namely the mapping ν_i above.[5] When the mappings $(\nu_i)_{i\in I}$ are injective the type space $(T_i, \phi_i)_{i\in I}$ is called *non-redundant*. In this case each ν_i is an embedding onto the image $\nu_i(T_i)$, which is a measurable set and constitutes a *belief-closed subspace* of $\mathcal{T}_i$, i.e. $\mu_i(\nu_i(t_i))[\Theta \times \nu_{-i}(T_{-i})] = 1$ for all $i \in I$ and $t_i \in T_i$. Conversely, every belief-closed subspace of $\mathcal{T}_i$ can be viewed as a non-redundant type space.

2.2 Bayesian games and interim correlated rationalizability.

A *game* is a tuple $G = (A_i, g_i)_{i\in I}$, where A_i is a finite set of *actions* for player i and $g_i : A_i \times A_{-i} \times \Theta \to [-M, M]$ is the *payoff* function of player i, with $M > 0$ an arbitrary bound on payoffs that we fix throughout. Let $\mathcal{G}$ denote the set of all games, and for each integer $m \geq 1$ let $\mathcal{G}^m$ denote the set of all games in which $|A_i| \leq m$ for all $i \in I$.

[5] To say that ν_i is belief-preserving means that $\mu_i(\nu_i(t_i))[\theta \times E] = \phi_i(t_i)[\theta \times (\nu_{-i})^{-1}(E)]$ for all $\theta \in \Theta$ and measurable $E \subseteq \mathcal{T}_{-i}$.

The solution concept of *interim correlated ε-rationalizability*, or ε-ICR, was introduced in Dekel, Fudenberg, and Morris (2007). Given a type space $(T_i, \phi_i)_{i \in I}$ and a game G, for each player $i \in I$, each integer $k \geq 0$, and each type $t_i \in T_i$, we let $R_i^k(t_i, G, \varepsilon) \subseteq A_i$ designate the set of *k-order ε-rationalizable actions* of t_i, writing just $R_i^k(t_i, \varepsilon)$ when G is clear from the context. These sets are defined as follows:

$$R_i^0(t_i, G, \varepsilon) = A_i,$$

and recursively for $k \geq 1$, $R_i^k(t_i, G, \varepsilon)$ is the set of all actions $a_i \in A_i$ for which there is a measurable function $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \varepsilon) \quad \text{for } \phi_i(t_i)\text{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i},$$

and for all $a_i' \in A_i$,

$$\int_{\Theta \times T_{-i}} [g_i(a_i, \sigma_{-i}(\theta, t_{-i}), \theta) - g_i(a_i', \sigma_{-i}(\theta, t_{-i}), \theta)]\, \phi_i(t_i)(d\theta \times dt_{-i}) \;\geq\; -\varepsilon.$$

For future reference, a mapping $\sigma_i : \Theta \times T_{-i} \to \Delta(A_{-i})$ satisfying the former condition will be called a *$(k-1)$-order ε-rationalizable conjecture*.

The set of *ε-rationalizable* actions of type t_i is then defined as

$$R_i(t_i, G, \varepsilon) = \bigcap_{k \geq 1} R_i^k(t_i, G, \varepsilon).$$

Dekel, Fudenberg, and Morris (2007) shows that the set of ε-ICR actions for a type is determined by the hierarchy of beliefs of the type. That is, if two types (possibly belonging to different type spaces) have the same hierarchy of beliefs then they will have the same set of ε-ICR actions in all games. Thus, without loss of generality we will identify all types with their corresponding hierarchies of beliefs and, accordingly, we will take the universal type space to be the domain of the correspondence $R_i(\cdot, G, \varepsilon) : \mathcal{T}_i \rightrightarrows A_i$.

A property of ICR that we will repeatedly use is that the k-order ε-rationalizability correspondence, $R_i^k(\cdot, G, \varepsilon) : \mathcal{T}_i \rightrightarrows A_i$, is measurable with respect to k-order beliefs, i.e. $R_i^k(t_i, G, \varepsilon) = R_i^k(s_i, G, \varepsilon)$ for every pair of types s_i and t_i with $t_i^k = s_i^k$. This implies that, in order to establish whether a certain action is k-order ε-rationalizable, we can restrict attention to $(k-1)$-order ε-rationalizable conjectures $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$

satisfying the additional restriction that $\sigma_{-i}(\theta, t_{-i}) = \sigma_{-i}(\theta, s_{-i})$ for all t_{-i}, s_{-i} with $t_{-i}^{k-1} = s_{-i}^{k-1}$.

Yet another property that will prove useful in what follows is that ICR can be alternatively defined using *best reply sets*. In particular, fix a measurable function $\varsigma_i : T_i \to 2^{A_i}$ for every player i. If, for every player i and every $t_i \in T_i$, each $a_i \in \varsigma_i(t_i)$ can be ε-rationalized by means of a conjecture $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ such that $\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq \varsigma_{-i}(t_{-i})$ for $\phi_i(t_i)$-almost every $(\theta, t_{-i}) \in \Theta \times T_{-i}$, then $\varsigma_i(t_i) \subseteq R_i(t_i, G, \varepsilon)$ for every i and t_i.

Finally, the following result shows that, just as for rationalizability in complete information games, interim rationalizability has a characterization in terms of iterated dominance, where the notion of dominance is now an interim one.

Proposition 1. *Let $G = (A_i, g_i)_{i \in I}$ be a game and let $(T_i, \phi_i)_{i \in I}$ be a type space. Let $\varepsilon \geq 0$, $k \geq 1$, $i \in I$, $t_i \in T_i$, and $a_i \in A_i$. Then $a_i \in R_i^k(t_i, G, \varepsilon)$ if and only if for every $\alpha_i \in \Delta(A_i)$ and $\eta > 0$ there exists a measurable $\mathbf{a}_{-i} : \Theta \times T_{-i} \to A_{-i}$ such that*

$$\mathbf{a}(\theta, t_{-i}) \in R_{-i}^{k-1}(t_{-i}, G, \varepsilon) \quad \textit{for } \phi_i(t_i)\textit{-almost every } (\theta, t_{-i}) \in \Theta \times T_{-i} \tag{1}$$

$$\textit{and} \quad \int_{\Theta \times T_{-i}} [g_i(a_i, \mathbf{a}_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}(\theta, t_{-i}), \theta)]\, \phi_i(t_i)(d\theta \times dt_{-i}) \geq -\varepsilon - \eta. \tag{2}$$

The proof of the proposition, relegated to Appendix A, uses a separation argument analogous to the familiar one establishing the equivalence between strictly dominated and never-best reply strategies in complete information games. Here, too, the usefulness of the result comes from the fact that, in order to check whether an action is rationalizable for a type, we are allowed to reverse the order of quantifiers by seeking a possibly different conjecture for each possible (mixed) deviation.

3 Topologies on types.

As introduced in Dekel, Fudenberg, and Morris (2006), the *strategic topology*, or S-*topology* for short, is the coarsest topology on the universal type space $\mathcal{T}_i$ under which the correspondence of interim correlated ε-rationalizable actions, $R_i(\cdot, G, \cdot) : \mathcal{T}_i \times [0, M] \rightrightarrows A_i$, is upper and lower hemi-continuous (along sequences $\varepsilon_n \to \varepsilon$ with $\varepsilon_n > \varepsilon$) in all games.

For $G \in \mathcal{G}$, $a_i \in A_i$ and $t_i \in \mathcal{T}_i$ let $h_i(t_i|a_i, G) = \min\{\varepsilon \geq 0 : a_i \in R_i(t_i, G, \varepsilon)\}$. Dekel, Fudenberg, and Morris (2006) show that the S-topology on $\mathcal{T}_i$ is metrizable by the distance d^{S} defined as

$$d_i^{\mathsf{S}}(t_i, s_i) = \sum_{m \geq 1} 2^{-m} \sup_{G=(A_i, g_i)_{i \in I} \in \mathcal{G}^m} \sup_{a_i \in A_i} |h_i(t_i|a_i, G) - h_i(s_i|a_i, G)|.$$

They also define the *uniform strategic topology* or US-*topology*, obtained by requiring uniform convergence over the number of actions, viz

$$d_i^{\mathsf{US}}(t_i, s_i) = \sup_{m \geq 1} \sup_{G=(A_i, g_i)_{i \in I} \in \mathcal{G}^m} \sup_{a_i \in A_i} |h_i(t_i|a_i, G) - h_i(s_i|a_i, G)|.$$

In this section we relate the S-topology and the US-topology to the *uniform-weak topology*, or UW-*topology*, defined as the metric topology generated by the distance

$$d_i^{\mathsf{UW}}(t_i, s_i) = \sup_{k \geq 1} d_i^k(t_i, s_i),$$

where d^0 is the discrete metric on Θ, and recursively for $k \geq 1$, d_i^k is the Prohorov distance on $\Delta(\Theta \times \mathcal{T}_{-i}^{k-1})$ induced by the metric $\max\{d^0, d_{-i}^{k-1}\}$ on $\Theta \times \mathcal{T}_{-i}^{k-1}$.[6] First we prove that the UW-topology is finer than the US-topology, in the strong sense that, for all $\gamma \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for every type t_i and every game G, all actions that are γ-rationalizable for t_i remain $(\gamma+\varepsilon)$-rationalizable for every type whose UW-distance from t_i is smaller than δ. Second, we prove a partial converse, namely, that around finite types — types belonging to a finite type space,— the S-topology (and hence the US-topology) is finer than the UW-topology.

3.1 UW-convergence implies US-convergence

Theorem 1. *The* UW*-topology is finer than the* US*-topology.*

The theorem is an immediate consequence of the following proposition.

[6]Recall that for a given metric space (S, d) the weak* topology on $\Delta(S)$ is metrizable with the Prohorov distance ρ defined as

$$\rho(\mu, \mu') = \inf\{\delta > 0 \ : \ \mu(E) \leq \mu'(E^\delta) + \delta \ \text{ for each measurable } E \subseteq S\} \qquad \forall \mu, \mu' \in \Delta(S),$$

where $E^\delta = \{s \in S \ : \ \inf_{s' \in S} d(s, s') < \delta\}$.

Proposition 2. *For every $\gamma \geq 0$, every $\varepsilon > 0$, every $G \in \mathcal{G}$, every $i \in I$, and every $k \geq 1$,*

$$d_i^k(s_i, t_i) < \varepsilon/(12M) \quad \Longrightarrow \quad R_i^k(t_i, G, \gamma) \subseteq R_i^k(s_i, G, \gamma + \varepsilon) \qquad \forall s_i, t_i \in \mathcal{T}_i.$$

Proof. Fix $\varepsilon > 0$ and let $\delta = \varepsilon/(12M)$. Fix $\gamma \geq 0$ and a game $G = (A_i, g_i)_{i \in I}$. The proof is by induction on k. For $k = 1$, let s_i and $t_i \in \mathcal{T}_i$ be such that $d_i^1(s_i, t_i) < \delta$. We now fix an arbitrary $a_i \in R_i^1(t_i, G, \gamma)$ and show that $a_i \in R_i^1(s_i, G, \gamma + \varepsilon)$ using Proposition 1. Let $\alpha_i \in \Delta(A_i)$ be a deviation and $\mathbf{a}_{-i} : \Theta \to A_{-i}$ a corresponding function satisfying the incentive constraint (2) with a $M\delta$ slack, i.e.

$$\sum_{\theta \in \Theta} [g_i(a_i, \mathbf{a}_{-i}(\theta, t_{-i}), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}(\theta, t_{-i}), \theta)] t_i^1(\theta) \geq -\gamma - 4M\delta. \tag{3}$$

(Note that condition (1) is trivial for $k = 1$.) Let $\mathbf{a}_{-i}^* : \Theta \to A_{-i}$ be any function such that

$$\mathbf{a}_{-i}^*(\theta) \in \operatorname*{arg\,max}_{a_{-i} \in A_{-i}} \Big[g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta) \Big] \qquad \forall \theta \in \Theta$$

and let $h(\theta) = g_i(a_i, \mathbf{a}_{-i}^*(\theta), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}^*(\theta), \theta)$ for all $\theta \in \Theta$. Since $|h(\theta)| \leq 2M$ for all $\theta \in \Theta$, it follows from $d_i^1(s_i, t_i) < \delta$ that $\sum_{\theta \in \Theta} h(\theta)[s_i^1(\theta) - t_i^1(\theta)] > -4M\delta$, and hence

$$\begin{aligned}
\sum_{\theta \in \Theta} h(\theta) s_i^1(\theta) &= \sum_{\theta \in \Theta} h(\theta)[s_i^1(\theta) - t_i^1(\theta)] + \sum_{\theta \in \Theta} h(\theta) t_i^1(\theta) \\
&> -4M\delta + \sum_{\theta \in \Theta} h(\theta) t_i^1(\theta) \\
&\geq -4M\delta + \sum_{\theta \in \Theta} [g_i(a_i, \mathbf{a}_{-i}(\theta), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}(\theta), \theta)] t_i^1(\theta) \\
&\geq -4M\delta - \gamma - M\delta > -\gamma - \varepsilon,
\end{aligned}$$

where the penultimate inequality follows from (3). This shows that $a_i \in R_i^1(s_i, G, \gamma + \varepsilon)$, which proves the result for $k = 1$.

Proceeding by induction, we now suppose the result is valid for $k - 1$ and show that is valid for k. Let $s_i, t_i \in \mathcal{T}_i$ be such that $d_i^k(s_i, t_i) < \delta$. Let $a_i \in R_i^k(t_i, G, \gamma)$ and let us show that $a_i \in R_i^k(s_i, G, \gamma + \varepsilon)$ using Proposition 1. Fix $\alpha_i \in \Delta(A_i)$ and let $\mathbf{a}_{-i} : \Theta \times \mathcal{T}_{-i}^{k-1} \to A_{-i}$ satisfy conditions (1) and (2) with $M\delta$ slack, i.e.

$$\int_{\Theta \times \mathcal{T}_{-i}^{k-1}} [g_i(a_i, \mathbf{a}_{-i}(\theta, t_{-i}^{k-1}), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}(\theta, t_{-i}^{k-1}), \theta)] t_i^k(d\theta \times dt_{-i}^{k-1}) \geq -\gamma - M\delta. \tag{4}$$

Let $\mathbf{a}^*_{-i} : \Theta \times \mathcal{T}^{k-1}_{-i} \to A_{-i}$ be a measurable function such that

$$\mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}) \in \underset{a_{-i} \in R^{k-1}_{-i}(t^{k-1}_{-i}, G, \gamma+\varepsilon)}{\arg\max} [g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)] \quad \forall (\theta, t^{k-1}_{-i}) \in \Theta \times \mathcal{T}^{k-1}_{-i}.$$

We thus have $\mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}) \in R^{k-1}_{-i}(t^{k-1}_{-i}, G, \gamma + \varepsilon)$ for all $(\theta, t^{k-1}_{-i}) \in \Theta \times \mathcal{T}^{k-1}_{-i}$, so it only remains to show that

$$\int_{\Theta \times \mathcal{T}^{k-1}_{-i}} [g_i(a_i, \mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}), \theta) - g_i(\alpha_i, \mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}), \theta)] s^k_i(d\theta \times dt^{k-1}_{-i}) \geq -\gamma - \varepsilon.$$

For ease of notation, consider an enumeration $\bar{A}_1, \ldots, \bar{A}_L$ of the non-empty subsets of A_{-i}, and for each $\ell = 1, \ldots, L$ define

$$\begin{aligned} P_\ell &= \{t^{k-1}_{-i} \in \mathcal{T}^{k-1}_{-i} \ : \ R^{k-1}_{-i}(t^{k-1}_{-i}, G, \gamma) = \bar{A}_\ell\}, \\ Q_\ell &= \{t^{k-1}_{-i} \in \mathcal{T}^{k-1}_{-i} \ : \ R^{k-1}_{-i}(t^{k-1}_{-i}, G, \gamma + \varepsilon) = \bar{A}_\ell\}, \end{aligned}$$

so that $\{P_1, \ldots, P_L\}$ and $\{Q_1, \ldots, Q_L\}$ are finite partitions of $\mathcal{T}^{k-1}_{-i}$ into measurable sets. For each $\theta \in \Theta$ and $\ell = 1, \ldots, L$ define

$$h_\ell(\theta) = \max_{a_{-i} \in \bar{A}_\ell} [g_i(a_i, a_{-i}, \theta) - g_i(\alpha_i, a_{-i}, \theta)].$$

We then have

$$\int_{\Theta \times \mathcal{T}^{k-1}_{-i}} [g_i(a_i, \mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}), \theta) - g_i(\alpha_i, \mathbf{a}^*_{-i}(\theta, t^{k-1}_{-i}), \theta)] s^k_i(d\theta \times dt^{k-1}_{-i}) = \\ = \sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta) s^k_i(\theta \times Q_\ell).$$

Also, it follows from condition (1) and the definition of P_ℓ that $\mathbf{a}_{-i}(\theta, t^{k-1}_{-i}) \in \bar{A}_\ell$ for all $(\theta, t^{k-1}_{-i}) \in \Theta \times P_\ell$, and hence

$$\int_{\Theta \times \mathcal{T}^{k-1}_{-i}} [g_i(a_i, \mathbf{a}_{-i}(\theta, t^{k-1}_{-i}), \theta) - g_i(\alpha_i, \mathbf{a}_{-i}(\theta, t^{k-1}_{-i}), \theta)] t^k_i(d\theta \times dt^{k-1}_{-i}) \leq \\ \leq \sum_{\theta \in \Theta} \sum_{\ell=1}^{L} h_\ell(\theta) t^k_i(\theta \times P_\ell).$$

Therefore,

$$\begin{aligned}
\int_{\Theta\times\mathcal{T}_{-i}^{k-1}} [g_i(a_i,\mathbf{a}^*_{-i}(\theta,t_{-i}^{k-1}),\theta) - g_i(\alpha_i,\mathbf{a}^*_{-i}(\theta,t_{-i}^{k-1}),\theta)]s_i^k(d\theta\times dt_{-i}^{k-1}) \quad &\geq \\
\geq \quad \int_{\Theta\times\mathcal{T}_{-i}^{k-1}} [g_i(a_i,\mathbf{a}_{-i}(\theta,t_{-i}^{k-1}),\theta) - g_i(\alpha_i,\mathbf{a}_{-i}(\theta,t_{-i}^{k-1}),\theta)]t_i^k(d\theta\times dt_{-i}^{k-1}) \\
+ \sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)\Big[s_i^k(\theta\times Q_\ell) - t_i^k(\theta\times P_\ell)\Big] \\
\geq \quad -\gamma - M\delta + \sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)\Big[s_i^k(\theta\times Q_\ell) - t_i^k(\theta\times P_\ell)\Big],
\end{aligned}$$

where the last inequality follows from (4).

To conclude the proof, we will now show that

$$\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)\Big[s_i^k(\theta\times Q_\ell) - t_i^k(\theta\times P_\ell)\Big] \geq -4M\delta = -\varepsilon + M\delta.$$

Let $N = |\Theta|L$ and consider an enumeration of $\Theta\times\{1,\dots,L\}$, denoted $\{(\theta_n,\ell_n)\}_{n=1}^N$, such that $h_{\ell_n}(\theta_n) \geq h_{\ell_{n+1}}(\theta_{n+1})$ for all $n = 1,\dots,N-1$. Next, define

$$E_n = \theta_n \times P_{\ell_n} \quad \text{and} \quad F_n = \theta_n\times Q_{\ell_n} \qquad \forall n = 1,\dots,N,$$

so that $\{E_1,\dots,E_N\}$ and $\{F_1,\dots,F_N\}$ are finite measurable partitions of $\Theta\times\mathcal{T}_{-i}^{k-1}$. Thus, we have

$$\begin{aligned}
\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)\Big[s_i^k(\theta\times Q_\ell) - t_i^k(\theta\times P_\ell)\Big] = \\
&= \sum_{n=1}^{N} h_{\ell_n}(\theta_n)\Big[s_i^k(F_n) - t_i^k(E_n)\Big] \\
&= \sum_{n=1}^{N-1}\left[h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})\right]\sum_{j=1}^{n}\left[s_i^k(F_j) - t_i^k(E_j)\right] \\
&= \sum_{n=1}^{N-1}\left[h_{\ell_n}(\theta_n) - h_{\ell_{n+1}}(\theta_{n+1})\right][s_i^k(\theta_n\times Q_{\ell_n}) - t_i^k(\theta_n\times P_{\ell_n})].
\end{aligned}$$

By the induction hypothesis we have $Q_{\ell_n} \supseteq P_{\ell_n}^\delta$, and so $d^k(s_i,t_i) < \delta$ implies

$$s_i^k(\theta_n\times Q_{\ell_n}) \geq s_i^k(\theta_n\times P_{\ell_n}^\delta) \geq t_i^k(\theta_n\times P_{\ell_n}) - \delta \qquad \forall n = 1,\dots,N,$$

and therefore

$$\sum_{\theta\in\Theta}\sum_{\ell=1}^{L} h_\ell(\theta)\Big[s_i^k(\theta\times Q_\ell)-t_i^k(\theta\times P_\ell)\Big] =$$

$$\begin{aligned}
&= \sum_{n=1}^{N-1} \overbrace{[h_{\ell_n}(\theta_n)-h_{\ell_{n+1}}(\theta_{n+1})]}^{\geq 0}\overbrace{[s_i^k(\theta_n\times Q_{\ell_n})-t_i^k(\theta_n\times P_{\ell_n})]}^{\geq -\delta}\\
&\geq -\delta\sum_{n=1}^{N-1}[h_{\ell_n}(\theta_n)-h_{\ell_{n+1}}(\theta_{n+1})]\\
&= -\delta[h_{\ell_1}(\theta_1)-h_{\ell_N}(\theta_N)] \geq -4M\delta,
\end{aligned}$$

as required. □

Corollary 1. *The Borel σ-algebras of the* UW-, US-, S- *and product topologies coincide.*

An implication of this corollary is that the Mertens-Zamir universal type space $(\mathcal{T}_i,\mu_i)_{i\in I}$ remains a universal type space when equipped with either the UW-, the US- or the S-topology, instead of the product topology. In effect, since these topologies leave the measurable structure unchanged, the function $\mu_i:\mathcal{T}_i\to\Delta(\Theta\times\mathcal{T}_{-i})$ remains the unique belief-preserving mapping and a Borel isomorphism, albeit no longer a homeomorphism.

3.2 S-convergence implies UW-convergence to finite types.

Here we prove a partial converse to Theorem 1: as far as convergence to *finite types* is concerned, convergence in the S-topology also implies convergence in the UW-topology.

Theorem 2. *Around finite types, the* S*-topology is finer than the* UW*-topology.*

The theorem is an immediate consequence of the following proposition, whose proof in turn relies on Lemma 3 in Appendix A.

Proposition 3. *For every finite type space $(T_i,\phi_i)_{i\in I}$ and every $\delta>0$ there exist $\varepsilon>0$ and $G\in\mathcal{G}$ such that, for every $i\in I$,*

$$d_i^k(s_i,t_i)\geq\delta \quad\Longrightarrow\quad R_i^k(t_i,G,0)\not\subseteq R_i^k(s_i,G,\varepsilon) \qquad \forall k\geq 1,\ \forall(t_i,s_i)\in T_i\times\mathcal{T}_i.$$

Proof. Let $(T_i, \phi_i)_{i\in I}$ be a finite type space and fix $\delta > 0$. By Lemma 3 there exist $\varepsilon > 0$ and a game $G = (A_i, g_i)_{i\in I} \in \mathcal{G}$ with the following properties for every $i \in I$. First, $A_i \supseteq T_i$. Second, for every $t_i \in T_i$,

$$t_i \in \operatorname*{arg\,max}_{a_i\in A_i} \sum_{\theta\in\Theta} \sum_{t_{-i}\in T_{-i}} \phi_i(t_i)[\theta, t_{-i}] g_i(a_i, t_{-i}, \theta). \tag{5}$$

Third, for every $t_i \in T_i$, every $s_i \in \mathcal{T}_i$, and every measurable $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$, if

$$\int_{\Theta\times\mathcal{T}_{-i}} \sigma_{-i}(\theta, t_{-i})[D]\mu_i(s_i)[d\theta \times dt_{-i}] < \mu_i(t_i)[D] - \delta$$

for any $D \subseteq \Theta \times T_{-i}$, then

$$\min_{a_i\in A_i} \sum_{\theta\in\Theta} \sum_{a_{-i}\in A_{-i}} \psi_i[\theta, a_{-i}][g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta)] < -\varepsilon. \tag{6}$$

We now prove that, for every $i \in I$,

$$t_i \in R_i^k(t_i, G, 0) \qquad \forall k \geq 1,\ \forall t_i \in T_i, \tag{7}$$

$$d_i^k(t_i, s_i) > \delta \quad \Rightarrow \quad t_i \notin R_i^k(s_i, G, \varepsilon) \qquad \forall k \geq 1,\ \forall (t_i, s_i) \in T_i \times \mathcal{T}_i. \tag{8}$$

Let $i \in I$ and $t_i \in T_i$. Let $\sigma_{-i} : \Theta \times T_{-i} \to \Delta(A_{-i})$ be defined by $\sigma_{-i}(\theta, t_{-i})[t_{-i}] = 1$ for all $(\theta, t_{-i}) \in \Theta \times T_{-i}$. Then t_i is a best reply to σ_{-i} by (5), and hence $t_i \in R_i(t_i, G, 0)$ by the best-reply sets definition of ICR, proving (7). To prove (8) for $k = 1$, pick any $s_i \in \mathcal{T}_i$ with $d_i^1(t_i, s_i) > \delta$. Then there exists $E \subseteq \Theta$ such that $s_i^1[E] < t_i^1[E] - \delta$, and hence for every $\sigma_{-i} : \Theta \to \Delta(A_{-i})$ we have

$$\sum_{\theta\in E} s_i^1[\theta]\sigma_{-i}(\theta)[T_{-i}] \leq s_i^1[E] < t_i^1[E] - \delta = \mu_i(t_i)[E \times T_{-i}] - \delta.$$

It follows from (6) that $t_i \notin R_i^1(\varepsilon, t_i'; G)$. Proceeding by induction, let $k \geq 2$ and assume that (8) holds for $k-1$. Fix $i \in I$ and $t_i \in T_i$ and pick any $s_i \in \mathcal{T}_i$ with $d_i^k(t_i, s_i) > \delta$. Then there exists some $E \subseteq \Theta \times T_{-i}^{k-1}$ such that $s_i^k[E^\delta] < t_i^k[E] - \delta$. For notational convenience, define $D = \{(\theta, t_{-i}) \in \Theta \times T_{-i} : (\theta, t_{-i}^{k-1}) \in E\}$ and note that $\mu_i(s_i)[D] = s_i^k[E]$. Let $\sigma_{-i} : \Theta \times \mathcal{T}_{-i} \to \Delta(A_{-i})$ be an arbitrary $(k-1)$-order ε-rationalizable conjecture, i.e.

$$\operatorname{supp} \sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}^{k-1}(t_{-i}, G, \varepsilon) \qquad \text{for } \mu_i(s_i)\text{-almost every } (\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i}. \tag{9}$$

By the induction hypothesis, for $\mu_i(s_i)$-almost every $(\theta, t_{-i}) \in \Theta \times \mathcal{T}_{-i}$ and for every $(\theta, s_{-i}) \in D$, we can have $\sigma_{-i}(\theta, t_{-i})[s_{-i}] > 0$ only if $d_{-i}^{k-1}(t_{-i}, s_{-i}) \leq \delta$. Thus,

$$\begin{aligned}\int_{\Theta\times\mathcal{T}_{-i}} \sigma_{-i}(\theta, t_{-i})[D]\mu_i(s_i)[d\theta \times dt_{-i}] &\leq \int_{(\pi_{-i}^{k-1})^{-1}(E^\delta)} \sigma_{-i}(\theta, t_{-i})[D]\mu_i(s_i)[d\theta \times dt_{-i}] \\ &\leq \int_{(\pi_{-i}^{k-1})^{-1}(E^\delta)} \mu_i(s_i)[d\theta \times dt_{-i}] \\ &= s_i^k[E^\delta] < t_i^k[E] - \delta = \mu_i(t_i)[D] - \delta,\end{aligned}$$

and it follows from (6) that $t_i \notin R_i^k(s_i, G, \varepsilon)$. □

Corollary 2. *Around finite types, the* UW-, US-, *and* S- *topologies coincide.*

4 Non-denseness of finite types.

In this section we first show by an example that finite types are not dense under the US-topology (and hence, by Theorem 1, not dense under the UW-topology). We achieve this by proving that the e-mail type $u_{1,0}$ who received no messages cannot be US-approximated by finite types. Based upon this, we go one step further to show that the set of finite types is nowhere dense, under both the US-topology and the UW-topology, in the universal type space, i.e. the complement of the US-closure (resp. UW-closure) of the set of finite types is open and dense under the US-topology (resp. UW-topology).[7] Finally, we remark that $u_{1,0}$ is a critical type in the sense of Ely and Peski (2008) and comment on the implications of our example on the relationship between the S-topology and the UW-topology around critical types.

Let $\Theta = \{\theta_0, \theta_1\}$ and consider the type space (U_1, U_2) thus defined:[8]

$$U_1 = \{u_{1,0}, u_{1,1}, u_{1,2}, \dots\}, \qquad U_2 = \{u_{2,0}, u_{2,1}, u_{2,2}, \dots\},$$

[7]By Theorem 1 the UW-topology is finer than the US-topology, so non-denseness in the US-topology implies non-denseness in the UW-topology. However, the analogous claim of nowhere denseness in the UW-topology, while true, is *not* a corollary of Theorem 1 and nowhere denseness in the US-topology. (In general, a set may be nowhere dense under a topology and not under a finer one.)

[8]This type space is an instance of the e-mail type space where the more informed player 1 who received k messages attaches probability $p = 2/3$ (resp. $1 - p = 1/3$) to player 2 having received $k - 1$ (resp. k) messages, and the less informed player 2 who received k messages attaches probability p (resp. $1 - p$) to player 1 having received k (resp. $k + 1$) messages. Our choice that $p = 2/3$ is unimportant; our results hold true, with a few unimportant changes, if we assume any other value for p.

and $u_{1,0}[\theta_0, u_{2,0}] = 1$, $u_{2,0}[\theta_0, u_{1,0}] = 2/3$, $u_{2,0}[\theta_1, u_{1,1}] = 1/3$,

$$u_{1,k}[\theta_1, u_{2,k-1}] = 2/3, \qquad u_{1,k}[\theta_1, u_{2,k}] = 1/3 \qquad \forall k \geq 1,$$
$$u_{2,k}[\theta_1, u_{2,k}] = 2/3, \qquad u_{2,k}[\theta_1, u_{2,k+1}] = 1/3 \qquad \forall k \geq 1.$$

The following proposition establishes existence of a US-neighborhood of $u_{1,0}$ with no finite types in it, thereby proving that the set of finite types is not dense, under the US-topology, in the universal type space.

Proposition 4. $d_1^{\text{us}}(t_1, u_{1,0}) \geq M/6$ *for every finite type* $t_1 \in \mathcal{T}_1$*.*

The proposition is a direct consequence of the following two lemmas, whose proofs are relegated to Appendix A.

Lemma 1. $d_1^{\text{uw}}(t_1, u_{1,0}) \geq 1/3$ *for every finite type* $t_1 \in \mathcal{T}_1$*.*

Lemma 2. $d_1^{\text{us}}(t_1, u_{1,0}) \geq (M/2) d_1^{\text{uw}}(t_1, u_{1,0})$ *for every* $t_1 \in \mathcal{T}_1$*.*

The proof of the latter actually establishes a slightly stronger claim, namely, that for every $n \geq 1$ there exist a game and an action $a_{1,0}$ for player 1 in that game such that, for every $\delta \geq 0$ and every type $t_1 \in \mathcal{T}_1$ for which $a_{1,0}$ is n-order $(M\delta/2)$-rationalizable, the n-order beliefs of t_1 cannot differ from those of $u_{1,0}$ by more than δ. Thus the game, and in particular the number of actions in the game, will depend on n, but not on the particular t_1 chosen. To better guide the reader through that proof, it is useful to illustrate the argument for a particular n and the particular case where $M = 2$.[9]

Consider the game in Figure 1. The key feature of this game, at the base of the idea of the proof, is that for each $\delta \geq 0$, if there exists a δ-rationalizable conjecture that δ-rationalizes the risky action $a_{i,k}$ simultaneously against the safe action s_i and both of the even riskier actions $b_{i,k}$ and $c_{i,k}$, then the beliefs (at some appropriate order) of player i cannot differ by more than δ from those of type $u_{i,k}$. (To see this more clearly, the relevant payoffs appear in bold in the figure.) Given the presence of s_1 and s_2, it is clear that $a_{1,0}$ is not δ-rationalizable for any t_1 with $t_1^1[\theta_0] < 1 - \delta$, while $a_{1,1}$ and $a_{1,2}$ are not δ-rationalizable for any t_1 with $t_1^1[\theta_1] < 1 - \delta$, and $a_{2,1}$ and $a_{2,2}$ are not

[9]The game in Figure 1 works for $n = 6$. As is evident from the proof of the lemma, for every even n the game has $3n/2 - 1$ actions for player 1 and $3n/2 + 1$ actions for player 2.

	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	$b_{2,1}$	$c_{2,1}$	$a_{2,2}$	$b_{2,2}$	$c_{2,2}$	s_2
$a_{1,0}$	**0**, **0**	-1, **1**	-1, **-1**	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$a_{1,1}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$b_{1,1}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$c_{1,1}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$a_{1,2}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$b_{1,2}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$c_{1,2}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
s_1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, **0**

$\theta = \theta_0$

	$a_{2,0}$	$b_{2,0}$	$c_{2,0}$	$a_{2,1}$	$b_{2,1}$	$c_{2,1}$	$a_{2,2}$	$b_{2,2}$	$c_{2,2}$	s_2
$a_{1,0}$	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$a_{1,1}$	**0**, **0**	-1, **-2**	-1, **2**	**0**, **0**	-1, **1**	-1, **-1**	-1, -1	-1, -1	-1, -1	-1, **0**
$b_{1,1}$	**1**, -1	-1, -1	-1, -1	**-2**, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$c_{1,1}$	**-1**, -1	-1, -1	-1, -1	**2**, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, -1	-1, **0**
$a_{1,2}$	-1, -1	-1, -1	-1, -1	**0**, **0**	-1, **-2**	-1, **2**	**0**, **0**	-1, **1**	-1, **-1**	-1, **0**
$b_{1,2}$	-1, -1	-1, -1	-1, -1	**1**, -1	-1, -1	-1, -1	**-2**, -1	-1, -1	-1, -1	-1, **0**
$c_{1,2}$	-1, -1	-1, -1	-1, -1	**-1**, -1	-1, -1	-1, -1	**2**, -1	-1, -1	-1, -1	-1, **0**
s_1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, -1	**0**, **0**	**0**, **-2**	**0**, **2**	**0**, **0**

$\theta = \theta_1$

Figure 1: Here $a_{1,0} \in R_1(u_{1,0}, 0)$ but $a_{1,0} \notin R_1(t_1, \delta)$ if $d_1^6(t_1, u_{1,0}) > \delta$.

δ-rationalizable for any t_2 with $t_2^1[\theta_1] < 1 - \delta$. Also, given the presence of s_2 and $b_{2,0}$ (resp. s_2 and $c_{2,0}$), action $a_{2,0}$ is not δ-rationalizable for any t_2 with $t_2^1[\theta_1] < 1/3 - \delta$ (resp. $t_2^1[\theta_0] < 2/3 - \delta$). Based on these facts, inductively using essentially the same arguments, we see e.g. that $a_{1,1}$ is not δ-rationalizable for any t_1 with $t_1^2[\theta_1 \times \{u_{2,0}^1\}^\delta] < 2/3 - \delta$ or $t_1^2[\theta_1 \times \{u_{2,1}^1\}^\delta] < 1/3 - \delta$, that $a_{2,0}$ is not δ-rationalizable for any t_2 with $t_2^2[\theta_1 \times \{u_{1,1}^1\}^\delta] < 1/3 - \delta$ or $t_2^2[\theta_0 \times \{u_{1,0}^1\}^\delta] < 2/3 - \delta$, and so on, eventually proving that $a_{1,0}$ is not δ-rationalizable for any t_1 with $t_1^6[\theta_0 \times \{u_{2,0}^5\}^\delta] < 1 - \delta$.

We are now ready to prove the main result in this section.

Theorem 3. *Finite types are nowhere dense under the* US*-topology and the* UW*-topology.*

Proof. It suffices to prove that every finite type can be UW-approximated by a sequence of infinite types, none of which is the US-limit of a sequence of finite types.[10] Fix a finite

[10]Indeed, by Theorem 1 the sequence will also US-approximate the finite type, hence nowhere denseness in the US-topology will follow; by the same theorem, none of the types in the sequence will be the UW-limit of a sequence of finite types, thus nowhere denseness in the UW-topology will also follow.

type space (T_1, T_2) and a type $t_2 \in T_2$. For each $\ell \geq 1$ let $\delta_\ell = 1/(\ell+1)$ and define the infinite type $t_{2,\ell}$ by the requirement that, for every $k \geq 1$ and every measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$,

$$t_{2,\ell}^k[E] = (1-\delta_\ell)t_2^k[E] + \delta_\ell u_{2,0}^k[E].$$

Note that for all $\ell \geq 1$, $k \geq 1$, and measurable $E \subseteq \Theta \times \mathcal{T}_1^{k-1}$ we have

$$t_{2,\ell}^k[E] = (1-\delta_\ell)t_2^k[E] + \delta_\ell u_{2,0}^k[E] \leq t_{2,\ell}^k[E^{\delta_\ell}] + \delta_\ell,$$

hence $d_2^{\mathsf{uw}}(t_{2,\ell}, t_2) \leq \delta_\ell$. It follows that the sequence $(t_{2,\ell})_{\ell \geq 1}$ UW-approximates t_2, and it only remains to prove that none of the types in the sequence is in the US-closure of the set of finite types. To this end, we will prove that $d_2^{\mathsf{us}}(t_{2,\ell}, s_2) \geq \min\{M/12, M/(3\ell+1)\}$ for every $\ell \geq 1$ and every finite type $s_2 \in \mathcal{T}_2$.

Fix $\ell \geq 1$, a finite type space (S_1, S_2), and a type $s_2 \in S_2$. Using Proposition 4, choose $n \geq 1$ large enough so that

$$d_1^{2(n+1)}(t_1, u_{1,0}) \geq 1/3 \qquad \forall t_1 \in T_1 \cup S_1. \tag{10}$$

Let $G_n = (A_{i,n}, g_{i,n})_{i=1,2}$ be the game defined in the proof of Lemma 2. Define another game $G_n' = (A_{i,n}', g_{i,n}')_{i=1,2}$ as follows:

$$A_{1,n}' = A_{1,n}, \qquad A_{2,n}' = A_{2,n} \times \{0,1\},$$

and, for all $a_1 \in A_{1,n}$, $a_2 \in A_{2,n}$, $x \in \{0,1\}$, and $\theta \in \Theta$,

$$g_{1,n}'(a_1, a_2, x, \theta) = \frac{1}{2} g_{1,n}(a_1, a_2, \theta) \tag{11}$$

$$g_{2,n}'(a_1, a_2, x, \theta) = \frac{1}{2} g_{2,n}(a_1, a_2, \theta) + \begin{cases} M/2 & \text{if } x = 1 \text{ and } a_1 = a_{1,0}, \\ -M/(3\ell+1) & \text{if } x = 1 \text{ and } a_1 \neq a_{1,0}, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note that, since all payoffs of G_n are between $-M$ and M, the same is true for all payoffs of G_n'. Moreover, for all $k \geq 0$ and all $\delta \geq 0$,

$$R_1^k(t_1, G_n, 2\delta) = R_1^k(t_1, G_n', \delta) \qquad \forall t_1 \in \mathcal{T}_1, \tag{13}$$

$$R_2^k(t_2, G_n, 2\delta) = \mathsf{proj}_{A_2} R_2^k(t_2, G_n', \delta) \qquad \forall t_2 \in \mathcal{T}_2, \tag{14}$$

which we prove below. Assuming for the moment that (13) and (14) hold, we now prove that $(a_2, 1) \in R_2(t_{2,\ell}, G'_n, 0)$ for some $a_2 \in A_2$, but $(a_2, 1) \notin R_2(s_2, G'_n, \delta)$ for all $a_2 \in A_2$ and all $\delta < \min\{M/12, M/(3\ell+1)\}$, reaching the desired conclusion.

Let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,n})$ be an arbitrary rationalizable conjecture for game G_n and define the conjecture $\sigma'_1 : \Theta \times \mathcal{T}_1 \to \Delta(A'_{1,n})$ for game G'_n as

$$\begin{aligned} \sigma'_1(\theta, t_1)[a_1] &= \sigma_1(\theta, t_1)[a_1] && \forall t_1 \in \mathcal{T}_1 \setminus U_1,\ \forall a_1 \in A'_1, \\ \sigma'_1(\theta, u_{1,k})[a_{1,k}] &= 1 && \forall k \geq 0. \end{aligned}$$

From the proof of Lemma 2 it follows — using (13) — that σ'_1 is a rationalizable conjecture and also — using (10) — that

$$a_{1,0} \notin R_1(t_1, G_n, \delta) \qquad \forall \delta < M/6,\ \forall t_1 \in T_1 \cup S_1, \tag{15}$$

so $\sigma'_1(\theta, t_1)[a_{1,0}] = 0$ for all $\theta \in \Theta$ and all $t_1 \in T_1$. Thus, for all $a_2 \in A_2$,

$$\int_{\Theta \times \mathcal{T}_1} \sigma'_1(\theta, t_1)[a_1]\Big[g'_2(a_1, a_2, 1, \theta) - g'_2(a_1, a_2, 0, \theta)\Big]\mu_2(t_{2,\ell})[d\theta, dt_1] =$$
$$= (2\delta_\ell/3)\frac{M}{2} - (1 - 2\delta_\ell/3)\frac{M}{3\ell+1} = 0,$$

i.e. $(a_2, 0)$ and $(a_2, 1)$ give type $t_{2,\ell}$ the same expected payoff under σ'_1. This implies that $(a_2, 1)$ is a best reply to σ'_1, hence that $(a_2, 1) \in R_2(t_{2,\ell}, G'_n, 0)$, for some $a_2 \in A_2$. However, $(a_2, 1) \notin R_2(s_2, G'_n, \delta)$ for all $a_2 \in A_2$ and all $\delta < \min\{M/12, M/(3\ell+1)\}$. Indeed, by (13) and (15), for every such δ and every δ-rationalizable conjecture σ'_1 for game G'_n, we must have $\sigma'_1(\theta, s_1)[a_{1,0}] = 0$ for all $\theta \in \Theta$ and all $s_1 \in S_1$, hence

$$\sum_{\theta \in \Theta} \sum_{s_1 \in S_1} s_2^{k+1}[\theta, s_1^k] \sum_{a_1 \in A_1} \sigma_1(\theta, s_1)[a_1]\Big[g'_1(a_1, a_2, 1, \theta) - g'_1(a_1, a_2, 0, \theta)\Big] = -\frac{M}{3\ell+1}$$

for all $a_2 \in A_2$, proving that $(a_2, 1)$ is not δ-rationalizable.

It remains to prove (13) and (14). They are trivially true for $k = 0$, so assume they hold for some $k \geq 0$. Then there exists a mapping $\xi : \mathcal{T}_2 \times A_2 \to \{0, 1\}$ satisfying

$$(a_2, \xi(t_2, a_2)) \in R_2^k(t_2, G'_n, \delta) \qquad \forall t_2 \in \mathcal{T}_2,\ \forall a_2 \in R_2^k(t_2, G_n, 2\delta). \tag{16}$$

Let $t_1 \in \mathcal{T}_1$ and $a_1 \in A_1$. Suppose $a_1 \in R_1^{k+1}(t_1, G_n, 2\delta)$ and let $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_2)$ be a corresponding k-order 2δ-rationalizable conjecture. Define $\sigma'_2 : \Theta \times \mathcal{T}_2 \to \Delta(A'_2)$ as

$$\sigma'_2(\theta, t_2)[a_2, \xi(t_2, a_2)] = \sigma_2(\theta, t_2)[a_2] \qquad \forall \theta \in \Theta,\ \forall t_2 \in \mathcal{T}_2,\ \forall a_2 \in A_2.$$

By (16), σ_2' is a k-order δ-rationalizable conjecture in game G_n'. Moreover,

$$\int_{\Theta\times\mathcal{T}_2^k}\sum_{(a_2,x)\in A_2'}\sigma_2'(\theta,t_2)[a_2,x]\Big[g_1'(a_1',a_2,x,\theta)-g_1'(a_1,a_2,x,\theta)\Big]t_1^{k+1}(d\theta,dt_2^k)$$
$$=\frac{1}{2}\int_{\Theta\times\mathcal{T}_2^k}\sum_{a_2\in A_2}\sigma_2(\theta,t_2)[a_2]\Big[g_1(a_1',a_2,\theta)-g_1(a_1,a_2,\theta)\Big]t_1^{k+1}(d\theta,dt_2^k)\leq\frac{1}{2}2\delta=\delta$$

for all $a_1'\in A_1$, hence $a_1\in R_1^{k+1}(t_1,G_n',\delta)$. Conversely, assume the latter is true and let $\sigma_2':\Theta\times\mathcal{T}_2\to\Delta(A_2')$ be a corresponding k-order δ-rationalizable conjecture. Define $\sigma_2:\Theta\times\mathcal{T}_2\to\Delta(A_2)$ as

$$\sigma_2(\theta,t_2)=\mathsf{marg}_{A_2}\,\sigma_2'(\theta,t_2)\qquad\forall\theta\in\Theta,\ \forall t_2\in\mathcal{T}_2.$$

By (14), σ_2 is k-order 2δ-rationalizable in G_n. Moreover,

$$\int_{\Theta\times\mathcal{T}_2^k}\sum_{a_2\in A_2}\sigma_2(\theta,t_2)[a_2]\Big[g_1(a_1',a_2,\theta)-g_1(a_1,a_2,\theta)\Big]t_1^{k+1}(d\theta,dt_2^k)$$
$$=2\int_{\Theta\times\mathcal{T}_2^k}\sum_{(a_2,x)\in A_2'}\sigma_2'(\theta,t_2)[a_2,x]\Big[g_1'(a_1',a_2,x,\theta)-g_1'(a_1,a_2,x,\theta)\Big]t_1^{k+1}(d\theta,dt_2^k)\leq 2\delta$$

for all $a_1'\in A_1$, hence $a_1\in R_1^{k+1}(t_1,G_n,2\delta)$. Thus (13) remains true for $k+1$. To prove that (14) also remains true for $k+1$, let $t_2\in\mathcal{T}_2$ and $a_2\in A_2$. Suppose $a_2\in R_2^{k+1}(t_2,G_n,2\delta)$ and let $\sigma_1:\Theta\times\mathcal{T}_1\to\Delta(A_1)$ be a corresponding k-order 2δ-rationalizable conjecture. Choose any

$$x^*\in\operatorname*{arg\,max}_{x\in\{0,1\}}\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1(\theta,t_1)[a_1]g_2'(a_1,a_2,x,\theta)t_2^{k+1}(d\theta,dt_1^k).$$

Then for every $(a_2',x)\in A_2'$ we have

$$\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1(\theta,t_1)[a_1]\Big[g_2'(a_1,a_2',x,\theta)-g_2'(a_1,a_2,x^*,\theta)\Big]t_2^{k+1}(d\theta,dt_1^k)\leq$$
$$\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1(\theta,t_1)[a_1]\Big[g_2'(a_1,a_2',x,\theta)-g_2'(a_1,a_2,x,\theta)\Big]t_2^{k+1}(d\theta,dt_1^k)=$$
$$\frac{1}{2}\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1(\theta,t_1)[a_1]\Big[g_2(a_1,a_2,\theta)-g_2(a_1,a_2',\theta)\Big]t_2^{k+1}(d\theta,dt_1^k)\leq\frac{1}{2}2\delta=\delta,$$

hence $(a_2,x^*)\in R_2^{k+1}(t_2,G_n',\delta)$. Conversely, pick any $x\in\{0,1\}$ and assume $(a_2,x)\in R_2^{k+1}(t_2,G_n',\delta)$. Let $\sigma_1':\Theta\times\mathcal{T}_1\to\Delta(A_1')$ be a corresponding k-order δ-rationalizable conjecture. Then

$$\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1'(\theta,t_1)[a_1]\Big[g_2(a_1,a_2',\theta)-g_2(a_1,a_2,\theta)\Big]t_2^{k+1}(d\theta,dt_1^k)$$
$$=2\int_{\Theta\times\mathcal{T}_1^k}\sum_{a_1\in A_1}\sigma_1'(\theta,t_1)[a_1]\Big[g_2'(a_1,a_2',x,\theta)-g_2'(a_1,a_2,x,\theta)\Big]t_2^{k+1}(d\theta,dt_1^k)\le 2\delta$$

for all $a_2'\in A_2$, hence $a_2\in R_2^{k+1}(t_2,G_n,2\delta)$, thus establishing (14) for $k+1$. □

A Omitted Proofs

A.1 Proof of Proposition 1

Fix $k\ge 1$ and $t_i\in T_i$ and let Σ_{-i} denote the set of measurable functions $\sigma_{-i}:\Theta\times T_{-i}\to\Delta(A_{-i})$ such that

$$\operatorname{supp}\sigma_{-i}(\theta,t_{-i})\subseteq R_{-i}^{k-1}(t_{-i},G,\varepsilon)\quad\text{for }\phi_i(t_i)\text{-almost every }(\theta,t_{-i})\in\Theta\times T_{-i},$$

where we identify any pair of functions that are equal $\phi(t_i)$-almost surely. The set Σ_{-i} can thus be viewed as a convex subset of the real vector space of $\mathbb{R}^{|A_{-i}|}$-valued measurable functions over $\Theta\times T_{-i}$.

Consider the function $f:\Delta(A_i)\times\Sigma_{-i}\to\mathbb{R}$ such that

$$f(\alpha_i,\sigma_{-i})=\int_{\Theta\times T_{-i}}[g_i(a_i,\sigma_{-i}(\theta,t_{-i}),\theta)-g_i(\alpha_i,\sigma_{-i}(\theta,t_{-i}),\theta)]\phi_i(t_i)(d\theta\times dt_{-i}).$$

Thus, f is the restriction of a bi-linear functional to the Cartesian product of the compact, convex set $\Delta(A_i)$ with the convex set Σ_{-i} (not topologized). By a minimax theorem of Fan (1953) we obtain

$$\min_{\alpha_i\in\Delta(A_i)}\sup_{\sigma_{-i}\in\Sigma_{-i}}f(\alpha_i,\sigma_{-i})=\sup_{\sigma_{-i}\in\Sigma_{-i}}\min_{\alpha_i\in\Delta(A_i)}f(\alpha_i,\sigma_{-i}).$$

Now $a_i\in R_i^k(t_i,G,\varepsilon)$ if and only if the right-hand side is greater than or equal to $-\varepsilon$. We have thus shown that $a_i\in R_i^k(t_i,G,\varepsilon)$ if and only if for every $\eta>0$ and $\alpha_i\in\Delta(A_i)$ there exists $\sigma_{-i}\in\Sigma_{-i}^{t_i}$ such that $f(\alpha_i,\sigma_{-i})>-\varepsilon-\eta$.

A.2 Proof of Lemma 1

First we prove by induction that

$$d_i^{\mathsf{uw}}(u_{i,k}, u_{i,\ell}) \geq 2/3 \qquad \forall i = 1,2,\ \forall k \geq 0,\ \forall \ell \geq 0\ \text{ s.t. } \ell \neq k. \tag{17}$$

For all $k \geq 1$, $u_{1,0}^1[\theta_0] = 1$ and $u_{1,k}^1[\theta_0] = 0$, hence $d_1^1(u_{1,0}, u_{1,k}) = 1 > 2/3$; moreover, $u_{2,0}^1[\theta_0] = 2/3$ and $u_{2,k}^1[\theta_0] = 0$, hence $d_2^1(u_{2,0}, u_{2,k}) \geq 2/3$. Assume that we have proved $d_i^k(u_{i,k-1}, u_{i,k+\ell}) \geq 2/3$ for all $i = 1,2$, some $n \geq 1$, all $1 \leq k \leq n$, and all $\ell \geq 0$. Then, for all $\ell \geq 1$, since $\mu_1(u_{1,n})[\theta_1 \times u_{2,n-1}] = 2/3$ and $\mu_1(u_{1,n+\ell})[\theta_1 \times u_{2,k}] = 0$ for all $k < n$, we obtain $u_{1,n}^{n+1}[\theta_1 \times u_{2,n-1}^n] = 2/3$ and $u_{1,n+\ell}^{n+1}[\theta_1 \times \{u_{2,n-1}^n\}^{2/3}] = 0$, hence $d_1^{n+1}(u_{1,n}, u_{1,n+\ell}) \geq 2/3$. Since $\mu_2(u_{2,n})[\theta_1 \times u_{1,n}] = 2/3$ and $\mu_2(u_{2,n+\ell})[\theta_1 \times u_{1,k}] = 0$ for all $k \leq n$, we also get $u_{2,n}^{n+1}[\theta_1 \times u_{1,n}^n] = 2/3$ and $u_{2,n+\ell}^{n+1}[\theta_1 \times \{u_{1,n}^n\}^{2/3}] = 0$, hence $d_2^{n+1}(u_{2,n}, u_{2,n+\ell}) \geq 2/3$. The proof of (17) is complete.

Now let (T_1, T_2) be a finite type space and for every $i = 1,2$ and every $k \geq 0$ define

$$T_{i,k} = \{t_i \in T_i \ :\ d_i^{\mathsf{uw}}(t_i, u_{i,k}) < 1/3\}.$$

Note that, by (17),

$$T_{i,k} \cap T_{i,\ell} = \emptyset \qquad \forall i = 1,2,\ \forall k \geq 0,\ \forall \ell \geq 0\ \text{ s.t. } \ell \neq k. \tag{18}$$

Assume, contrary to our claim, that $T_{1,0}$ is nonempty. Pick any $t_{1,0} \in T_{1,0}$ and any $1/3 > \delta > d_1^{\mathsf{uw}}(t_{1,0}, u_{1,0})$. Then

$$t_{1,0}^n[\theta_0 \times \{u_{2,0}^{n-1}\}^\delta] \geq u_{1,0}^n[\theta_0 \times u_{2,0}^{n-1}] - \delta = 1 - \delta \qquad \forall n \geq 0$$

and, therefore,

$$\begin{aligned} 0 < 1 - \delta &\leq \lim_{n\to\infty} t_{1,0}^n[\theta_0 \times \{u_{2,0}^{n-1}\}^\delta] \\ &= \mu_1(t_{1,0})[\theta_0 \times \{t_2 \in \mathcal{T}_2 \ :\ d_2^{\mathsf{uw}}(t_2, u_{2,0}) < \delta\}] = \mu_1(t_{1,0})[\theta_0, T_{2,0}], \end{aligned}$$

implying that $T_{2,0}$ is also nonempty. Now assume $T_{2,k}$ is nonempty for some $k \geq 0$ and pick any $t_{2,k} \in T_{2,k}$ and any $1/3 > \delta > d_2^{\mathsf{uw}}(t_{2,0}, u_{2,0})$. Then

$$t_{2,k}^n[\theta_1 \times \{u_{1,k+1}^{n-1}\}^\delta] > u_{2,k}^n[\theta_1 \times u_{1,k+1}^{n-1}] - \delta = 1/3 - \delta \qquad \forall n \geq 0$$

and hence, as before,

$$0 < 1/3 - \delta \le \lim_{n\to\infty} t^n_{2,k}[\theta_1 \times \{u^{n-1}_{1,k+1}\}^\delta]$$
$$= \mu_2(t_{2,k})[\theta_1 \times \{t_1 \in \mathcal{T}_1 \,:\, d^{\mathsf{uw}}_1(t_1, u_{1,k+1}) < \delta\}] = \mu_2(t_{2,k})[\theta_1 \times T_{1,k+1}],$$

so $T_{1,k+1}$ is also nonempty. Finally, assume $T_{1,k}$ is nonempty for some $k \ge 1$ and pick any $t_{1,k} \in T_{1,k}$ and any $1/3 > \delta > d^{\mathsf{uw}}_1(t_{1,k}, u_{1,k})$. Then

$$t^n_{1,k}[\theta_1 \times \{u^{n-1}_{2,k}\}^\delta] > u^n_{1,k}[\theta_1 \times \{u^{n-1}_{2,k}\}] - \delta = 1/3 - \delta \qquad \forall n \ge 0,$$

hence again

$$0 < 1/3 - \delta \le \lim_{n\to\infty} t^n_{1,k}[\theta_1 \times \{u^{n-1}_{2,k}\}^\delta]$$
$$= \mu_1(t_{1,k})[\theta_1 \times \{t_2 \in \mathcal{T}_2 \,:\, d^{\mathsf{uw}}_2(t_2, u_{2,k}) < \delta\}] = \mu_1(t_{1,k})[\theta_1 \times T_{2,k}],$$

so $T_{2,k}$ is also nonempty. We have reached the conclusion that $T_{i,k}$ is nonempty for every $i = 1, 2$ and every $k \ge 0$. By (18), this contradicts the finiteness of T_1 and T_2.

A.3 Proof of Lemma 2

Fix $n \ge 1$. The proof is divided into three steps. In Step 1 we construct a game $G_n = (A_{i,n}, g_{i,n})_{i\in I}$ and prove that a certain action of player 1 is rationalizable for type $u_{1,0}$. In Step 2 and Step 3 we prove by induction that, for all $\delta \ge 0$ and $t_1 \in \mathcal{T}_1$, if that action is δ-rationalizable for t_1, then $d^{2(n+1)}_1(t_1, u_{1,0}) \le 2\delta/M$.

Step 1. The action sets in game G_n are as follows:

$$A_{1,n} = \Big\{a_{1,0}, a_{1,1}, b_{1,1}, c_{1,1}, \ldots, a_{1,n}, b_{1,n}, c_{1,n}, s_1\Big\};$$
$$A_{2,n} = \Big\{a_{2,0}, b_{2,0}, c_{2,0}, a_{2,1}, b_{2,1}, c_{2,1} \ldots, a_{2,n}, b_{2,n}, c_{2,n}, s_2\Big\}.$$

Payoffs are as follows. Actions s_1 and s_2 are safe actions that give constant payoffs:

$$g_{1,n}(\theta, s_1, a_2) = g_{2,n}(\theta, a_1, s_2) = 0 \qquad \text{for every } \theta \in \Theta,\ a_1 \in A_{1,n}, \text{ and } a_2 \in A_{2,n}.$$

Actions $a_{1,0}, \ldots, a_{1,n}$ and $a_{2,0}, \ldots, a_{2,n}$ are risky actions, weakly dominated by the safe actions above:

$$g_{1,n}(\theta, a_{1,\ell}, a_2) = \begin{cases} 0 & \text{if } \ell = 0,\ \theta = \theta_0, \text{ and } a_2 = a_{2,0}, \\ 0 & \text{if } \ell > 0,\ \theta = \theta_1, \text{ and } a_2 \in \{a_{2,\ell-1}, a_{2,\ell}\}, \\ -M/2 & \text{otherwise;} \end{cases}$$

$$g_{2,n}(\theta, a_1, a_{2,\ell}) = \begin{cases} 0 & \text{if } \ell = 0 \text{ and } (\theta, a_1) \in \{(\theta_0, a_{1,0}), (\theta_1, a_{1,1})\}, \\ 0 & \text{if } \ell > 0,\ \theta = \theta_1, \text{ and } a_1 \in \{a_{1,\ell}, a_{1,\ell+1}\}, \\ -M/2 & \text{otherwise.} \end{cases}$$

Actions $b_{1,1}, c_{1,1}, \dots, b_{1,n}, c_{1,n}$ and $b_{2,0}, c_{2,0}, \dots, b_{2,n}, c_{2,n}$ are even riskier, though not weakly dominated actions: for every $\ell = 1, \dots, n$,

$$g_{1,n}(\theta, b_{1,\ell}, a_2) = \begin{cases} M/2 & \text{if } \theta = \theta_1 \text{ and } a_2 = a_{2,\ell-1}, \\ -M & \text{if } \theta = \theta_1 \text{ and } a_2 = a_{2,\ell}, \\ -M/2 & \text{otherwise,} \end{cases}$$

$$g_{1,n}(\theta, c_{1,\ell}, a_2) = \begin{cases} M & \text{if } \theta = \theta_1 \text{ and } a_2 = a_{2,\ell}, \\ -M/2 & \text{otherwise.} \end{cases}$$

For player 2,

$$g_{2,n}(\theta, a_1, b_{2,0}) = \begin{cases} M/2 & \text{if } \theta = \theta_0 \text{ and } a_1 = a_{1,0}, \\ -M & \text{if } \theta = \theta_1 \text{ and } a_1 = a_{1,1}, \\ -M/2 & \text{otherwise,} \end{cases}$$

$$g_{2,n}(\theta, a_1, c_{2,0}) = \begin{cases} M & \text{if } \theta = \theta_1 \text{ and } a_1 = a_{1,1}, \\ -M/2 & \text{otherwise,} \end{cases}$$

and, for every $\ell = 1, \dots, n$,

$$g_{2,n}(\theta, a_1, b_{2,\ell}) = \begin{cases} M/2 & \text{if } \theta = \theta_1 \text{ and } a_1 = a_{1,\ell}, \\ -M & \text{if } \theta = \theta_1 \text{ and } a_1 = a_{1,\ell+1}, \\ -M/2 & \text{otherwise,} \end{cases}$$

$$g_{2,n}(\theta, a_1, c_{2,\ell}) = \begin{cases} M & \text{if } \theta = \theta_1 \text{ and } a_1 = a_{1,\ell+1}, \\ -M/2 & \text{otherwise.} \end{cases}$$

It is immediate to verify that $a_{1,0}$ is rationalizable for $u_{1,0}$. To see this, just note that the conjectures $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,n})$ and $\sigma_2 : \Theta \times \mathcal{T}_2 \to \Delta(A_{2,n})$ such that

$\sigma_1(\theta, t_{1,\ell})[a_{1,\ell}] = 1$ and $\sigma_2(\theta, t_{2,\ell})[a_{2,\ell}] = 1$ for every $\theta \in \Theta$ and every $\ell = 0, \dots, n$ clearly satisfy the requirement in the best reply set definition of ICR.

Step 2. For convenience, in this step we define $a_{1,n+1}$ to be s_1. Also, in this step and the next, for any $2 \le \ell \le n$, we define θ_ℓ to be θ_1. In this step we prove that, for all $\ell = 0, \dots, n$,

$$a_{2,\ell} \in R_2^1(t_2, \delta) \quad \Longrightarrow \quad d_2^1(t_2, t_{2,\ell}) \le 2\delta/M \qquad \forall t_2 \in \mathcal{T}_2 \tag{19}$$

and

$$a_{1,\ell} \in R_1^2(t_1, \delta) \quad \Longrightarrow \quad d_1^2(t_1, t_{1,\ell}) \le 2\delta/M \qquad \forall t_1 \in \mathcal{T}_1. \tag{20}$$

Fix any $t_2 \in \mathcal{T}_2$ and $0 \le \ell \le n$, assume that $a_{2,\ell} \in R_2^1(t_2, \delta)$ and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,n})$ be a corresponding 0-order δ-rationalizable conjecture. Since $a_{2,\ell}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 instead of $a_{2,\ell}$ under σ_1 must be no greater than δ, i.e.

$$\frac{M}{2} \int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[A_{1,n} \setminus \{a_{1,\ell}\}]\mu_2(t_2)[\theta_\ell, dt_1] + \\ + \frac{M}{2} \int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[A_{1,n} \setminus \{a_{1,\ell+1}\}]\mu_2(t_2)[\theta_1, dt_1] \le \delta.$$

Rearranging the latter, we get

$$\int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] + \\ + \int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1, dt_1] \ge 1 - \frac{2\delta}{M}. \tag{21}$$

Similarly, the difference in expected payoff when choosing $b_{2,\ell}$ or $c_{2,\ell}$ instead of $a_{2,\ell}$ under σ_1 must be no greater than δ, that is, respectively,

$$\frac{M}{2} \int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] - M \int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1, dt_1] \le \delta$$

and

$$-\frac{M}{2} \int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] + M \int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1, dt_1] \le \delta.$$

The latter two inequalities together with (21) imply

$$\int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] \ge \frac{2}{3} - \frac{2\delta}{M} \tag{22}$$

and

$$\int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1,t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1,dt_1] \geq \frac{1}{3}-\frac{2\delta}{M}. \tag{23}$$

Since $\sigma_1(\theta,t_1)[a_1] \leq 1$ for all $(\theta,t_1)\in\Theta\times\mathcal{T}_1$ and all $a_1\in A_{1,n}$, inequalities (21), (22), and (23) respectively imply

$$\mu_2(t_2)[\theta_\ell]+\mu_2(t_2)[\theta_1] \geq 1-\frac{2\delta}{M}, \quad \mu_2(t_2)[\theta_\ell] \geq \frac{2}{3}-\frac{2\delta}{M}, \quad \text{and} \quad \mu_2(t_2)[\theta_1] \geq \frac{1}{3}-\frac{2\delta}{M},$$

hence $d_2^1(t_2,t_{2,\ell}) \leq 2\delta/M$, as required by (19).

To prove (20), fix any $t_1\in\mathcal{T}_1$ and $0\leq\ell\leq n$, assume that $a_{1,\ell}\in R_1^2(t_1,\delta)$ and let $\sigma_2:\Theta\times\mathcal{T}_2\to\Delta(A_{2,n})$ be a corresponding 1-order δ-rationalizable conjecture. First consider the case $\ell=0$. Since $a_{1,0}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, i.e.

$$\frac{M}{2}\int_{\{\theta_0\}\times\mathcal{T}_2} \sigma_2(\theta_0,t_2)[A_{2,n}\setminus\{a_{2,0}\}]\mu_1(t_1)[\theta_0,dt_2]+\frac{M}{2}\mu_1(t_1)[\theta_1] \leq \delta.$$

Rearranging, and using (19) and the fact that σ_2 is 1-order δ-rationalizable, we get

$$\mu_1(t_1)\Big[\{\theta_0\}\times\Big\{t_2\in\mathcal{T}_2 \ : \ d_2^1(t_2,u_{2,0}) \leq 2\delta/M\Big\}\Big] \geq 1-2\delta/M,$$

as required by (20) when $\ell=0$. Next, consider the case $\ell>0$. Since $a_{1,\ell}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, i.e.

$$\frac{M}{2}\mu_1(t_1)[\theta_0]+\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[A_{2,n}\setminus\{a_{2,\ell-1},a_{2,\ell}\}]\mu_1(t_1)[\theta_1,dt_2] \leq \delta.$$

Rearranging the latter, we get

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]+\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \geq 1-\frac{2\delta}{M}.$$

Similarly, comparing $a_{1,\ell}$ to $b_{1,\ell}$ and $c_{1,\ell}$, we must have, respectively,

$$\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]-M\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \leq \delta$$

and

$$-\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]+M\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \leq \delta.$$

The latter three inequalities together imply

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2] \geq \frac{2}{3}-\frac{2\delta}{M}$$

and

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1, t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1, dt_2] \geq \frac{1}{3} - \frac{2\delta}{M}.$$

Since σ_2 is a 1-order δ-rationalizable conjecture, by (19) we have $\sigma_2(\theta_1, t_2)[a_{2,\ell-1}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^1(t_2, t_{2,\ell-1}) > 2\delta/M$, and $\sigma_2(\theta_1, t_2)[a_{2,\ell}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^1(t_2, u_{2,\ell}) > 2\delta/M$. Thus, we also have

$$\mu_1(t_1)\Big[\{\theta_1\} \times \Big\{t_2 \in \mathcal{T}_2 \ : \ d_2^1(t_2, u_{2,\ell-1}) \leq 2\delta/M\Big\}\Big] \geq \frac{2}{3} - \frac{2\delta}{M}$$

and

$$\mu_1(t_1)\Big[\{\theta_1\} \times \Big\{t_2 \in \mathcal{T}_2 \ : \ d_2^1(t_2, u_{2,\ell}) \leq 2\delta/M\Big\}\Big] \geq \frac{1}{3} - \frac{2\delta}{M},$$

as required by (20) when $\ell > 0$.

Step 3. Let $1 \leq k \leq n$ and assume that for all $\ell = 0, \ldots, n-k+1$ we have proved both

$$a_{2,\ell} \in R_2^{2k-1,\delta}(t_2) \quad \Longrightarrow \quad d_2^{2k-1}(t_2, t_{2,\ell}) \leq 2\delta/M \qquad \forall t_2 \in \mathcal{T}_2 \tag{24}$$

and

$$a_1^\ell \in R_1^{2k,\delta}(t_1) \quad \Longrightarrow \quad d_1^{2k}(t_1, u_{1,\ell}) \leq 2\delta/M \qquad \forall t_1 \in \mathcal{T}_1. \tag{25}$$

We now prove that these are again true, for all $\ell = 0, \ldots, n-k$, when k is replaced by $k+1$. This induction step will conclude the proof of the lemma.

Pick any $t_2 \in \mathcal{T}_2$ and any $0 \leq \ell \leq n-k$, assume that $a_{2,\ell} \in R_2^{2k+1,\delta}(t_2)$ and let $\sigma_1 : \Theta \times \mathcal{T}_1 \to \Delta(A_{1,n})$ be a corresponding $2k$-order δ-rationalizable conjecture. Since $a_{2,\ell}$ is a δ-best reply to σ_1, the difference in expected payoff when choosing s_2 instead of $a_{2,\ell}$ under σ_1 must be no greater than δ, i.e.

$$\frac{M}{2}\int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[A_{1,n} \setminus \{a_{1,\ell}\}]\mu_2(t_2)[\theta_\ell, dt_1] + \\ + \frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[A_{1,n} \setminus \{a_{1,\ell+1}\}]\mu_2(t_2)[\theta_1, dt_1] \leq \delta.$$

Rearranging the latter, we get

$$\int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] + \int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1, dt_1] \geq 1 - 2\delta/M.$$

Similarly, the difference in expected payoff when choosing $b_{2,\ell}$ or $c_{2,\ell}$ instead of $a_{2,\ell}$ under σ_1 must be no greater than δ, that is, respectively,

$$\frac{M}{2}\int_{\{\theta_\ell\}\times\mathcal{T}_1} \sigma_1(\theta_\ell, t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell, dt_1] - M\int_{\{\theta_1\}\times\mathcal{T}_1} \sigma_1(\theta_1, t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1, dt_1] \leq \delta$$

and

$$-\frac{M}{2}\int_{\{\theta_\ell\}\times\mathcal{T}_1}\sigma_1(\theta_\ell,t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell,dt_1]+M\int_{\{\theta_1\}\times\mathcal{T}_1}\sigma_1(\theta_1,t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1,dt_1]\leq\delta.$$

The latter three inequalities together imply

$$\int_{\{\theta_\ell\}\times\mathcal{T}_1}\sigma_1(\theta_\ell,t_1)[a_{1,\ell}]\mu_2(t_2)[\theta_\ell,dt_1]\geq\frac{2}{3}-\frac{2\delta}{M}$$

and

$$\int_{\{\theta_1\}\times\mathcal{T}_1}\sigma_1(\theta_1,t_1)[a_{1,\ell+1}]\mu_2(t_2)[\theta_1,dt_1]\geq\frac{1}{3}-\frac{2\delta}{M}.$$

Since σ_1 is a $2k$-order δ-rationalizable conjecture, by the induction hypothesis (25) we have $\sigma_1(\theta_\ell,t_1)[a_{1,\ell}]=0$ for all $t_1\in\mathcal{T}_1$ such that $d_1^{2k}(t_1,u_{1,\ell})>2\delta/M$, and $\sigma_1(\theta_1,t_1)[a_{1,\ell+1}]=0$ for all $t_1\in\mathcal{T}_1$ such that $d_1^{2k}(t_1,u_{1,\ell+1})>2\delta/M$. Thus, we also have

$$\mu_2(t_2)\Big[\{\theta_\ell\}\times\Big\{t_1\in\mathcal{T}_1\ :\ d_1^{2k}(t_1,u_{1,\ell})\leq 2\delta/M\Big\}\Big]\geq\frac{2}{3}-\frac{2\delta}{M}$$

and

$$\mu_2(t_2)\Big[\{\theta_1\}\times\Big\{t_1\in\mathcal{T}_1\ :\ d_1^{2k}(t_1,u_{1,\ell+1})\leq 2\delta/M\Big\}\Big]\geq\frac{1}{3}-\frac{2\delta}{M}.$$

This proves that for all $\ell=0,\ldots,n-k$ we have

$$a_{2,\ell}\in R_2^{2k+1}(t_2,\delta)\quad\Longrightarrow\quad d_2^{2k+1}(t_2,u_{2,\ell})\leq 2\delta/M\qquad\forall t_2\in\mathcal{T}_2,\tag{26}$$

i.e. that (24) remains true with $k+1$ instead of k.

Now pick any $t_1\in\mathcal{T}_1$ and any $0\leq\ell\leq n-k$, assume that $a_{1,\ell}\in R_1^{2k+2}(t_1,\delta)$ and let $\sigma_2:\Theta\times\mathcal{T}_2\to\Delta(A_{2,n})$ be a corresponding $(2k+1)$-order δ-rationalizable conjecture. First consider the case $\ell=0$. Since $a_{1,0}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, i.e.

$$\frac{M}{2}\int_{\{\theta_0\}\times\mathcal{T}_2}\sigma_2(\theta_0,t_2)[A_{2,n}\setminus\{a_{2,0}\}]\mu_1(t_1)[\theta_0,dt_2]+\frac{M}{2}\mu_1(t_1)[\theta_1]\leq\delta.$$

Rearranging, and using (26) and the fact that σ_2 is $(2k+1)$-order δ-rationalizable,

$$\mu_1(t_1)\Big[\{\theta_0\}\times\Big\{t_2\in\mathcal{T}_2\ :\ d_2^{2k+1}(t_2,u_{2,0})\leq 2\delta/M\Big\}\Big]\geq 1-2\delta/M.\tag{27}$$

Next consider the case $\ell>0$. Since $a_{1,\ell}$ is a δ-best reply to σ_2, it must give an expected payoff within δ of the one from s_1, i.e.

$$\frac{M}{2}\mu_1(t_1)[\theta_0]+\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2}\sigma_2(\theta_1,t_2)[A_{2,n}\setminus\{a_{2,\ell-1},a_{2,\ell}\}]\mu_1(t_1)[\theta_1,dt_2]\leq\delta.$$

Rearranging the latter, we get

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]+\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \geq 1-2\delta/M.$$

Similarly, comparing $a_{1,\ell}$ to $b_{1,\ell}$ and $c_{1,\ell}$, we must have, respectively,

$$\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]-M\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \leq \delta$$

and

$$-\frac{M}{2}\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2]+M\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \leq \delta.$$

The latter three inequalities together imply

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell-1}]\mu_1(t_1)[\theta_1,dt_2] \geq \frac{2}{3}-\frac{2\delta}{M}$$

and

$$\int_{\{\theta_1\}\times\mathcal{T}_2} \sigma_2(\theta_1,t_2)[a_{2,\ell}]\mu_1(t_1)[\theta_1,dt_2] \geq \frac{1}{3}-\frac{2\delta}{M}.$$

Since σ_2 is $(2k+1)$-order δ-rationalizable, by (26) we have $\sigma_2(\theta_1,t_2)[a_{2,\ell-1}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2,u_{2,\ell-1}) > 2\delta/M$, and $\sigma_2(\theta_1,t_2)[a_{2,\ell}] = 0$ for all $t_2 \in \mathcal{T}_2$ such that $d_2^{2k+1}(t_2,u_{2,\ell}) > 2\delta/M$. Thus, we also have

$$\mu_1(t_1)\Big[\{\theta_1\}\times\Big\{t_2 \in \mathcal{T}_2 \,:\, d_2^{2k+1}(t_2,u_{2,\ell-1}) \leq 2\delta/M\Big\}\Big] \geq \frac{2}{3}-\frac{2\delta}{M}$$

and

$$\mu_1(t_1)\Big[\{\theta_1\}\times\Big\{t_2 \in \mathcal{T}_2 \,:\, d_2^{2k+1}(t_2,u_{2,\ell}) \leq 2\delta/M\Big\}\Big] \geq \frac{1}{3}-\frac{2\delta}{M}.$$

Together with (27), this implies that, for all $\ell = 0,\ldots,n-k$,

$$a_{1,\ell} \in R_1^{2k+2}(t_1,\delta) \quad\Longrightarrow\quad d_1^{2k+2}(t_1,u_{1,\ell}) \leq 2\delta/M \qquad \forall t_1 \in \mathcal{T}_1,$$

i.e. that (25) remains true with $k+1$ instead of k, as was to be proved.

A.4 Lemma 3 and its proof

Lemma 3. *Let $(T_i,\phi_i)_{i\in I}$ be a finite type space. For every $\delta > 0$ there exist $\varepsilon > 0$ and a game $G = (A_i,g_i)_{i\in I}$, with $A_i \supseteq T_i$ for all $i \in I$, such that, for every $i \in I$ and $t_i \in T_i$,*

$$t_i \in \arg\max_{a_i\in A_i} \sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} \phi_i(t_i)[\theta,t_{-i}]g_i(a_i,t_{-i},\theta) \qquad (28)$$

and, for every $\psi \in \Delta(\Theta \times A_{-i})$ *such that* $\psi[D] < \phi[D] - \delta$ *for some* $D \subseteq \Theta \times T_{-i}$*,*

$$\min_{a_i \in A_i} \sum_{\theta \in \Theta} \sum_{a_{-i} \in A_{-i}} \psi[\theta, a_{-i}][g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta)] < -\varepsilon. \tag{29}$$

Proof. For each $i \in I$ let ρ_i and $\|\cdot\|_i$ denote the Prohorov distance on $\Delta(\Theta \times T_{-i})$ and the Euclidean norm on $\mathbb{R}^{\Theta \times T_{-i}}$, respectively. Also, let $f_i : \Theta \times T_{-i} \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$(\theta, t_{-i}, \psi) \mapsto 2\psi[\theta, t_{-i}] - \|\psi\|_i^2,$$

and let $F_i : \Delta(\Theta \times T_{-i}) \times \Delta(\Theta \times T_{-i}) \to \mathbb{R}$ be the function defined by

$$(\psi', \psi) \mapsto \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi[\theta, t_{-i}] f_i(\theta, t_{-i}, \psi').$$

Note that $F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|_i^2$ for all $\psi, \psi' \in \Delta(\Theta \times T_{-i})$, hence

$$\eta = \frac{1}{2} \min \left\{ F_i(\psi, \psi) - F_i(\psi', \psi) \, : \, \psi', \psi \in \Delta(\Theta \times T_{-i}), \; \rho_i(\psi, \psi') \geq \frac{\delta}{2} \right\}$$

is well defined and positive, and moreover,[11]

$$\rho_i(\psi, \psi') < \eta/2 \quad \Longrightarrow \quad F_i(\psi, \psi) - F_i(\psi', \psi) < \eta \qquad \forall \psi, \psi' \in \Delta(\Theta \times T_{-i}).$$

The compact set $\Delta(\Theta \times T_{-i})$ can be covered by a finite union of open balls of radius $\eta/2$. (These balls are taken according to the metric ρ_i.) Choose one point in each of these balls and let $A_i \subseteq \Delta(\Theta \times T_{-i})$ denote the finite set of chosen points. Enlarge A_i, if necessary, to ensure $A_i \supseteq T_i$. (We identify each $t_i \in T_i$ with $\phi_i(t_i)$.) Thus, for every $\psi \in \Delta(\Theta \times T_{-i})$ there exists $a_i \in A_i$ such that $F_i(\psi, \psi) - F_i(a_i, \psi) < \eta$.

Now define the payoff function $g_i : \Theta \times A_i \times A_j \to \mathbb{R}$, as follows:

$$g_i(\theta, a_i, a_j) = \begin{cases} (M\delta/4) f_i(\theta, a_{-i}, a_i) & \text{if } a_{-i} \in T_{-i}, \\ -M & \text{if } a_i \in T_i \text{ and } a_{-i} \notin T_{-i}, \\ -M\delta/4 & \text{if } a_i \notin T_i \text{ and } a_{-i} \notin T_{-i}. \end{cases}$$

[11] Letting $h : \Theta \times T_{-i} \to [-1, 1]$ be the mapping $(\theta, t_{-i}) \mapsto \psi[\theta, t_{-i}] - \psi'[\theta, t_{-i}]$, for all $\zeta \geq 0$ we have

$$F_i(\psi, \psi) - F_i(\psi', \psi) = \|\psi - \psi'\|^2 = \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi[\theta, t_{-i}] h(\theta, t_{-i}) - \sum_{(\theta, t_{-i}) \in \Theta \times T_{-i}} \psi'[\theta, t_{-i}] h(\theta, t_{-i}) \leq 2\zeta$$

whenever $\rho_i(\psi, \psi') \leq \zeta$, as we explained in Section 2 when we introduced the Prohorov distance.

It follows directly from the definition of g_i and the fact that $\phi_i(t_i)[\Theta \times T_{-i}] = 1$ that each $a_i \in A_i$ yields an expected payoff of $(M\delta/4)F_i(a_i, \phi_i(t_i))$. Since $F_i(\phi_i(t_i), \phi_i(t_i)) \geq F_i(a_i, \phi_i(t_i))$ for all $a_i \in A_i$, (28) follows.

Fix any $0 < \varepsilon < (M\delta/4)\min\{\eta(1-\delta/2), \delta/2\}$. We shall prove (29) now. Fix $t_i \in T_i$ and $\psi \in \Delta(\Theta \times A_{-i})$, and assume that there exists $D \subseteq \Theta \times T_{-i}$ such that $\psi[D] < \phi_i(t_i)[D] - \delta$. First suppose $\psi[\Theta \times T_{-i}] < 1 - \delta/2$. Pick any $a_i \in A_i \setminus T_i$. Since f_i maps into $[-1, 1]$,

$$\sum_{\theta \in \Theta} \sum_{a_{-i} \in A_{-i}} \psi[\theta, a_{-i}][g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta)] \leq$$

$$\leq 2(M\delta/4)(1-\delta/2) + (\delta/2)(-M + M\delta/4) = -M\delta^2/8 < -\varepsilon.$$

This establishes (29) for the case $\psi[\Theta \times T_{-i}] < 1 - \delta/2$. Now suppose instead that $\psi[\Theta \times T_{-i}] \geq 1 - \delta/2$. Consider the conditional probability $\bar{\psi}(\cdot) \equiv \psi(\cdot | \Theta \times T_{-i})$. Then

$$\bar{\psi}[D] \geq \psi[D] = \bar{\psi}[D]\psi[\Theta \times T_{-i}] \geq \bar{\psi}[D] - \delta/2, \tag{30}$$

hence

$$\begin{aligned} \left|\bar{\psi}[D] - \phi_i(t_i)[D]\right| &\geq \left|\psi[D] - \phi_i(t_i)[D]\right| - \left|\psi[D] - \bar{\psi}[D]\right| \\ &> \delta - \delta/2 = \delta/2, \end{aligned}$$

which implies $F_i(\bar{\psi}, \bar{\psi}) - F_i(t_i, \bar{\psi}) \geq 2\eta$, by the definition of η. Now pick any $a_i \in A_i$ with $\rho_i(\bar{\psi}, a_i) < \gamma$, so that $F_i(a_i, \bar{\psi}) - F_i(\bar{\psi}, \bar{\psi}) > -\eta$. Then

$$F_i(a_i, \bar{\psi}) - F_i(\phi_i(t_i), \bar{\psi}) > \eta$$

and hence

$$\sum_{\theta \in \Theta} \sum_{a_{-i} \in A_{-i}} \psi[\theta, a_{-i}][g_i(t_i, a_{-i}, \theta) - g_i(a_i, a_{-i}, \theta)] \leq$$

$$\leq (M\delta/4)(-\eta)(1-\delta/2) + (\delta/2)(-M + M\delta/4) < (M\delta/4)(-\eta)(1-\delta/2) < -\varepsilon,$$

proving (29) for the case $\psi[\Theta \times T_{-i}] \geq 1 - \delta/2$ and thus concluding the proof. □

References

DEKEL, E., D. FUDENBERG, AND S. MORRIS (2006): "Topologies on Types," *Theoretical Economics*, 1, 275–309.

——— (2007): "Interim Correlated Rationalizability," *Theoretical Economics*, 2, 15–40.

ELY, J. C., AND M. PESKI (2008): "Critical Types," mimeo., Northwestern University.

FAN, K. (1953): "Minimax theorems," *Proceedings of the National Academy of Sciences of the United States of America*, 39, 42–47.

HARSANYI, J. (1967-68): "Games with Incomplete Information Played by Bayesian Players, I-III," *Management Science*, 14, 159–182, 320–334, 486–502.

KAJII, A., AND S. MORRIS (1997): "Refinements and Higher Order Beliefs: A Unified Survey," mimeo., Princeton University.

MERTENS, J.-F., AND S. ZAMIR (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14(1), 1–29.

MONDERER, D., AND D. SAMET (1989): "Approximating Common Knowledge with Common Beliefs," *Games and Economic Behavior*, 1, 170–190.

MORRIS, S. (2002): "Typical Types," mimeo., Princeton University.

RUBINSTEIN, A. (1989): "The Electronic Mail Game: A Game with Almost Common Knowledge," *American Economic Review*, 79, 389–391.

WEINSTEIN, J., AND M. YILDIZ (2007): "A Structure Theorem for Rationalizability with Application to Robust Predictions of Refinements," *Econometrica*, 75(2), 365–400.