Generic Characterization of the Set of Belief-Free Review-Strategy Equilibrium Payoffs

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Abstract

We study repeated games with imperfect private monitoring. We obtain the characterization, which only depends on the parameters of the stage game, of the set of payoffs that can be implemented as the belief-free review-strategy equilibrium in the limit as the discount factor converges to one. Our characterization is valid for the generic monitoring technology if the number of private signals is sufficiently large and the number of players is no less than four.

We show the characterized set is large enough to attain the folk theorem for the games with prisoners'-dilemma structure.

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1 Introduction

The main finding in the theory of repeated games is that long term relationships enhance the possibility of cooperation. In fact, the central result is probably the folk theorem; any feasible and individually rational payoff can be sustained in the equilibrium when players are sufficiently patient. Fudenberg and Maskin (1986) establish the folk theorem under perfect monitoring, that is, when players can observe the opponents' action directly. Fudenberg, Levine, and Maskin (1994) extend the folk theorem to imperfect public monitoring, where players cannot observe the opponents' action directly, but observe public noisy signals about the opponents' action. Recently, it is shown that these results are robust to the introduction of private monitoring. Hörner and Olszewski (2006) show the robustness of the folk theorem in perfect monitoring to almost perfect monitoring, where players can observe neither the opponents' action nor public signals, but only private noisy signals that are close to perfect monitoring. Hörner and Olszewski (2008) establish the robustness of the folk theorem in public monitoring to almost public monitoring, where players can observe neither the opponents' action nor public signals, but only private noisy signals that are close to public monitoring.

On the other hand, when the monitoring is neither almost perfect nor almost public, almost all the results are attained only with conditionally independent monitoring; players can obtain no information on what their opponents have observed by observing their own private signals conditional on an action profile. Under this assumption, Matsushima (2004) shows the folk theorem for two-player prisoner's dilemma; Ely, Hörner, and Olszewski (2005) characterize the set of limit equilibrium payoffs implementable by the similar strategy as the discount factor converges to one in general two-by-two games; Yamamoto (2007) shows the efficiency result in non-degenerate N-player games; and finally, Yamamoto (2008b) obtains the characterization of the equilibrium payoff set implementable by so-called belief-free review-strategy equilibria for general N-player games.

The idea of the above papers is as follows. The infinite periods are regarded as a sequence of review rounds. In each review round, players take a constant action and any action taken on the equilibrium path is indifferent at the beginning of the round regardless of the opponents' history up to the present action. For example, in Matsushima's equilibrium strategies, the infinite periods are divided into T-period review rounds. Then, however noisy the monitoring technology is, when T is sufficiently large, players can statistically infer the opponents' action with arbitrarily high power, using the information pooled in a T-period review round. Since a player's continuation strategy is a best reply regardless of the history at the beginning of every review round, a player does not need to know the history in the past review rounds to compute her best reply. Thus, the continuation game at the beginning of each review round is identical to each other and the history of the opponents is a sufficient statistic for a player's payoffs. Therefore, this class of equilibria can be analyzed using recursive techniques with the history of the opponents being a state variable in a dynamic programming formulation of a player's optimization.

Unfortunately, there are two difficulties to be solved to attain the folk theorem for general N-player games with the generic monitoring structure by applying their methodology. Firstly, their method establishes the folk theorem only for games with prisoner's-dilemma structure. Secondly, conditional independence is not a generic property.

This paper concentrates on the second problem, that is, establishes the characterization of the set of equilibrium payoffs implementable as the belief-free review-strategy equilibrium for the generic monitoring structure. Therefore, this paper is the first to establish a nontrivial lower bound of the set of sequential equilibrium payoffs for the generic monitoring structure. In addition, we show that this lower bound is large enough to attain the folk theorem for the games with prisoner's-dilemma structure.

The necessity of the conditional independence for Matsushima (2004), Ely, Hörner, and Olszewski (2005), and Yamamoto (2007 and 2008b) is explained as follows. To statistically infer the opponents' action by pooling the information, it is important to create an equilibrium where each player take a constant action in each review round. Consider the prisoner's-dilemma example, in which player j is likely to be punished in the following review round if a lot of signals indicating a noncooperative action are pooled in the current review round by player $i \neq j$. Suppose player j takes a cooperative action initially in the current review round. If the signals are conditionally independent, after any realization of the signals, it is optimal to stick to the cooperative action since the observed signals have no information about whether player j will be punished or not. With conditionally dependent signals, however, at the periods near to the end of the current review round, under some realization of the signals, player j considers that she will be punished regardless of actions in the remaining periods of the current review round, which destroys the incentive to continue cooperation. Let us call this problem the statistical inference problem.

Specifically, in Matushima(2004), Ely, Hörner, and Olszewski(2005), and Yamamoto(2007 and 2008b), the signals are pooled as follows. Consider the situation where player i reviews player j. If player i plays a_i during the review round, she reviews player j's action by a random event whose possible realizations are $\{0, 1\}$ and the probability of taking 1 depends on (a_i, ω_i), where ω_i is player i's private signal. Note that the realization indirectly depends on player j's action through the conditional distribution of ω_i. Player i considers that player j takes cooperation if the sum of the realized random events is sufficiently high. From player j's perspective, statistical inference on whether player i believes that player j cooperates or not depends on the expectation of player i's random events conditional on player j's signals and actions. If the conditional probability of the random event does not depend on player j's signals but only on her actions, the statistical inference problem does not emerge. However, so that the conditional probability only depends on player j's actions for the generic monitoring structure, it is necessary to have $|A_j| \times |\Omega_j| \leq |\Omega_i|$ in order to prevent the statistic inference problem for player j, which cannot be satisfied for all (i, j). Here, $|\Omega_i|$ is the number of player i's private signals and $|A_i|$ is the number of player i's actions.

The idea of this paper is, if player i can also use the signal $\omega_{-(i,j)}$ from players $-(i, j)$ to create the random event to review player j, it is sufficient to have $|A_{-i}| \times |\Omega_j| \leq |\Omega_{-j}|$, which can be satisfied for all (i, j). Therefore, if communication is perfect, instantaneous, and costless and player $-(i, j)$ has an incentive to tell the truth to player i, we can generate the same situation as conditionally independent signals.

However, there are three remaining difficulties. Firstly, since we do not assume explicit communication, players $-(i,j)$ have to send message about $\omega_{-(i,j)}$ to player i by taking actions. Since this is time consuming and players have to take inefficient actions, if players communicate the realized signals of each period, it destroys too much payoff. Therefore, after each review round, players choose about which periods are used for reviewing. Specifically, player i randomly chooses which periods are used for reviewing player j. Player i sends a message about these randomly picked periods by taking specific actions. After receiving this message from player i, players $-(i,j)$ will send the message $\omega_{-(i,j)}$ observed in these periods to player i. Since the randomization takes place after the review round, this does not destroy the incentive to take a constant action. If players are sufficiently patient, only small probability of being reviewed is enough to give an enough incentive not to deviate. Thus, if the number of periods used for reviewing is sufficiently small compared to the review round, this does not destroy so much efficiency.

The second difficulty is to give an incentive for player i to randomly pick the periods to review player j and for players $-(i,j)$ to tell the truth about $\omega_{-(i,j)}$. Although these messages are used to directly punish player j, if the punishment is costly or beneficial to players i or $-(i,j)$, they may not have an incentive to tell the truth. It is well know in mechanism design that if there are at least three players, it is easy to implement the truthtelling equilibrium. Thus, we assume there are at least four players so that at least three players can review player j. Specifically, when player i reviews player j, one player l does not send a message when the other $-(i,j,l)$ send the message. Player i uses the message from players $-(i,j,l)$ to review player j. Player i decides to suggest the punishment of player j only if according to every $(N-2)$-tuple $-(i,j,l)$, player j's deviation is inferred. Players will punish player j only if all the players but player j suggest the punishment of player j. Further, if some player n's message about either the periods for reviewing or the signals has an impact on whether player j will be punished or not, we give player n some constant continuation value. Thus, there exists the (weak) incentive to tell the truth. In addition, since no one deviates on the equilibrium path, if all the others tell the truth, any single player's message has an impact only with very low probability.

Finally, to restore the key properties of the conditionally independent signals, we have to make sure that player l cannot infer from ω_l whether player i infers player j's deviation according to $-(i,j,l)$.[1] This poses $|A_{-i}| \times |\Omega_l|$ restriction when players $-(i,j,l)$ send the message to player i. At the same time, the restriction from inference problem for player j puts $|A_{-i}| \times |\Omega_j|$ restrictions. On the other hand, the number of information available to player i is $\left|\Omega_{-(j,l)}\right|$. Therefore, to satisfy these two restrictions generically, it is sufficient to have $|A_{-i}| \times (|\Omega_j| + |\Omega_l|) \leq \left|\Omega_{-(i,j)}\right|$, which can be generically satisfied.

Let me comment on two related papers, Yamamoto (2008b) and Fong, Gossner, Hörner, and Sannikov (2007). One may argue that our equilibrium strategy is close to one analyzed by Yamamoto (2008b), which assumes conditionally independent monitoring although we have to substantially modify the strategy to attain the characterization for the generic monitoring structure. However, we consider this closeness is not the weakness of our paper but the strength. This closeness means our paper offers the direct way to extend the results with conditionally independent monitoring to conditionally dependent monitoring. Since the games with conditionally independent signals are easy to analyze, we offer the important method to sidestep the difficulty caused by the conditionally dependent monitoring and show the sufficient condition for analyses of the conditionally independent monitoring to be enough.

Recently, Fong, Gossner, Hörner, and Sannikov (2007) attain the efficiency result in two-player prisoner's dilemma with not-nongeneric monitoring structures. Neither their result nor ours contains the other as a special case. Fong, Gossner, Hörner, and Sannikov (2007) require some restriction on the monitoring structure, but they attain efficiency result for two-player prisoner's dilemma. We require that the number of players is no less than four, but attain the folk theorem with the generic monitoring structure. Moreover, the main ideas are different. Fong, Gossner, Hörner, and Sannikov (2007) derive the condition that the statistical inference problem emerges only with small probability on the equilibrium path.

[1] As we will see, player l's continuation value depends on whether another player j's deviation is inferred by another player i or not.

On the other hand, this paper constructs a strategy that is completely free from the statistical inference problem.

In fact, many economic situations should be analyzed with repeated games with private monitoring. The most famous example is Stigler's (1964) story of secret price cutting in cartel oligopoly. In the trade of intermediate goods, the price is set via face-to-face negotiation of a buyer and a seller. Thus, a seller's action (price setting) cannot be observed and a seller can get only private signals about the opponents' action such as its own profit. It is important to verify whether each seller's option of setting price below cartel price can prevent the cartel agreement from being self-enforcing or not by the analyses of repeated games with private monitoring as Matsushima (2004) argues.

In contrast to Stigler's argument, our folk theorem result implies that the full cartel agreement can be self-enforcing for the generic monitoring structure. In addition, our results have implications for the effects of anti-trust laws. Our model is applicable to the situation where communication is possible, but noisy or costly. For example, there is the risk of being arrested by the anti-trust committee. Therefore, if firms are sufficiently patient, the effect of anti-trust law is limited. Further, to the best of our knowledge, the generic cooperation result can be attained only for $N \geq 4$. While we have to wait for the full characterization of all sequential equilibrium payoffs, it is not so obvious that more competition prevents noncooperative collusion.

Let us finish the introduction by reviewing other approaches to attaining efficiency in repeated games with private monitoring. Several papers such as Sekiguchi (1997) and Bhaskar and Obara (2002) focus on belief-based techniques. In such equilibria, players' strategies involve statistical inference about the past history of the play. Results in those papers are limited to prisoner's dilemma with almost perfect monitoring.

Another approach is to introduce explicit communication. Versions of the folk theorem have been proven by Compte (1998), Kandori and Matsushima (1998), Aoyagi (2002), Fudenberg and Levine (2002), and Obara (2009). Introducing a public element allows these papers to sidestep the inherent issues unresolved in games with private monitoring. However, the analyses do not apply to some practical economic settings in which communication is not possible or costly. For example, in Stigler's oligopoly example above, anti-trust laws make communication illegal. Note that this paper does not introduce explicit communication.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 defines the belief-free review-strategy equilibrium. Section 4 states the main result. Section 5 briefly overviews the basic theoretical ideas behind the equilibrium construction. Sections 6 and 7 explains the equilibrium strategy. Section 8 presents the folk theorem for N-player prisoner's dilemma as an application. Section 9 concludes.

2 Model

The stage game is given by $\left\{I, (A_i, \Omega_i, g_i)_{i\in I}, q\right\}$. $I = \{1, 2, \ldots, N\}$ is the set of players, A_i is the finite set of player i's pure actions, Ω_i is the finite set of player i's private signals, $g_i : A_i \times \Omega_i \to \mathbb{R}$ is player i's utility function, and q is the probability distribution of the signals. We assume $\#\{i \in I \mid |A_i| \geq 2\} \geq 4$. Let $A \equiv \prod_{i\in I} A_i$ and $\Omega \equiv \prod_{i\in I} \Omega_i$ be the set of action profiles and signal profiles, respectively. For $\mathcal{I} \in 2^I$, $A_{-\mathcal{I}}$, $\Omega_{-\mathcal{I}}$, $a_{-\mathcal{I}} \in A_{-\mathcal{I}}$, $\omega_{-\mathcal{I}} \in \Omega_{-\mathcal{I}}$ are defined as usual.

In every stage game, players choose an action profile $a \equiv (a_1, \ldots, a_N) \in A$, and then a signal profile $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$ is distributed according to the conditional probability function $q(\cdot \mid a)$. Given an action $a_i \in A_i$ and a private signal $\omega_i \in \Omega_i$, player i receives a profit $g_i(a_i, \omega_i)$. Thus, her expected payoff conditional on an action profile $a \in A$ is denoted by $\pi_i(a) = \sum_{\omega\in\Omega} q(\omega \mid a)\, g_i(a_i, \omega_i)$. For each $a \in A$, let $\pi(a)$ represent the payoff vector $(\pi_i(a))_{i\in I}$.

Consider the infinitely repeated game in which the discount factor is $\delta \in (0, 1)$. Let $a_{i,\tau}$ and $\omega_{i,\tau}$ denote the performed action and the observed private signal in period τ by player i, respectively. Player i's private history up to period $t \geq 2$ is given by $h_{i,t} \equiv (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1}$. Let $h_{i,1} = \emptyset$ and for each $t \geq 2$, let $H_{i,t}$ be the set of all $h_{i,t}$. Then, a strategy for player i is defined to be a mapping $\sigma_i : \bigcup_{t=1}^{\infty} H_{i,t} \to \triangle A_i$. Let Σ_i be the set of all strategies for player i and let $\Sigma \equiv \prod_{i \in I} \Sigma_i$. Let $w_i(\sigma)$ represent player i's expected average payoff by a strategy profile $\sigma \in \Sigma$, that is, $w_i(\sigma) = (1-\delta) E\left[\sum_{t=1}^{\infty} \delta^{t-1} \pi_i(a_t) \mid \sigma\right]$. For any strategy $\sigma_i \in \Sigma_i$ and any history $h_{i,t} \in H_{i,t}$, let $\sigma_i \mid h_{i,t}$ be player i's continuation strategy after $h_{i,t}$. Also, for any $\sigma_i \in \Sigma_i$, $h_{i,t} \in H_{i,t}$, and $a_i \in A_i$, let $\sigma_i \mid (h_{i,t}, a_i)$ represent player i's strategy $\tilde{\sigma}_i \in \Sigma_i$ such that $\tilde{\sigma}_i(h_{i,1}) = a_i$ and such that for any $h_{i,2} \in H_{i,2}$, $\tilde{\sigma}_i \mid h_{i,2} = \sigma_i \mid h_{i,t+1}$ where $h_{i,t+1} = (h_{i,t}, h_{i,2})$. In words, $\sigma_i \mid (h_{i,t}, a_i)$ denotes the continuation strategy after history $h_{i,t}$ but the play in period t is replaced with the pure action a_i.

This paper imposes three assumptions on the monitoring technology q. Firstly, monitoring satisfies the *full support condition* if $q(\omega \mid a) > 0$ for all $a \in A$ and $\omega \in \Omega$.

Secondly, monitoring satisfies the *identifiability condition* if

$$rank Q_i(\mathcal{A}_l \mid a_i) = |A_{-i}|$$

for all $i, l \in I$ and any $a_i \in A_i$ and $\mathcal{A}_l \subset A_l$. Here, $Q_i(\mathcal{A}_l \mid a_i)$ is defined as follows. $Q_i\left(a_i a_l a_{-(i,l)} \mid a_i\right)$ is a matrix with rows $\left(q\left(\omega_i \mid a_i a_l a_{-(i,l)}\right)\right)_{\omega_i \in \Omega_i}$. $Q_i(a_l \mid a_i)$ is a matrix stacking $Q_i\left(a_l a_{-(i,l)} \mid a_i\right)$ for all $a_{-(i,l)}$ vertically. $Q_i(\mathcal{A}_l \mid a_i)$ is a matrix stacking $Q_i(a_l \mid a_i)$ vertically with the first $|\mathcal{A}_l|$ matrices satisfying $a_l \in \mathcal{A}_l$.

Thirdly, monitoring satisfies the $(N-2)$-*identifiability condition* if

$$rank Q_{-(i,l)}(\mathcal{A}_j \mid a_i) = |A_{-i}| \times (|\Omega_j| + |\Omega_l|)$$

for all $i, j, l \in I$ with $i \neq j \neq l \neq i$ and any $a_i \in A_i$ and $\mathcal{A}_j \subset A_j$. Here, $Q^j_{-(i,l)}(\mathcal{A}_j \mid a_i)$ is defined as follows. $Q^j_{-(i,l)}\left(a_j a_{-(i,j)} \omega_j \mid a_i\right)$ is a matrix with rows $\left(q\left(\omega_{-(j,l)} \mid a_i a_j a_{-(i,j)} \omega_j\right)\right)_{\omega_{-(j,l)} \in \Omega_{-(i,j)}}$. $Q^j_{-(i,l)}\left(a_j a_{-(i,j)} \mid a_i\right)$ is a matrix stacking $Q^j_{-(i,l)}\left(a_j a_{-(i,j)} \omega_j \mid a_i\right)$ for all ω_j vertically. $Q^j_{-(i,l)}(a_j \mid a_i)$ is a matrix stacking $Q^j_{-(i,l)}\left(a_j a_{-(i,j)} \mid a_i\right)$ for all $a_{-(i,j)}$ vertically. $Q^j_{-(i,l)}(\mathcal{A}_j \mid a_i)$ is a matrix stacking $Q^j_{-(i,l)}(a_j \mid a_i)$ vertically with the first $|\mathcal{A}_j|$ matrices satisfying $a_j \in \mathcal{A}_j$. Similarly, $Q^l_{-(i,l)}\left(a_j a_{-(i,j)} \omega_l \mid a_i\right)$ is a matrix with rows $\left(q\left(\omega_{-(j,l)} \mid a_i a_j a_{-(i,j)} \omega_l\right)\right)_{\omega_{-(j,l)} \in \Omega_{-(i,j)}}$ for all ω_l and $Q^l_{-(i,l)}\left(a_j a_{-(i,j)} \mid a_i\right)$ is a matrix stacking $Q^l_{-(i,l)}\left(a_j a_{-(i,j)} \omega_l \mid a_i\right)$ for all ω_l vertically. $Q^l_{-(i,l)}(a_j \mid a_i)$ is a matrix stacking $Q^l_{-(i,l)}\left(a_j a_{-(i,j)} \mid a_i\right)$ for all $a_{-(i,j)}$ vertically.

$Q^{l}_{-(i,l)}(\mathcal{A}_j \mid a_i)$ is a matrix stacking $Q^{i}_{-(i,l)}(a_j \mid a_i)$ with the first $|\mathcal{A}_j|$ matrices satisfying $a_j \in \mathcal{A}_j$. Finally, $Q_{-(i,l)}(\mathcal{A}_j \mid a_i)$ is a matrix stacking $Q^{j}_{-(i,l)}(\mathcal{A}_j \mid a_i)$ and $Q^{l}_{-(i,l)}(\mathcal{A}_j \mid a_i)$ vertically.

We show that if the number of private signals is sufficiently large, the above three assumptions are generically satisfied.

Lemma 1 *If $|A_{-i}| \leq |\Omega_i|$ for all i and $|A_{-i}| \times (|\Omega_j| + |\Omega_l|) \leq \left|\Omega_{-(j,l)}\right|$ for all $i, j, l \in I$ with $i \neq j \neq l \neq i$, then the full support condition, the identifiability condition, and the $(N-2)$-identifiability condition are generically satisfied.*

Proof. Note that the full support condition is generic. The identifiability condition is generically satisfied if $|A_{-i}| \leq |\Omega_i|$. The $(N-2)$-identifiability condition is generically satisfied for $(i,j,l) \in I^3$ with $i \neq j \neq l \neq i$ if $|A_{-i}| \times (|\Omega_j| + |\Omega_l|) \leq \left|\Omega_{-(j,l)}\right|$. ■

3 Belief-Free Review-Strategy Equilibrium

This section states a notion of belief-free review-strategy equilibrium, which is the extension of belief-free equilibrium studied by Ely and Välimäki (2002), Matsushima (2004), Ely, Hörner, and Olszewski (2005), and Yamamoto (2007).

Firstly, we define a *review strategy profile.*

Definition 2 *Let $(t_l)_{l=0}^{\infty}$ be a sequence of integers with $t_0 = 1$, $t_l > t_{l-1}$ for all $l \geq 1$. $(t_l)_{l=0}^{\infty}$ is divided into T_{ca} and T_{bf}. A strategy profile $\sigma \in \Sigma$ is a review strategy profile with $(t_l)_{l=0}^{\infty}$ if for all $t_l \in T_{ca}$, $\sigma_i(h_{i,t})[a_{i,t-1}] = 1$ for each $t \in \{t_{l-1}+1, \ldots, t_l - 1\}$ for each i and for each $h_{i,t} = (a_{i,\tau}, \omega_{i,\tau}) \in H_{i,t}$.*

In this definition, a infinitely repeated game is divided into infinitely repeated review rounds. The lth review round is from t_{l-1} to $t_l - 1$ and each review round is classified as T_{ca} and T_{bf}, which stand for the constant-action rounds and the completely belief-free rounds, respectively. The requirement for a strategy profile to be a review strategy is that every agent takes a constant action in each constant-action round.

Secondly, we define a *belief-free review-strategy equilibrium.*

Definition 3 *Let $\sigma \in \Sigma$ be a review strategy profile with a sequence $(t_l)_{l=0}^{\infty}$. σ is belief-free if for all $i \in I$ and $l \geq 1$,*

$$\sigma_i \mid h_{i,t_{l-1}} \in BR(s_{-i} \mid h_{-i,t_{l-1}}, a_{-i}) \text{ for all } a_{-i} \in \text{supp}\{\sigma_{-i}(h_{-i,t_{l-1}})\}$$
$$\text{for all } h_{t_{l-1}} \in H_{t_l-1} \text{ if } t_l \in T_{ca},$$
$$\sigma_i \mid h_{i,t} \in BR(s_{-i} \mid h_{-i,t}, a_{-i}) \text{ for all } a_{-i} \in \text{supp}\{\sigma_{-i}(h_{-i,t})\}$$
$$\text{for all } h_i \in H_t \text{ and } t \in (t_{l-1}, \ldots, t_l - 1) \text{ with } t_l \in T_{bf}.$$

Note that the requirement for a review strategy profile to be belief-free is that after any history of past review rounds and after any action the other players choose in the first period of the current round, player i's continuation strategy is a best response to the others in every constant-action round and that after any history, after any action the other players choose in the current period, player i's continuation strategy is a best response to the others in every completely belief-free round. $\sigma \in \Sigma$ is a *belief-free review-strategy equilibrium with* $(t_l)_{l=0}^{\infty}$ if $\sigma \in \Sigma$ is a review strategy profile with a sequence $(t_l)_{l=0}^{\infty}$ and belief-free. Note that a belief-free review-strategy profile is a Nash equilibrium by definition.

My definition of a belief-free review-strategy equilibrium is the same as Yamamoto (2008b) since the completely belief-free round is the same as the consecutive one-period constant-action round. We introduce completely belief-free round for the notational convenience.

4 Characterization

Yamamoto(2008b) attains the characterization of belief-free review-strategy equilibrium with $T_{bf} = \varnothing$, which is valid when the signals are conditionally independent. Our objective is to show that the same characterization works even for the generic monitoring structure.

To state the characterization, the following notation is useful. A non-empty set $\mathcal{A} \subset A$

is a *regime generated from* A if $\mathcal{A}$ has a product structure, that is, $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ and $\mathcal{A}_i \subset A_i$ for all $i \in I$. Let $\mathcal{J}$ be the set of all regimes generated from A. Then, for any probability distribution $p \in \triangle \mathcal{J}$, define $V(p)$ as

$$V(p) \equiv co \left\{ \sum_{\mathcal{A} \in \mathcal{J}} p(\mathcal{A}) \pi(a(\mathcal{A})) \mid a(\mathcal{A}) \in \mathcal{A}, \forall \mathcal{A} \in \mathcal{J} \right\}.$$

For each i and $\mathcal{A} \in \mathcal{J}$, let $\underline{v}_i(\mathcal{A})$ and $\overline{v}_i(\mathcal{A})$ be

$$\underline{v}_i(\mathcal{A}) \equiv \min_{a_{-i} \in \mathcal{A}_{-i}} \max_{a_i \in A_i} \pi_i(a), \overline{v}_i(\mathcal{A}) \equiv \max_{a_{-i} \in \mathcal{A}_{-i}} \min_{a_i \in \mathcal{A}_i} \pi_i(a).$$

Also, for each i and $\mathcal{A} \in \mathcal{J}$, let $\underline{a}^i(\mathcal{A}) \in \mathcal{A}$ and $\overline{a}^i(\mathcal{A}) \in \mathcal{A}$ be such that $\underline{a}^i_{-i}(\mathcal{A}) \in \mathcal{A}_{-i}$ and $\overline{a}^i_{-i}(\mathcal{A}) \in \mathcal{A}_{-i}$ solve the above problem, that is,

$$\underline{v}_i(\mathcal{A}) = \max_{a_i \in A_i} \pi_i\left(a_i, \underline{a}^i_{-i}(\mathcal{A})\right), \overline{v}_i(\mathcal{A}) = \min_{a_i \in \mathcal{A}_i} \pi_i\left(a_i, \overline{a}^i_{-i}(\mathcal{A})\right). \tag{1}$$

Note that $\underline{a}^i_i(\mathcal{A}) \in \mathcal{A}_i$ and $\overline{a}^i_i(\mathcal{A}) \in \mathcal{A}_i$ are arbitrary.

For all $i \in I$, let $\underline{v}_i$ be a column vector with the components $\underline{v}_i(\mathcal{A})$ for all $\mathcal{A} \in \mathcal{J}$, that is, $\underline{v}_i \equiv^\top (\underline{v}_i(\mathcal{A}))_{\mathcal{A} \in \mathcal{J}}$. Similarly, let $\overline{v}_i^\top \equiv (\overline{v}_i(\mathcal{A}))_{\mathcal{A} \in \mathcal{J}}$. The *full dimensional condition* is said to be satisfied if

$$\dim \bigcup_{p \in \triangle \mathcal{J}} (V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]) = N.$$

Our main result is that if the full support condition, the identifiability condition, $(N-2)$-identifiability condition, and the full dimensional condition are satisfied, then $\dim \bigcup_{p \in \triangle \mathcal{J}} (V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i])$ is the limit set of belief-free review-strategy equilibrium payoffs as the discount factor converges to one.

Theorem 4 *If the full support condition, the identifiability condition, the* $(N-2)$*-identifiability condition, and the full dimensional condition are satisfied, then*

$$\bigcup_{p \in \triangle \mathcal{J}} (V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]) = \lim_{\delta \to 1} E(\delta),$$

where $E(\delta)$ is the set of belief-free review-strategy equilibrium payoffs with discount factor δ.

Proof. The proof of $\bigcup_{p \in \triangle \mathcal{J}} (V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]) \supset \lim_{\delta \to 1} E(\delta)$ is analogous to **Proposition 1** of Yamamoto (2008b) since our definition of belief-free review strategy is the same. See Sections 7 and 8 for the proof of the other direction. ■

5 Basic Ideas

5.1 Recovery of Belief-Free situation

The idea of the review strategies in Matsushima (2004), Ely, Hörner, and Olszewski (2007), and Yamamoto (2007 and 2008b) is as follows. Even if the monitoring is far from perfect, if we construct the blocks called review rounds and make players take constant actions, we can statistically infer the opponents' action by pooling the information during the review round. However, to make a constant action optimal on the equilibrium path after any history, it is important to prevent the following statistical inference problem. Consider the prisoner's-dilemma example, in which player j is likely to be punished in the following review round if a lot of bad events indicating a noncooperative action are observed in the current review round by player $i \neq j$. Suppose player j takes a cooperative action initially in the current review round. If player j's signal has information about the number of bad events observed by player i, at the periods near to the end of the current review round, after some realization of the signals, player j considers that she will be punished regardless of actions in the remaining periods of the current review round with very high probability, which destroys the incentive to continue cooperation. We call this problem the statistical inference problem.

To prevent the statistical inference problem, it is enough to show the existence of the solution for

$$\begin{bmatrix} q(\omega_i^1 \mid C_j a_i \omega_j^1) & \cdots & q(\omega_i^{|\Omega_i|} \mid C_j a_i \omega_j^1) \\ \vdots & & \vdots \\ q(\omega_i^1 \mid C_j a_i \omega_j^{|\Omega_j|}) & \cdots & q(\omega_i^{|\Omega_i|} \mid C_j a_i \omega_j^{|\Omega_j|}) \\ q(\omega_i^1 \mid D_j a_i \omega_j^1) & \cdots & q(\omega_i^{|\Omega_i|} \mid D_j a_i \omega_j^1) \\ \vdots & & \vdots \\ q(\omega_i^1 \mid D_j a_i \omega_j^{|\Omega_j|}) & \cdots & q(\omega_i^{|\Omega_i|} \mid D_j a_i \omega_j^{|\Omega_j|}) \end{bmatrix} \begin{bmatrix} \psi(a_i \omega_i^1) \\ \vdots \\ \psi(a_i \omega_i^{|\Omega_i|}) \end{bmatrix} = \begin{bmatrix} q_C \\ \vdots \\ q_C \\ q_D \\ \vdots \\ q_D \end{bmatrix}$$

with $q_C > q_D$ for all $a_i \in A_i$. Then, if player i who plays a_i during the review phase reviews player j's action by counting a random event which takes 1 with probability $\psi(a_i\omega_i)$ and 0 with $1 - \psi(a_i\omega_i)$, player j cannot infer player i's counting from ω_j. However, so that the above system has a solution generically, it is necessary to have $|A_j| \times |\Omega_j| \leq |\Omega_i|$, which cannot be satisfied for all (i, j).

The idea of this paper is, suppose player i can also use the signal $\omega_{-(j,l)}$ with $l \neq i, j$ to review player j and

$$\begin{bmatrix} q(\omega^1_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^1_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^1_j) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^{|\Omega_j|}_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^{|\Omega_j|}_j) \\ & \vdots & \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_j) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_j|}_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_j|}_j) \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^1_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^1_j) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^{|\Omega_j|}_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^{|\Omega_j|}_j) \\ & \vdots & \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_j) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_j|}_j) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_j|}_j) \end{bmatrix} \begin{bmatrix} \psi(a_i \omega^1_{-(j,l)}) \\ \vdots \\ \psi(a_i \omega^{|\Omega_{-(j,l)}|}_{-(j,l)}) \end{bmatrix} = \begin{bmatrix} q_C \\ \vdots \\ q_C \\ q_D \\ \vdots \\ q_D \end{bmatrix} \tag{2}$$

has a solution for all a_i and player i. If player i who plays a_i during the review round reviews player j's action by counting a random event which takes 1 with probability $\psi(a_i \omega_{-(j,l)})$ and 0 with $1 - \psi(a_i \omega_{-(j,l)})$, player j cannot infer player i's counting from ω_j.

As we see below, it is important to have one player l whose signal is not used when player i reviews player j and player l cannot infer player i's counting about player j, which is expressed by the condition that

$$\begin{bmatrix} q(\omega^1_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^1_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^1_l) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^{|\Omega_l|}_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^1_{-(i,j)} \omega^{|\Omega_l|}_l) \\ & \vdots & \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_l) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_l|}_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid C_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_l|}_l) \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^1_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^1_l) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^{|\Omega_l|}_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^1_{-(i,j)} \omega^{|\Omega_l|}_l) \\ & \vdots & \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^1_l) \\ \vdots & & \vdots \\ q(\omega^1_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_l|}_l) & \cdots & q(\omega^{|\Omega_{-(j,l)}|}_{-(j,l)} \mid D_j a_i a^{|A_{-i,j}|}_{-(i,j)} \omega^{|\Omega_l|}_l) \end{bmatrix} \begin{bmatrix} \psi(a_i \omega^1_{-(j,l)}) \\ \vdots \\ \psi(a_i \omega^{|\Omega_{-(j,l)}|}_{-(j,l)}) \end{bmatrix} = \begin{bmatrix} q_C \\ \vdots \\ q_C \\ q_D \\ \vdots \\ q_D \end{bmatrix}. \tag{3}$$

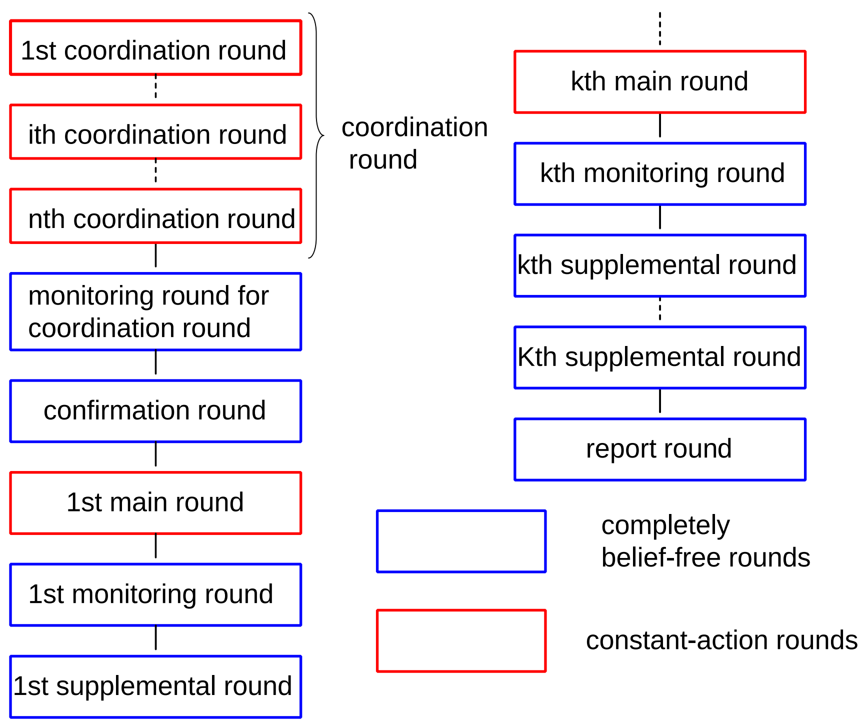
If the $(N-2)$-identifiability is satisfied, there exists $\psi : A_i \times \Omega_{-(i,l)} \to [0,1]$ satisfying (2) and (3).

5.2 Rounds and Recommended Actions

Let $p \in \triangle \mathcal{J}$ be such that v is included in the interior of the intersection of $V(p)$ and $\prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]$. Let $(\underline{w}_i)_{i \in I}$ and $(\overline{w}_i)_{i \in I}$ be vectors of real numbers such that $\underline{w}_i < v_i < \overline{w}_i$ and such that the hyperrectangle $\prod_{i \in I} [\underline{w}_i, \overline{w}_i]$ is included in the interior of $V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]$. It suffices to show that $\prod_{i \in I} [\underline{w}_i, \overline{w}_i]$ is sustained in a belief-free review-strategy equilibrium for sufficiently large δ. The construction is similar to Yamamoto (2008b).

Let $X_i \equiv \{G, B\}$ be the set of player i's possible states and $X \equiv \prod_{i \in I} X_i$ be the set of state profiles. We want to construct an equilibrium such that for each player i, if $x_{i-1} = B$, her expected utility is given by $\underline{w}_i$ and if $x_{i-1} = G$, her expected utility is given by $\overline{w}_i$. In the rest of the paper, let "player $i-1$" refer to player $i-1$ for each $i \in \{2, \ldots, N\}$ and to player N for $i = 1$. Likewise, let "player $i+1$" refer to player $i-1$ for each $i \in \{1, \ldots, N-1\}$ and to player 1 for $i = N$.

Given an integer K, we consider a belief-free review-strategy equilibrium where for each integer $n \geq 1$, review rounds from $l = 1 + (n-1)(3K + 3 + N)$ to $l = n(3K + 3 + N)$ form one review phase. Thus, a review phase consists of $(3K + 3 + N)$ review rounds as shown below.



The coordination round and the main rounds are constant-action rounds and the others are completely-belief-free rounds. For notational convenience, we also classify the coordination round, the supplemental round, and the report round as communication rounds and the monitoring rounds for the coordination round and the main rounds as the monitoring rounds. Assume that the length of main rounds is sufficiently large compared to the other rounds.

The actions taken in the main rounds are determined by the state profile. Let $\tilde{K}$ be an integer and $(\mathcal{A}^1, \ldots, \mathcal{A}^{\tilde{K}})$ be a sequence of regimes such that

$$\frac{1}{\tilde{K}} \sum_{k=1}^{\tilde{K}} \underline{v}_i \left(\mathcal{A}^k\right) < \underline{w}_i < v_i < \overline{w}_i < \frac{1}{\tilde{K}} \sum_{k=1}^{\tilde{K}} \overline{v}_i \left(\mathcal{A}^k\right) \tag{4}$$

for all $i \in I$, and such that for each $x \in X$, there exists a sequence of action profiles $(a^{x,1}, \ldots, a^{x,\tilde{K}})$ such that

$$a^{x,k} \in \mathcal{A}^k$$

for all $k \in \{1, \ldots, \tilde{K}\}$ and letting $w^x = (w_i^x)_{i \in I}$ be the time-average payoff vector over the sequence $(a^{x,1}, \ldots, a^{x,\tilde{K}})$,

$$w_i^x \begin{cases} < \underline{w}_i & \text{if } x_{i-1} = B \\ > \overline{w}_i & \text{if } x_{i-1} = G \end{cases}$$

for all $i \in I$. Given a natural number K, let $\left(\mathcal{A}^1, \ldots, \mathcal{A}^K\right)$ be a cyclic sequence of $(\mathcal{A}^1, \ldots, \mathcal{A}^{\tilde{K}})$ with length K, that is, $\mathcal{A}^{k+n\tilde{K}} = \mathcal{A}^k$ for all $k \in \{1, \ldots, \tilde{K}\}$ and $n \geq 0$. Also, let $(a^{x,1}, \ldots, a^{x,K})$ be a cyclic sequence of $(a^{x,1}, \ldots, a^{x,\tilde{K}})$ for all x. Let us call $a^{x,k}$ a recommended action in the kth main round.

Suppose players take recommended actions. Then, since the length of the main rounds are sufficiently large, the time-average of the instantaneous utilities for each review phase is lower than $\underline{w}_i$ if $x_{i-1} = B$ and higher than $\overline{w}_i$ if $x_{i-1} = G$. To show that $\prod_{i \in I} [\underline{w}_i, \overline{w}_i]$ is sustained in a belief-free review-strategy equilibrium for sufficiently large δ, it suffices to show that a constant action is optimal, that the time-average of the instantaneous utilities for the review phase is lower than $\underline{w}_i$ if $x_{i-1} = B$ and higher than $\overline{w}_i$ if $x_{i-1} = G$ even if players take other actions from $\mathcal{A}^k$, and that they do not have any incentive to take actions that are not included in $\mathcal{A}^k$.

5.3 Description of the Strategy with Cheap Talk

In this subsection, for the simple explanation, we assume players can explicitly communicate. It is useful to see each review phase as a block game. The length of each review phase and so that of the block game is given by $T_b \equiv t_{n(3K+3+N)} - t_{(n-1)(3K+3+N)}$. The block strategy is defined as the mapping from $\bigcup_{t=1}^{T_b} h_{i,t}$ to $\Delta(A_i)$. Let Σ_{i,T_b} denote the set of all block strategies. In what follows, we specify two important block strategies, a good strategy $\sigma_i^G \in \Sigma_{i,T_b}$ and a bad strategy $\sigma_i^B \in \Sigma_{i,T_b}$. Let $\sigma^x = (\sigma_i^{x_i})_{i \in I}$ with $x \in \{G, B\}^N$ denote the generic strategy profile consisting of $\sigma_i^{x_i}$ with $x_i \in \{G, B\}$. For each $i \in I$, pick two elements of A_i arbitrarily and call each of them a_i^G and a_i^B, respectively. The behavior under σ^x is described as follows.

***i*th Coordination Round** $[T]$ This round is regarded as a communication round in which each player i sends her message from $x_i \in \{G, B\}$. Player i sends message x_i by constantly choosing the action $a_i^{x_i}$ for T periods. Players $-i$ takes a_{-i}^G.

Monitoring Round for the Coordination Round This round is regarded as a monitoring round. In this subsection, since we assume the existence of cheap talk, this phase is instantaneous.

Players form $(N-1)$-tuples $(i, j, -(i, j, l))_{l \neq i,j}$. Players $-(i, j, l)$ report what signals are observed in the coordination round to player i so that player i can use that information to review player j's action in the coordination round. As mentioned in Subsection 5.1, we can restore the key properties of the conditional independent signals by allowing player i to use $\omega_{-(i,j,l)}$.

Player i infers the message in the coordination round from player j by using the messages in the monitoring round for the coordination round from players $-(i, j, l)$. If player i infers that player j's message is x_j by using the messages from players $-(i, j, l)$, we say that player i infers that player j's message is x_j according to players $-(i, j, l)$ and let $x^0(i, j, -(i, j, l)) = x_j$ denote this situation. The details of the inference will be specified later.

Confirmation Round This round is regarded as a communication round. In this subsection, since we assume the existence of cheap talk, this phase is instantaneous.

Every player i tells $(x_i, (x^0(i, j, -(i, j, l))_{l \neq i,j})_{j \neq i})$ to the other players. Then, player n

receives the message $(\hat{x}_i, (\hat{x}^0(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n}$, where $\hat{x}$ denotes the inference of the message x by player n.[2] Let $m_n^0 \equiv ((\hat{x}_i, (\hat{x}^0(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n} \cup (x_n, (x^0(n,j,-(n,j,l))_{l\neq n,j})_{j\neq n})$ be the message profile of player n. Let M_n^0 be the set of message profiles m_n^0.

We say that $\tilde{x}$ is confirmed for player n if according to the message m_n^0, for each $j \neq n$, $\tilde{x}_j$ and $\tilde{x}_j' \neq \tilde{x}_j$,

$$\begin{aligned} &((N-2)(N-3)+1) \times 1_{\tilde{x}_j}\{\hat{x}_j\} + \sum_{i\neq n}\sum_{l\neq i,j} 1_{\tilde{x}_j}\left\{\hat{x}^0(i,j,-(i,j,l))\right\} + \sum_{l\neq n} 1_{\tilde{x}_j}\left\{x^0(n,j,-(n,j,l))\right\} \\ > \; &((N-2)(N-3)+1) \times 1_{\tilde{x}_j'}\{\hat{x}_j\} + \sum_{i\neq n}\sum_{l\neq i,j} 1_{\tilde{x}_j'}\left\{\hat{x}^0(i,j,-(i,j,l))\right\} + \sum_{l\neq n} 1_{\tilde{x}_j'}\left\{x^0(n,j,-(n,j,l))\right\}, \end{aligned}$$

and for $\tilde{x}_n$ and $\tilde{x}_n' \neq \tilde{x}_n$,

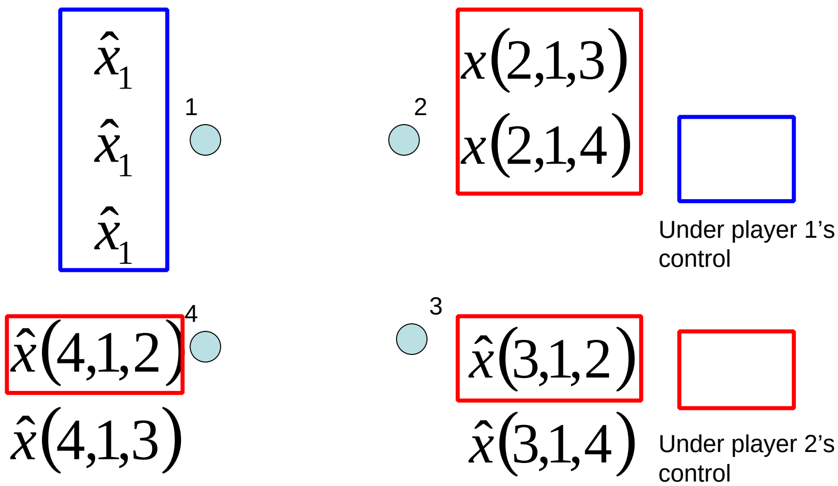
$$\begin{aligned} &((N-2)(N-3)+1) \times 1_{\tilde{x}_n}\{x_n\} + \sum_{i\neq n}\sum_{l\neq i,n} 1_{\tilde{x}_n}\left\{\hat{x}^0(i,n,-(i,n,l))\right\} \\ > \; &((N-2)(N-3)+1) \times 1_{\tilde{x}_n'}\{x_n\} + \sum_{i\neq n}\sum_{l\neq i,n} 1_{\tilde{x}_n'}\left\{\hat{x}^0(i,n,-(i,n,l))\right\}. \end{aligned}$$

Then, for each $\tilde{x} \in X$, let $M_n^0(\tilde{x})$ denote the set of all $m_n^0 \in M_n^0$ such that $\tilde{x}$ is confirmed. Since $((N-2)(N-3)+1) + \#\{i \mid i \neq j\}\#\{l \mid l \neq i,j\}$ is an odd number for any j, $M_n^0(\tilde{x})$ is a partition of $m_n^0 \in M_n^0$.

Consider the confirmation process of $\tilde{x}_j$. The total number of the messages is $((N-2)(N-3)+1) + (N-1)(N-2)$. The number of the messages under player j's control is $(N-2)(N-3)+1$, which is less than the half of the number of the total messages. On the other hand, the number of votes under player $i \neq j$'s control is $(N-2) + (N-2)(N-3)$ since player i can affect $x^0(n,j,-(n,j,l))$ with $n, l \neq i$ by manipulating the message in the monitoring round and $\hat{x}^0(i,j,-(i,j,l))_{l\neq i,j}$ by manipulating the message in the confirmation round. Again, $(N-2)+(N-2)(N-3)$ is less than the half of the number of the total messages. Therefore, any player cannot have an impact on the confirmation if the inference up to the confirmation round is perfect and the others tell the truth.

[2] In this subsection, with perfect cheap talk, $\hat{x} = x$.

See the figure below for the case with $N = 4$, $n = 2$, and $j = 1$. Note that since the number of votes under any single player's control is less than the half of the number of the total messages.



kth Main Round $[KT]$ Players' behavior in the main rounds is determined by the history in the confirmation round and the previous supplemental rounds. Thus, we postpone the explanation.

kth Monitoring Round This round is regarded as a monitoring round. In this subsection, since we assume the existence of cheap talk, this round is instantaneous.

Players form the same $(N-1)$-tuples $(i, j, -(i, j, l))_{l \neq i,j}$ as in the monitoring round for the coordination round.

Player i randomly picks $t_{-(i,j,l)} \in \{1, \ldots, KT\}$. Let $T_\varepsilon = \{t_{-(i,j,l)}, t_{-(i,j,l)}+1, \ldots, t_{-(i,j,l)} + \lfloor \varepsilon KT \rfloor - 1\}$ where we identify τ and $\tau + nKT$ for some $n \in \mathbb{N}$, that is, player i randomly picks one period $t_{-(i,j,l)}$ from KT periods in the kth main round and T_ε is the consecutive

$\lfloor \varepsilon KT \rfloor$ periods from $t_{-(i,j,l)}$. Player i sends a message about $t_{-(i,j,l)}$. Note that if the others know $t_{-(i,j,l)}$, it pins down which periods are included in T_ε.

Players $-(i,j,l)$ report what signals are observed in the periods included in T_ε to player i so that player i can use that information to review player j's action in the kth main round. As mentioned in Subsection 5.1, we can restore the key properties of the conditional independent signals by allowing player i to use $\omega_{-(i,j,l)}$.

Player i with $m_i^0 \in M_i^0(x)$ infers the action of player j in the kth main round by using the messages in the kth monitoring round from players $-(i,j,l)$. If player i infers that player j deviate from $a^{x,k}$ in the kth main round, we say that player i infers player j's deviation according to player $-(i,j,l)$ and let $x^k(i,j,-(i,j,l)) = 1$ denote this situation. Otherwise $x^k(i,j,-(i,j,l)) = 0$. The details of the inference is specified later.

kth Supplemental Round This round is regarded as a communication round. In this subsection, since we assume the existence of cheap talk, this round is instantaneous.

Every player i tells $(x^k(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i}$ to the other players. Then, player n receives the message $((\hat{x}^k(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n}$, where $\hat{x}$ denotes the inference of message x by player n.[3] Let $m_n^k \equiv ((\hat{x}^k(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n} \cup ((x^k(n,j,-(n,j,l))_{l\neq n,j})_{j\neq n}$ be the message profile of player n. Let M_n^k be the set of message profiles m_n^k.

We say that player j's deviation with $j \neq n$ is confirmed for player n if according to the messages m_n^k,

$$\hat{x}^k(i,j,-(i,j,l)) = x^k(n,j,-(n,j,l)) = 1 \text{ for all } i \neq j, n \text{ and } l \neq i, j.$$

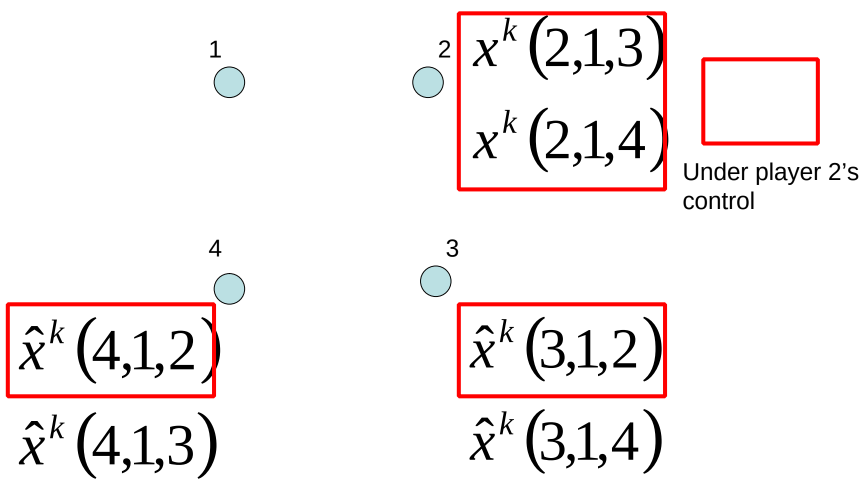
Let $M_n^k(0)$ be the set of m_n^k with which no unilateral deviation is inferred and $M_n^k(i)$ be the set of m_n^k with which player i's deviation is inferred.

Since $(N-2)(N-3)$ messages are under player i's control for any player $i \neq j$ in neither the kth monitoring round nor kth supplemental round, any player cannot have an impact on the review of the players on the equilibrium path if the inference up to the kth supplemental

[3] In this subsection, with perfect cheap talk, $\hat{x} = x$.

round is perfect.

See the figure below for the case with $N = 4$, $n = 2$, and $j = 1$. Note that the message under any single player's control is no more than 4.



kth main round With the explanation of the supplemental round, we can fully specify the action in the kth main round. For player i with $m_i^0 \in M_i^0(x)$, in the first main round, player i takes $a_i^{x,1}$. For $k \geq 2$, if there exists $\tilde{k} \leq k$ such that $(m_i^0, \ldots, m_i^{\tilde{k}-1}) \in M_i^0(x) \times M_i^1(0) \times \cdots \times M_i^{\tilde{k}-1}(0)$ and $m_i^{\tilde{k}} \in M_i^{\tilde{k}}(j) \setminus \bigcup_{l \neq j} M_i^{\tilde{k}}(l)$ with $j \neq i$, then player i takes $\underline{a}_i^j(\mathcal{A})$ and $\overline{a}_i^j(\mathcal{A})$ if $x_{j-1} = B$ and G, respectively. Note that the first condition means no deviation is inferred up to the $\tilde{k}$th round and the second condition means player j's unilateral deviation is inferred at the $\tilde{k}$th round. Otherwise, player i takes $a_i^{x,k}$.

Report round This round is regarded as a communication round. In this subsection, since we assume the existence of cheap talk, this phase is instantaneous. Player i sends a message about $J_i \equiv (M_i^0 \times M_i^1 \times \cdots \times M_i^K)$.

5.4 Optimality of the Block Strategies

In this subsection, we briefly explain why the above two strategies are optimal in all $x \in X$. For notational convenience, fix $\overline{u}$ such that $\max_i \max_a |\pi_i(a)| < \overline{u}$.

Incentive to Tell the Truth in the Coordination Round Since player i's state controls player $i+1$'s payoff, there is no direct incentive to tell a lie by taking an action different from $a_i^{x_i}$. As implied from the discussion in Subsection 5.1, it is verified in the Appendix that it is optimal for players to take a constant action to send the message x_i.

Incentive to Tell the Truth in the Monitoring Round for the Coordination Round and Confirmation Round Player i's expected payoff in the block game is higher when player $i-1$'s state is G than when it is B. Thus, player i has an incentive to tell a lie and try $x_{i-1} = G$ to be confirmed. To prevent this problem, whenever player i's message has an impact for players $-i$'s confirmation of the states, we give $K^2T^2\overline{u}$ and $-K^2T^2\overline{u}$ as the non-averaged expected payoff of player i during the current review phase if player $i-1$'s state is G and B, respectively. Then, player i's message has no impact on player i's payoff. As explained in 5.3, this never occurs with cheap talk and this occurs only with small probability even without perfect cheap talk assumption on the equilibrium. Thus, $\overline{u}$ does not destroy too much value.

Incentive to Take the Prescribed Action in the Main Round Suppose players have to take a constant action in the kth main round. As implied from the discussion in Subsection 5.1, it is verified in the Appendix that it is optimal for players to take a constant action in the kth main round. Pick arbitrary player i.

Firstly, consider the case where there exists x such that $m_n^0 \in M_n^0(x)$ for all $n \neq i$ and $m_n^k \in M_n^k(i)$ if and only if $m_{n'}^k \in M_{n'}^k(i)$ for all $n' \neq i$. In this case, the inference up to the kth supplemental round allows players $-i$ to perfectly coordinate their actions. Note that from (4), since player i's unilateral deviation is inferred with high probability if T is sufficiently large and the other players take $\underline{a}_{-i}^i(\mathcal{A})$ and $\overline{a}_{-i}^i(\mathcal{A})$ when $x_{i-1} = B$ and G, respectively, for any constant action,

$$\begin{aligned}\frac{1}{K}\sum_{k=1}^{K} v_i^k &< \underline{w}_i \text{ if } x_{i-1}=B,\\ \frac{1}{K}\sum_{k=1}^{K} v_i^k &> \overline{w}_i \text{ if } x_{i-1}=G \text{ and } a_i^k \in \mathcal{A}^k,\end{aligned}$$

where v_i^k is the expected value of the time-average of the instantaneous utility in the kth main round. Note that even if $m_i^0 \in M_i^0(x')$ with $x \neq x'$, that is, player i's state x' is different from the other players' state, the above inequality still hold from the following reason. After the first main round, player i's deviation will be inferred with high probability and the other players take $\underline{a}_{-i}^i(\mathcal{A})$ and $\overline{a}_{-i}^i(\mathcal{A})$ when $x_{i-1}=B$ and G, respectively. From (1), above inequalities hold. Therefore, as Ely, Hörner, and Olszewski (2005) show, we can make any constant action prescribed in the regime attain the same value $\underline{w}_i$ if $x_{i-1}=B$ and $\overline{w}_i$ if $x_{i-1}=G$ by properly determining the state profile of the following review phase.

Secondly, consider the case where $m_n^0 \in M_n^0(x)$ and $m_{n'}^0 \in M_{n'}^0(x')$ with $n \neq n'$ and $x \neq x'$ or $m_n^k \in M_n^k(i)$ and $m_{n'}^k \in M_{n'}^k(i')$ with $n \neq n'$ and $i \neq i'$. Note that for any strategy, $-\overline{u} < \frac{1}{K}\sum_{k=1}^{K} v_i^k < \overline{u}$. Thus, in this second case, we give $K^2T^2\overline{u}$ and $-K^2T^2\overline{u}$ as the non-averaged payoff during the the review phase for player i if $x_{i-1}=B$ and G, respectively. Then as Ely, Hörner, and Olszewski (2005) show, we can make any constant action prescribed in the regime attain the same value. Here, the role of the report round is important. When we infer this second case happens based on the report round, we give $K^2T^2\overline{u}$ and $-K^2T^2\overline{u}$ by determining the state profile of the following review phase properly. As we will show later, this case occurs only with small probability even without perfect cheap talk, $\overline{u}$ does not destroy too much value.

Incentive to Tell the Truth in the kth Monitoring Round and Supplemental Round Suppose player i's deviation is inferred. Then, the other players punish her by taking $\underline{a}_{-i}^i(\mathcal{A})$ when $x_{i-1}=B$ and reward her by taking $\overline{a}_{-i}^i(\mathcal{A})$ when $x_{i-1}=G$. Since (1) does not impose any restriction on the value of player $-i$ when players $-i$ punish or reward player i. Thus, it may be the case that

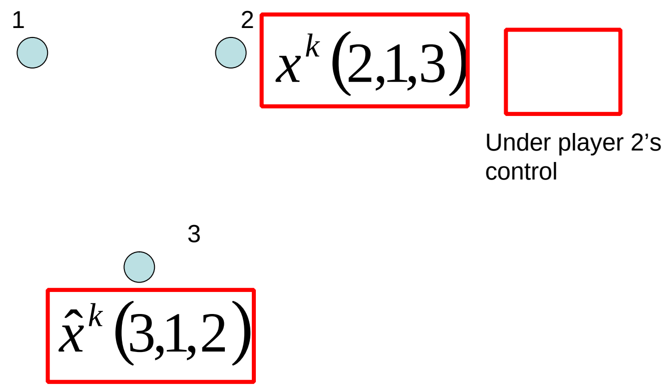
$$\frac{1}{K}\sum_{k=1}^{K} v_j^k > \underline{w}_j \text{ even if } x_{j-1} = B,$$
$$\frac{1}{K}\sum_{k=1}^{K} v_j^k < \overline{w}_j \text{ even if } x_{i-1} = G,$$

where v_j^k is the time average of the instantaneous utility in the kth main round when players $-i$ punish or reward player i. Then, it is impossible to make players $-i$'s belief irrelevant. To overcome this difficulty, whenever player $i \neq j$'s deviation is inferred, we give $K^2T^2\overline{u}$ and $-K^2T^2\overline{u}$ as the non-averaged expected payoff of player j during the current review phase if $x_{j-1} = B$ and G, respectively. Then, as Ely, Hörner, and Olszewski (2005) show, we can make players $-i$'s belief irrelevant.

However, this makes the incentive to tell a lie when player j sends the message used for the review of player i since if player $j-1$'s state is B, it is beneficial for player j that player i's deviation is inferred, and if player $j-1$'s state is G, it is costly for player j that player i's deviation is inferred.

Therefore, whenever player j's message has an impact on the inference of player i's deviation, we give $K^2T^2\overline{u}$ and $-K^2T^2\overline{u}$ as the non-averaged expected payoff of player j during the current review phase if player $j-1$'s state is G and B, respectively. Then, player j's message has no impact on player j's payoff. As explained in 5.3, this never occurs with cheap talk and this occurs only with small probability even without perfect cheap talk on the equilibrium. Thus, $\overline{u}$ does not destroy too much value.

Note that with $N = 3$, since player j's message always has an impact, it is impossible to give an incentive to tell the truth with our equilibrium construction. This is why we have the assumption that $N \geq 4$.



Incentive to Tell the Truth in the Report Round For the report round, player i's message in the report round is used to control players $-i$'s utility. Therefore, player i is indifferent for any message.

5.5 Modification without Cheap Talk

There is one remaining difficulty. Since we do not assume explicit communication, player i has to send message about x^0, x^k, m_i^0, m_i^k, ω_i, and t to the other players by taking actions during infinitely repeated games. Note that the message that player i has to send is $\{G, B\}$, $\{0, 1\}$, and $\omega_{i,t}$, and $t \in \{1, \ldots, KT\}$.

To send a message $x_i = G$ or 0, player i can make almost perfect message by taking a_i^G for long time. To send a message $x_i = B$ or 1, player i can make almost perfect message by taking a_i^B for long time.

To send a message $\omega_{i,t} = \omega_i$, consider the following procedure. Let us attach the sequence of actions $a_i(\omega_i) = (a_{i,1}(\omega_i), \ldots, a_{i,T_i}(\omega_i))$ to each $\omega_i \in \Omega_i$ with $a_{i,t}(\omega_i) \in \{a_i^G, a_i^B\}$ for all $i \in I$, $\omega_i \in \Omega_i$, and t. $a_{i,1}(\omega_i) = a_i^G$ if $\omega_i \in \{\omega_{i,1}, \ldots, \omega_{i,\lfloor|\Omega_i|/2\rfloor}\}$ and $a_{i,1}(\omega_i) = a_i^B$ otherwise. For ω_i with $a_{i,1}(\omega_i) = a_i^G$, $a_{i,2}(\omega_i) = a_i^G$ if $\omega_i \in \{\omega_{i,1}, \ldots, \omega_{i,\lfloor\lfloor|\Omega_i|/2\rfloor/2\rfloor}\}$ and $a_{i,2}(\omega_i) = a_i^B$ otherwise. Likewise, for ω_i with $a_{i,1}(\omega_i) = a_i^B$, $a_{i,2}(\omega_i) = a_i^G$ if $\omega_i \in \{\omega_{i,\lfloor|\Omega_i|/2\rfloor+1}, \ldots, \omega_{i,\lfloor(|\Omega_i|-\lfloor|\Omega_i|/2\rfloor)/2\rfloor}\}$ and $a_{i,2}(\omega_i) = a_i^B$ otherwise. Keep this procedure until we can identify ω_i uniquely from $a_i(\omega_i)$. Without loss of generality, we can assume $T_i = T_\omega$ for all i. To send a message $\omega_{i,t} = \omega_i$, player i takes $a_{i,1}(\omega_i)$ for sufficiently long time, and then takes $a_{i,2}(\omega_i)$ for sufficiently long time, and so forth. Note that T_ω is fixed and $a_{i,t}(\omega_i) \in \{a_i^G, a_i^B\}$.

For send a message about $t \in \{1, \ldots, KT\}$, consider the following procedure. Let us attach the sequence of actions $a_i(t) = (a_{i,1}(t), \ldots, a_{i,\lfloor\log_2 KT\rfloor+1}(t))$ to each $t \in \{1, \ldots, KT\}$ with $a_{i,\tau}(t) \in \{a_i^G, a_i^B\}$ for all $\tau \in \{1, \ldots, \lfloor\log_2 KT\rfloor + 1\}$. $a_{i,1}(t) = a_i^G$ if $t \in \{1, \ldots, \lfloor KT/2\rfloor\}$ and $a_{i,1}(t) = a_i^B$ otherwise. For t with $a_{i,1}(t) = a_i^G$, $a_{i,2}(t) = a_i^G$ if $t \in \{1, \ldots, \lfloor\lfloor KT/2\rfloor/2\rfloor\}$ and $a_{i,2}(t) = a_i^B$ otherwise. Likewise, for t with $a_{i,1}(t) = a_i^B$, $a_{i,2}(t) = a_i^G$ if $\omega_i \in \{\lfloor KT/2\rfloor, \ldots, \lfloor(KT - \lfloor KT/2\rfloor)/2\rfloor\}$ and $a_{i,2}(t) = a_i^B$ otherwise. Keep this procedure until we can identify t uniquely from $a_i(t)$. Note that we can make sure that $|a_i(t)| \leq \lfloor\log_2 KT\rfloor + 1$.

Note that if ε is sufficiently small, since we construct the equilibrium where players send the message about $\omega_{i,t}$ only ε fraction of KT, this does not destroy efficiency. In addition, since every period is chosen randomly and symmetrically, this does not destroy the incentive to take a constant action. If players are sufficiently patient, only small probability of being reviewed is enough to give an enough incentive not to deviate. Since $(\lfloor\log_2(KT)\rfloor + 1)/T^{\frac{1}{k}} \to 0$ for any $k \in \mathbb{N}$ as $T \to \infty$, the communication about $t_{-(i,j,l)}$ does not destroy efficiency if T is sufficiently large.

6 Formal Description of the Strategy without Cheap Talk

6.1 Random Events and Stochastic Characteristics

For the monitoring, confirmation, and supplemental rounds, we want to create the random variable to infer the message so that when player i sends a message, players $-i$ cannot manipulate player i's message. Non-manipulability is important since otherwise a reviewed player has an incentive to mix up the message.

Lemma 5 *Suppose that monitoring satisfies the identifiability condition. Then, there exist* $0 < p_1 < p_2 < p_3 < 1$ *such that for all* $i, l \in I$*,* $a_i \in A_i$*, and* $\mathcal{A}_l \subset A_l$*, there exist vectors* $\psi_i^{2,1}\left(\left\{a_l^G\right\} \mid a_i\right) : \Omega_i \to [0,1]$ *and* $\psi_i^{3,2}\left(\mathcal{A}_l \mid a_i\right) : \Omega_i \to [0,1]$ *such that*[4]

$$Q_i\left(\left\{a_l^G\right\} \mid a_i\right)\psi_i^{2,1}\left(\left\{a_l^G\right\} \mid a_i\right) = \begin{bmatrix} p_2\mathbf{1}_{\left|A_{-(i,l)}\right|} \\ p_1\mathbf{1}_{(|A_l|-1)\left|A_{-(i,l)}\right|} \end{bmatrix},$$

$$Q_i\left(\mathcal{A}_l \mid a_i\right)\psi_i^{3,2}\left(\mathcal{A}_l \mid a_i\right) = \begin{bmatrix} q_3\mathbf{1}_{\left|A_{-(i,l)}\right||\mathcal{A}_l|} \\ q_2\mathbf{1}_{\left|A_{-(i,l)}\right|(|A_l|-|\mathcal{A}_l|)} \end{bmatrix}.$$

Proof. Analogous to **Lemma 1** in Yamamoto (2008b). ■

Let $\psi_i^{2,1}\left(\left\{a_l^G\right\}\right)$ be the random variable with following features. If player i plays a_i and observes ω_i, player i draws a random variable from uniform distribution on $[0,1]$. If the random variable is no greater than $\psi_i^{2,1}\left(\left\{a_l^G\right\} \mid a_i\right)(\omega_i)$, the realization of random variable is $\psi_i^{2,1}\left(\left\{a_l^G\right\}\right) = 1$ and 0 otherwise. $\psi_i^{3,2}\left(\mathcal{A}_l\right)$ is analogously defined.

The way to send a message in the monitoring round is specified as follows. Let $\tilde{S}(T)$ be the function from $\mathbb{N}$ to $\mathbb{N}$. When player i wants to send the message about $t_{-(i,j,l)}$, she takes $a_{i,1}\left(t_{-(i,j,l)}\right)$ for $\tilde{S}(T)$ periods, then $a_{i,2}\left(t_{-(i,j,l)}\right)$ for $\tilde{S}(T)$ periods, and so forth. Therefore, to send a message about $t_{-(i,j,l)}$, it takes $(\lfloor\log_2 KT\rfloor + 1)\,\tilde{S}(T)$ periods. Remember

[4] In this paper, we identify functions as vectors whose elements correspond to the value of the functions if it causes no confusion.

$a_i(t) \in \{a_i^G, a_i^B\}$. Player n infers that player i sends the message a_i^G if $\psi_n^{2,1}(\{a_i^G\})$ occurs $\frac{p_1+p_2}{2}\tilde{S}(T)$ times or more during a $\tilde{S}(T)$-period interval. Otherwise, she infers player i sends a_i^B. Let $\Pr(t_{-(i,j,l)} = \mathbf{t} \mid t)$ represent the probability that each player $n \in (i,j,l)$ infers player i sends message t_n when player i tries to send a message t. Here, $\mathbf{t}$ is the vector consists of $(t_n)_{n\in(i,j,l)}$.

When player n wants to send the message ω_n with $a_n(\omega_n)$, she takes $a_{n,1}(\omega_n)$ for S periods, then $a_{n,2}(\omega_n)$ for S periods, and so forth. Remember $a_n(\omega_n) \in \{a_n^G, a_n^B\}$. Player i infers that player n sends the message a_n^G if $\psi_i^{2,1}(\{a_n^G\})$ occurs $\frac{p_1+p_2}{2}S$ times or more during a S-period interval. Otherwise, she infers player n sends a_n^B. Let $\Pr(\omega'_{-(i,j,l)} \mid \omega_{-(i,j,l)})$ represent the probability that player i infers players $-(i,j,l)$ send the message $\omega'_{-(i,j,l)}$ to i when $-(i,j,l)$ try to send message $\omega_{-(i,j,l)}$.

As explained in the basic idea, we want to create the random variable for $(i,j,-(i,j,l))$ to review the deviation of player j so that neither player j's signal nor player l's signal has any information about the counting.

Lemma 6 *Suppose that monitoring satisfies identifiability condition and $(N-2)$-identifiability condition. Suppose $\tilde{S}(T)$ satisfies*

$$o(\tilde{S}(T)^L)(\lfloor \log_2 KT \rfloor + 1) = o(T^2). \tag{5}$$

for some $L \geq 1$. Then, there exists S such that there exist $0 < q_1 < q_2 < q_3 < 1$ such that there exists $\overline{T}$ such that for all $T \geq \overline{T}$, for all $i,j,l \in I$, $a_i \in A_i$ and $\mathcal{A}_j \subset A_j$, there exist $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i) : \Omega_i \times \Omega_{-(i,j,l)} \to [0,1]$, $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j \mid a_i) : \Omega_i \times \Omega_{-(i,j,l)} \to [0,1]$, $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i, t) : \Omega_i \times \Omega_{-(i,j,l)} \to [0,1]$, and $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j \mid a_i, t) : \Omega_i \times \Omega_{-(i,j,l)} \to [0,1]$ such that

$$\hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i)\,\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i) = q_{3,2} \equiv \begin{bmatrix} q_3 \mathbf{1}_{|\Omega_j||A_{-(i,j)}||\mathcal{A}_j|} \\ q_2 \mathbf{1}_{|\Omega_j||A_{-(i,j)}|(|A_j|-|\mathcal{A}_j|)} \\ q_3 \mathbf{1}_{|\Omega_l||A_{-(i,j)}||\mathcal{A}_j|} \\ q_2 \mathbf{1}_{|\Omega_l||A_{-(i,j)}|(|A_j|-|\mathcal{A}_j|)} \end{bmatrix},$$

$$\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right) = q_{2,1} \equiv \begin{bmatrix} q_2 \mathbf{1}_{|\Omega_j|\left|A_{-(i,j)}\right||\mathcal{A}_j|} \\ q_1 \mathbf{1}_{|\Omega_j|\left|A_{-(i,j)}\right|(|A_j|-|\mathcal{A}_j|)} \\ q_2 \mathbf{1}_{|\Omega_l|\left|A_{-(i,j)}\right||\mathcal{A}_j|} \\ q_1 \mathbf{1}_{|\Omega_l|\left|A_{-(i,j)}\right|(|A_j|-|\mathcal{A}_j|)} \end{bmatrix},$$

$$\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right)\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = q_{3,2} \textit{ for all } t \in \{1, \ldots, KT\},$$

$$\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right)\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = q_{2,1} \textit{ for all } t \in \{1, \ldots, KT\},$$

where $\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ *is the matrix with each element* $q\left(\omega_{-(j,l)} \mid a_i a_{-i} \omega_j\right)$ *of* $Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ *replaced with*

$$\hat{q}\left(\omega_{-(j,l)} \mid a_i a_{-i} \omega_j\right) = \sum_{\omega'_{-(i,j,l)}} \Pr(\omega_{-(i,j,l)} \mid \omega'_{-(i,j,l)}) q(\omega_i, \omega'_{-(i,j,l)} \mid a_i a_{-i} \omega_j),$$

and $\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right)$ *is the matrix with each element* $q\left(\omega_{-(j,l)} \mid a_i a_{-i} \omega_j\right)$ *of* $Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ *replaced with*

$$\begin{aligned} & \hat{q}\left(\omega_{-(j,l)} \mid a_i a_{-i} \omega_j\right) \\ = \; & \Pr\left(t\mathbf{1} \mid t\right) \sum_{\omega'_{-(i,j,l)}} \Pr(\omega_{-(i,j,l)} \mid \omega'_{-(i,j,l)}) q(\omega_i, \omega'_{-(i,j,l)} \mid a_i a_{-i} \omega_j) \\ & + \sum_{\mathbf{t} \neq t\mathbf{1}} \Pr\left(\mathbf{t} \mid t\right) \left(\sum_{\omega'_{-(i,j,l)}} \Pr(\omega_{-(i,j,l)} \mid \omega'_{-(i,j,l)}) q((\omega'_n)_{n \in I(t)} \mid a_i a_{-i} \omega_j) q((\omega'_n)_{n \notin I(t)} \mid a_i a_{-i}) \right), \end{aligned}$$

where $I\left(t\right) = \{n \in -\left(i, j, l\right) \mid t_n = t\}$.

Note that the element of $\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ corresponds to $Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ but we take the probability of miscommunication about $\omega_{-(i,j,l)}$ into account. Note that the element of $\hat{Q}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right)$ corresponds to $Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ but we take the probability of miscommunication about t and $\omega_{-(i,j,l)}$ into account. Intuitively, $(N-2)$-identifiability condition guarantees the existence for $Q_{-(i,l)}$. Then, since the probability of the error in communication is sufficiently small if S and T are sufficiently large from the assumption $o(\tilde{S}\left(T\right)^L)(\lfloor \log_2 KT \rfloor +$

$1) = o(T^2)$, the existence is guaranteed for $\hat{Q}_{-(i,l)}$. For the rest of the paper, we fix S such that **Lemma 6** is satisfied.

Let $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, \varepsilon)$ be the random variable with following features. Player i randomly selects $t_{-(i,j,l)}$, and so T_ε. Assume player i can make $t_{-(i,j,l)}$ public. Players $-(i,j,l)$ sends the message about $\omega_{-(i,j,l)}$ in the periods included in T_ε. For each $t \in T_\varepsilon$, if player i plays $a_{i,t}$, observes $\omega_{i,t}$, and gets the message $\omega_{-(i,j,l)}$ from players $-(i,j,l)$, player i draws a random variable from uniform distribution on $[0,1]$. If the random variable is no greater than $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i)(\omega_{i,t}, \omega_{-(i,j,l)})$, the realization of random variable is $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, \varepsilon) = 1$ and 0 otherwise. For $t \notin T_\varepsilon$, $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, \varepsilon) = 0$. Let $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j, \varepsilon)$ be analogously defined by replacing $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i)$ by $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j \mid a_i)$. Note that for $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, \varepsilon)$ and $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j, \varepsilon)$, the periods used for the review are common knowledge.

Let $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon)$ with $k \in \{1, \ldots, K\}$ and $t \in \{1, \ldots, KT\}$ be the random variable with following features. Player i randomly select T_ε. Player i sends the message about $t_{-(i,j,l)}$ by taking actions. For each $t \in T_\varepsilon$, if player i plays $a_{i,t}$, observes $\omega_{i,t}$, and gets the message $\omega_{-(i,j,l)}$ from players $-(i,j,l)$, player i draws a random variable from uniform distribution on $[0,1]$. If the random variable is no greater than $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i, t)(\omega_{i,t}, \omega_{-(i,j,l)})$, the realization of random variable is $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon) = 1$ and 0 otherwise. For $t \notin T_\varepsilon$, $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon) = 0$. Let $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon)$ be analogously defined by replacing $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i, t)$ by $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j \mid a_i, t)$. Note that for $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon)$ and $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j, t, \varepsilon)$, the periods used for the review are not common knowledge.

Let $\varepsilon\psi^{3,2}_{-(i,l)}(\mathcal{A}_j)$ be the random variable with following features. If player i plays a_i, observes ω_i, and gets the message $\omega_{-(i,j,l)}$ from players $-(i,j,l)$, player i draws a random variable from uniform distribution on $[0,1]$. If the random variable is no greater than $\varepsilon\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i)(\omega_i, \omega_{-(i,j,l)})$, the realization of random variable is $\varepsilon\psi^{3,2}_{-(i,l)}(\mathcal{A}_j) = 1$ and 0 otherwise. Let $\varepsilon\psi^{2,1}_{-(i,l)}(\mathcal{A}_j)$ be analogously defined by replacing $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j \mid a_i)$ by $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j \mid a_i)$.

From now on, we assume ε is a rational number and $\varepsilon KT \in \mathbb{N}$. Then, the probability that $\psi^{3,2}_{-(i,l)}(\mathcal{A}_j, \varepsilon)$ occurs r times during a T-period interval when player j chooses $a_j \notin \mathcal{A}_j$ for the first τ period and then $a_j \in \mathcal{A}_i$ for the rest of the periods is the same as $\varepsilon\psi^{3,2}_{-(i,l)}(\mathcal{A}_j)$ and independent of (i,j,l). Therefore, let $\tilde{F}^{3,2}(\tau,T,r)$ represent this probability. Let $\tilde{F}^{2,1}(\tau,T,r)$ be analogously defined for $\psi^{2,1}_{-(i,l)}(\mathcal{A}_j,\varepsilon)$ and $\varepsilon\psi^{2,1}_{-(i,l)}(\mathcal{A}_j)$.

Let $F^{3,2}_{-(i,j,l)}\left(\mathcal{A}_j, KT, r \mid a^{KT}\right)$ and $F^{2,1}_{-(i,j,l)}\left(\mathcal{A}_j, KT, r \mid a^{KT}\right)$ represent the probability that $\sum_{t=1}^{KT}\psi^{3,2}_{-(i,l)}(\mathcal{A}_j,t,\varepsilon) = r$ and $\sum_{t=1}^{KT}\psi^{2,1}_{-(i,l)}(\mathcal{A}_j,t,\varepsilon) = r$, respectively, when players play $a^{KT} = (a_1,\ldots,a_{KT})$.

Lemma 7 *For all $\varepsilon > 0$ and K, there exists $\{Z_T\}_{T=1}^{\infty}$, $\{Z'_T\}_{T=1}^{\infty}$, and $\{Z''_T\}_{T=1}^{\infty}$ such that,*

$$Z_T \leq \varepsilon q_3 T,$$

$$Z''_T < \varepsilon q_2 T < Z'_T$$

$$\lim_{T\to\infty}\sum_{r=Z''_T+1}^{Z'_T}\tilde{F}^{3,2}(T,T,r) = \lim_{T\to\infty}\sum_{r=Z''_T+1}^{Z'_T}\tilde{F}^{2,1}(0,T,r) = \lim_{T\to\infty}\sum_{r>Z_T}\tilde{F}^{3,2}(0,T,r) = 1,$$

$$\lim_{T\to\infty} T\tilde{F}^{3,2}(0,T-1,Z_T) = \infty,$$

$$\lim_{T\to\infty}\left|\frac{Z''_T}{T} - \varepsilon q_2\right| = \lim_{T\to\infty}\left|\frac{Z'_T}{T} - \varepsilon q_2\right| = \lim_{T\to\infty}\left|\frac{Z_T}{T} - \varepsilon q_3\right| = 0.$$

Proof. Analogous to Yamamoto (2008b). ■

6.2 Strategy without Cheap Talk

This section re-specifies two strategies in the block game, $\sigma_i^G \in \Sigma_{i,T_b}$ and $\sigma_i^B \in \Sigma_{i,T_b}$ without cheap talk. In what follows, we explain the behavior of the players under $\sigma^x = (\sigma_i^{x_i})_{i\in I}$. Let S be a fixed integer such that **Lemma 6** is satisfied. In addition, take ε, K, T as given in this subsection so that $\varepsilon KT \in \mathbb{N}$ and **Lemma 6** is satisfied. The existence of $\tilde{S}(T)$ and L such that $o(\tilde{S}(T)^L)(\lfloor\log_2 KT\rfloor + 1) = o(T^2)$ is shown in the Appendix.

Coordination Round $[NT]$ The behavior in this round is the same as without cheap talk.

Monitoring Round for the Coordination Round $[N(N-1)(N-2)(N-3)T_\omega ST]$ The behavior in this round should be modified without cheap talk.

Players form $(N-1)$-tuples $(i, j, -(i, j, l))_{l \neq i,j}$ as before. Each player included in $-(i, j, l)$ reports what signals were observed in the periods in the coordination round to player i so that player i can use that information to review player j's action in the coordination round.

$(N-1)$-tuples exchange their messages alternately here. $(N-3) T_\omega ST$ periods are assigned for each $(N-1)$-tuple. When players $n \in -(i, j, l)$ sends her messages to player i, player n firstly sends the message $\omega_{n,1}$ for ST_ω periods as explained in Subsection 6.1, where $\omega_{n,1}$ being the signal of the first period of the jth coordination round. Then, player n sends a message $\omega_{n,2}$ for ST_ω periods with $\omega_{n,2}$ being the signal of the second period of the jth coordination round, and so forth. When the other players $-n$ send a message, player n takes a_n^G. Note that this round lasts for $N(N-1)(N-2)(N-3) T_\omega ST$ periods since $|-(i, j, l)| = N-3$ for each $(N-1)$-tuple and there exists $N(N-1)(N-2)$ $(N-1)$-tuples.

Inference in the Monitoring Round for the Coordination Round Player i infers the message in the coordination round from player j by using the messages from players $-(i, j, l)$ in the monitoring round for the coordination round. Player i infers that player j's message is $x_j = G$ by using the messages from players $-(i, j, l)$ if $\varepsilon\psi^{3,2}_{-(i,l)}(\{a_j^G\})$ occurs more than $\left\lfloor \frac{q_2+2q_3}{3}\varepsilon T \right\rfloor$ times. Player i infers that player j's message is $x_j = B$ otherwise. Let $x^0(i, j, -(i, j, l)) = x_j$ denote the situation where player i infers that player j's message is x_j by using the messages from players $-(i, j, l)$.

Confirmation Round $[(N + N(N-1)(N-2)(N-3)) T]$ The behavior in this round should be modified without cheap talk.

Every player i tells $(x_i, (x^0(i, j, -(i, j, l))_{l \neq i,j})_{j \neq i})$ to the other players. Note that the number of the elements of this vector is $1 + (N-1)(N-2)(N-3)$.

Players exchange their messages alternately here. $(1 + (N-1)(N-2)(N-3)) T$ periods are assigned for each player. When player i wants to send the message x_i, player i takes $a_i^{x_i}$ for T periods. Similarly, when player i wants to send the message $x^0(i, j, -(i, j, l)) = x_j$, player i takes $a_i^{x_j}$ for T periods. As before, when players $-i$ send their messages, player i takes a_i^G. Note that this round lasts for $(N + N(N-1)(N-2)(N-3)) T$ periods.

Inference in the Confirmation Round Player n receives the message $(\hat{x}_i, (\hat{x}^0(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n}$ where $\hat{x}$ denotes the inference of message x by player n. Player n infers $\hat{x}_i = G$ and $\hat{x}^0(i,j,-(i,j,l)) = G$ if $\varepsilon\psi_n^{3,2}(\{a_i^G\})$ occurs more than $\lfloor \frac{q_2+2q_3}{3}\varepsilon T \rfloor$ times during the periods when player i sends a message about x_i and $x^0(i,j,-(i,j,l))$, respectively. Player n infers $\hat{x}_i = B$ and $\hat{x}^0(i,j,-(i,j.l)) = B$ otherwise. The definitions of m_n^0 and $M_n^0(x)$ are the same as before.

k**th Main Round** $[KT]$ The behavior in this round is the same as without cheap talk.

k**th Monitoring Round** $[N(N-1)(N-2)((\lfloor \log_2 T \rfloor + 1)\tilde{S}(T)+(N-3)ST_\omega\varepsilon KT)]$ The behavior in this round should be modified without cheap talk.

Players form the same $(N-1)$-tuples $(i,j,-(i,j,l))_{l\neq i,j}$ as before. Player i randomly select $T_\varepsilon = \{t_{-(i,j,l)}, \ldots, t_{-(i,j,l)} + \varepsilon KT - 1\}$ and sends the message about $t_{-(i,j,l)}$. Each player included in $-(i,j,l)$ reports what signals were observed in the periods included in T_ε so that player i can use that information to review player j's action in the kth main round.

$(N-1)$-tuples exchange their messages alternately here. $(\lfloor \log_2 T \rfloor + 1)\tilde{S}(T)+(N-3)ST_\omega\varepsilon KT$ periods are assigned for each $(N-1)$-tuple. Firstly, player i sends the message about $t_{-(i,j,l)}$ for $(\lfloor \log_2 T \rfloor + 1)\tilde{S}(T)$ periods as explained in Subsection 6.1. Each player $n \in -(i,j,l)$ infers $t_{-(i,j,l)}$ as explained in Subsection 6.1. When players $n \in -(i,j,l)$ sends her messages to player i, player n firstly sends a message $\omega_{n,1}$ for ST_ω periods as explained in Subsection 6.1, where $\omega_{n,1}$ being the signal of the $\hat{t}_{-(i,j,l)}$th period in the main round, where $\hat{t}_{-(i,j,l)}$ is the inference of $t_{-(i,j,l)}$by player n. Then, player n sends a message $\omega_{n,2}$ for ST_ω periods with $\omega_{n,2}$ being the signal of the $(\hat{t}_{-(i,j,l)} + 1)$th period in the main round, and so forth. When the other players $-n$ send a message, player n takes a_n^G. Note that this round lasts for $N(N-1)(N-2)((\lfloor \log_2 T \rfloor + 1)\tilde{S}(T) + (N-3)ST_\omega\varepsilon KT)$ periods since there are $N(N-1)(N-2)$ $(N-1)$-tuples, and for each $(N-1)$-tuple $(i,j-(i,j,l))$, player i sends a message about $t_{-(i,j,l)}$ for $(\lfloor \log_2 T \rfloor + 1)\tilde{S}(T)$ periods and each $n \in -(i,j,l)$ sends a message for $ST_\omega\varepsilon KT$ periods .

Inference in the kth Monitoring Round Player i infers player j's action in the kth main round by using the message in the kth monitoring round from players $-(i,j,l)$. Player i with $m_i^0 \in M_i^0(x)$ infers that player j deviated from the prescribed action $a^{x,k}$ by using the messages from player $-(i,j,l)$ if $\sum_{t=1}^{KT} \psi_{-(i,l)}^{2,1}\left(\{a_j^{x,k}\}, t, \varepsilon\right) > Z''_{KT}$ and let $x^k(i,j,-(i,j,l)) = 1$ denote this situation. Otherwise, player i infers that player j did not deviate and let $x^k(i,j,-(i,j,l)) = 0$ denote this situation.

kth Supplemental Round $[N(N-1)(N-2)T]$ The behavior in this round should be modified without cheap talk.

Every player i tells $(x^k(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i}$ to the other players. Note that the number of the elements of this vector is $(N-1)(N-2)$.

Players exchange their messages alternately here. $(N-1)(N-2)T$ periods are assigned for each player. When player i wants to send the message $x^k(i,j,-(i,j,l)) = 0$, player i takes a_i^G for T periods. Similarly, when player i wants to send the message $x^k(i,j,-(i,j.l)) = 1$, player i takes a_i^B for T periods. Note that this round lasts for $N(N-1)(N-2)T$ periods.

Inference in the Supplemental Round Player n receives the message $((\hat{x}^k(i,j,-(i,j,l))_{l\neq i,j})_{j\neq i})_{i\neq n}$, where $\hat{x}$ denotes the inference of message x by player n. Player n infers $\hat{x}^k(i,j,-(i,j,l)) = 0$ if $\varepsilon\psi_n^{3,2}(\{a_i^G\})$ occurs more than $\left\lfloor \frac{q_2+2q_3}{3}\varepsilon T \right\rfloor$ times during the periods when player i sends a message about $x^k(i,j,-(i,j,l))$. Player n infers $\hat{x}^k(i,j,-(i,j,l)) = 1$ otherwise.

The definitions of m_n^k, $M_n^k(0)$, and $M_n^k(i)$ are the same as before.

Report Round $[(N^2 + N^2(N-1)(N-2)(N-3))T + N^2(N-1)(N-2)KT]$ Player i sends the message about $J_i \equiv (M_i^0 \times M_i^1 \times \cdots \times M_i^K)$. In this round, players announce their message alternately and when players $-i$ send their messages, player i takes a_i^G. The way to send the message and the inference are the same as the confirmation round and supplemental round. Therefore, report round lasts for $(N^2 + N^2(N-1)(N-2)(N-3))T + N^2(N-1)(N-2)KT$ periods.

Determinant of T_b For notational convenience, we define

$$\begin{aligned}
T_{-1} &= NT + N(N-1)(N-2)(N-3)T_\omega ST, \\
T_0 &= (N + N(N-1)(N-2)(N-3))T, \\
T_k &= KT + N(N-1)(N-2)((\lfloor \log_2 T \rfloor + 1)\tilde{S}(T) + (N-3)ST_\omega \varepsilon KT) + N(N-1)(N-2)T, \\
T_r &= \left(N^2 + N^2(N-1)(N-2)(N-3)\right)T + N^2(N-1)(N-2)KT.
\end{aligned}$$

In addition,

$$T_m = N(N-1)(N-2)((\lfloor \log_2 T \rfloor + 1)\tilde{S}(T) + (N-3)ST_\omega \varepsilon KT.$$

Note that this is the length of each kth monitoring round.

From the above discussion, $T_b = T_{-1} + T_0 + \sum_k T_k + T_r$. For notational simplicity, let

$$\Lambda \equiv \frac{T_{-1} + T_0 + N(N-1)(N-2)KT + T_r}{T} = o\left(K^2\right).$$

Note that ΛT is the length of the rounds except for the main round and the monitoring rounds for the main rounds.

7 Equilibrium Construction without Cheap Talk

7.1 Transfers

Before the analysis of infinitely repeated games, it is useful to consider the following T_b-period repeated game with transfers. The objective of this subsection is to show the existence of the transfers satisfying the similar conditions with **Lemmas 6** and **7** in Yamamoto (2008b). Suppose that player i receives a transfer $U_i : H_{i-1,T_b+1} \to \mathbb{R}$ after T_b-period block game. Let $w_i^A(\sigma_{T_b}, U_i)$ denote player i's average payoff in this *auxiliary scenario* when the players perform a block strategy profiles $\sigma_{T_b} \in \Sigma_{T_b}$, that is,

$$w_i^A\left(\sigma_{T_b}, U_i\right) \equiv \frac{1-\delta}{1-\delta^{T_b}}\left[\sum_{t=1}^{T_b} \delta^{t-1} E\left[\pi_i\left(a_t\right) \mid \sigma_{T_b}\right]+\delta^{T_b} E\left[U_i\left(h_{i-1, T_b+1}\right) \mid \sigma_{T_b}\right]\right].$$

Let $\sigma_{i,T_b} \mid h_{i,t}$ denote player i's continuation strategy after history $h_{i,t} \in H_{i,t}$ induced by $\sigma_{i,T_b} \in \Sigma_{i,T_b}$. Also, let $BR^A\left(\sigma_{-i,T_b} \mid h_{-i,t}, U_i\right)$ be the set of player i's best replies in the auxiliary scenario continuation game from period t on, given that opponents play $\sigma_{-i,T_b} \in \Sigma_{-i,T_b}$ in the block game and that their past history is $h_{-i,t} \in H_{-i,t}$.

Let U_i^B and U_i^G be as in **Lemmas 8** and **9**.

Lemma 8 *Suppose that the monitoring satisfies the full support condition, identifiability condition, and $(N-2)$-identifiability condition. Then, there exist K and $\bar{T}$ such that for all $T > \bar{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(\overline{\delta}, 1\right)$ and for all $i \in I$, there exists $U_i^B : H_{i-1,T_b} \to \mathbb{R}$ such that for all $l \geq 0$, $h_{t_l} \in H_{t_l}$, $h_{i-1,T_b+1} \in H_{i-1,T_b+1}$, and $x \in X$ with $x_{i-1} = B$,*

$$\sigma_i^{x_i} \mid h_{i,t_l} \in BR^A\left(\sigma_{-i}^{x_{-i}} \mid h_{-i,t_l}, U_i^B\right), \tag{6}$$

$$\sigma_i^{x_i} \mid h_{i,t} \in BR^A(s_{-i} \mid h_{-i,t}, a_{-i}) \text{ for all } h_{i,t} \in H_{i,t} \text{ and } a_{-i} \in \operatorname{supp}\left(\sigma_{-i}^{x_{-i}} \mid h_{i,t}\right)$$
$$\text{for } t \in (t_{l-1}, \ldots, t_l - 1) \text{ with } t_l \in T_{bf} \tag{7}$$

$$w_i^A\left(\sigma^x, U_i^B\right) = \underline{w}_i, \tag{8}$$

$$0 \leq U_i^B\left(h_{i-1,T_b}\right) < \tfrac{\overline{w}_i - \underline{w}_i}{1-\delta}. \tag{9}$$

Lemma 9 *Suppose that the monitoring satisfies the full support condition, identifiability condition, and $(N-2)$-identifiability condition. Then, there exist K and $\bar{T}$ such that for all $T > \bar{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(\overline{\delta}, 1\right)$ and for all $i \in I$, there exists $U_i^G : H_{i-1,T_b} \to \mathbb{R}$ such that for all $l \geq 0$, $h_{t_l} \in H_{t_l}$, $h_{i-1,T_b+1} \in H_{i-1,T_b+1}$, and $x \in X$ with*

$x_{i-1} = G$,

$$\sigma_i^{x_i} \mid h_{i,t_l} \in BR^A \left(\sigma_{-i}^{x_{-i}} \mid h_{-i,t_l}, U_i^G\right), \tag{10}$$

$$\sigma_i^{x_i} \mid h_{i,t} \in BR(s_{-i} \mid h_{-i,t}, a_{-i}) \textit{ for all } h_{i,t} \in H_{i,t} \textit{ and } a_{-i} \in \text{supp}\left(\sigma_{-i}^{x_{-i}} \mid h_{i,t}\right)$$

$$\textit{for } t \in (t_{l-1}, \ldots, t_l - 1) \textit{ with } t_l \in T_{bf}, \tag{11}$$

$$w_i^A \left(\sigma^x, U_i^G\right) = \overline{w}_i, \tag{12}$$

$$-\tfrac{\overline{w}_i - \underline{w}_i}{1-\delta} < U_i^G \left(h_{i-1,T_b+1}\right) \leq 0. \tag{13}$$

7.2 Equilibrium Construction

Now consider the infinitely repeated game. The purpose of this section is to show that for any payoff vector $v \in \prod_{i \in I} [\underline{w}_i, \overline{w}_i]$, there exists an equilibrium achieving v. The construction is the same as Yamamoto (2008b). For each player $i \in I$, player $(i-1)$'s strategy $\sigma_{i-1}(v)$ in the infinitely repeated game is specified by the following automaton with the initial state $v \in \prod_{i \in I} [\underline{w}_i, \overline{w}_i]$.

State w_i Go to phase B with probability p_{i-1} and go to phase G with probability $1 - p_{i-1}$ such that $w_i = p_{i-1}\underline{w}_i + (1 - p_{i-1}) \overline{w}_i$.

Phase B Play the block strategy s_{i-1}^B for T_b periods. After that, go to state w_i given by $w_i = \underline{w}_i + (1 - \delta) U_i^B(h_{i-1,T_b+1})$.

Phase G Play the block strategy s_{i-1}^G for T_b periods. After that, go to state w_i given by $w_i = \overline{w}_i + (1 - \delta) U_i^G(h_{i-1,T_b+1})$.

From **Lemmas 8** and **9** and the one shot deviation principle, $\sigma(v) = (\sigma_i(v))_{i \in I}$ is a Nash equilibrium. From Sekiguchi (1997), there exists a realization equivalent sequential equilibrium.

8 Folk Theorem

Definition 10 *The stage game is an N-player prisoner's dilemma if $|I| = N$, $A_i = \{C_i, D_i\}$ for all $i \in I$, $\pi_i(D_i, a_{-i}) \geq \pi_i(C_i, a_{-i})$ for all $i \in I$ and $a_{-i} \in A_{-i}$, $\pi_i(C_j, a_{-j}) \geq \pi_i(D_j, a_{-j})$ for all $j \neq i$ and $a_{-j} \in A_{-j}$ and $\pi_i(C_1, \ldots, C_N) > \pi_i(D_1, \ldots, D_N)$ for all $i \in I$.*

Theorem 11 *Suppose that the stage game is an N-player prisoner's dilemma with $N \geq 4$, that the full support condition, the identifiability condition, and the $(N-2)$-identifiability condition are satisfied. Then, $\lim_{\delta \to 1} E(\delta)$ is equal to the feasible and individually rational payoff set.*

Proof. Proposition 3 in Yamamoto (2008b) shows that $\bigcup_{p \in \triangle \mathcal{J}} \left(V(p) \cap \prod_{i \in I} [p\underline{v}_i, p\overline{v}_i]\right)$ includes all the feasible and individually rational payoffs. Then, **Theorem 4** establishes the result. ■

9 Conclusion

We characterize the set of payoffs that can be implemented as belief-free review-strategy equilibrium in the limit as the discount factor converges to one for the generic monitoring technology if the number of private signals are sufficiently large and the number of players is no less than four and show that the set is large enough to attain the folk theorem for the games with prisoner's-dilemma structure. Therefore, this paper is the first to attain the non-trivial characterization and to establish the folk theorem for the generic monitoring structure.

A true novelty lies in the construction of private counts that restore the key feature of the conditional independence of the signals in the brief-free review strategy. If the signals are conditionally independent, a player's own observation of the signals has no information about the opponents' counts, which is the key to show the optimality of a constant action during the review round. If the signals are conditionally dependent and each player counts a random event whose realization depends only on her own private signals and actions, a player's own observation of the signals has some information about the opponents' counts, which destroys the incentive to take a constant action. However, if each player counts a random event whose realization depends on the private signals of no less than two players, we can construct the equilibrium where a player's own observation of the signals has no information about the opponents' counts under the generic monitoring structure.

Let us comment on the key assumption in this paper, that is, the number of players is no less than four. We have to give an incentive to tell the truth about the choice of the periods for reviewing another player and the observation of the signals since we use the information from a player to punish another player. However, if the punishment is costly or beneficial for the other players, players may not have an incentive to tell the truth. To give an incentive to tell the truth, we require that no less than three players can monitor one player. The basic message from mechanism design is that if there are at least three players, it is easy to give an incentive to tell the truth. Therefore, we assume the number of players is no less than four. We do not know whether it is possible to construct an equilibrium with three players by properly specifying the continuation payoff after punishing one player.

In spite of this assumption, we conclude that the idea of this paper is quite robust if there exist no less than four players. Therefore, if we construct the belief-free review-strategy equilibrium with conditionally independent signals, we conjecture that we can directly extend the equilibrium to the generic monitoring structure. Although proving the folk theorem for general games with a not-almost-perfect or not-almost-public monitoring structure is difficult even with conditional independence as Hörner and Olszewski (2006) point out, we hope that this paper offers a key idea for the future research to extend results with conditional independence.

10 Appendix

10.1 Average Payoff with Perfect Monitoring

Let Σ_{i,T_b} be the set of player i's strategies in the T_b-period block game. Also, let $\mathcal{S}_{i,T_b}$ be the set of all $\sigma_{i,T_b} \in \Sigma_{i,T_b}$ such that player i chooses a constant action from the recommended set $\mathcal{A}_i^k$ in the kth main round, for each $k \in \{1, \ldots, K\}$ and for each history up to the beginning of the kth main round.

This subsection specifies $\sigma_i^G \in \Sigma_{i,T_b}$ and $\sigma_i^B \in \Sigma_{i,T_b}$ under perfect monitoring. Here, $h_{i,t}$ is expressed by a sequence of action profiles, since monitoring is perfect.

Coordination Round This round is the same as before.

Monitoring Round for the Coordination Round This round is irrelevant since players can observe actions directly. Thus, we leave the behavior in this round open.

Inference in the Monitoring Round for the Coordination Round Player i directly infers player j sends a message $x_j = G$ if $a_{j,\tau} = a_j^G$ for all τ included in the coordination round and $x_j = B$ otherwise.

Confirmation Round This round is irrelevant since players can observe actions directly. Thus, we leave the behavior in this round open.

Inference in the Confirmation Round The message profile $x_j = G$ is confirmed for each player n if $a_{j,\tau} = a_j^G$ for all τ included in the coordination round. Otherwise $x_j = B$ is confirmed. Note that the same profile is confirmed for every player.

kth Main Round This round is the same as before.

kth Monitoring Round This round is irrelevant since players can observe actions directly. Thus, we leave the behavior in this round open.

kth Supplemental Round This round is irrelevant since players can observe actions directly. Thus, we leave the behavior in this round open.

Inference in the kth Supplemental Round For each player n, $m_n^k \in M_n^k(i)$ if and only if player i does not take a constant action $a_i^{x,k}$ in the kth supplemental round.

Report Round This round is irrelevant since players can observe actions directly. Thus, we leave the behavior in this round open.

Let $w_i^P(\sigma_{T_b} \mid \delta)$ denote player i's average payoff in the block game with discount factor $\delta \in [0,1]$ when a strategy profile $\sigma_{T_b} \in \Sigma_{T_b}$ is performed under perfect monitoring and when payoffs in the periods other than the main rounds are replaced with 0.[5]

Lemma 12 *There exists $\overline{K}$ such that for all $K \geq \overline{K}$, for all $\tilde{S}(T)$ with $\tilde{S}(T)(\lfloor \log_2(KT) \rfloor + 1)/KT = o(T)$, there exists $\overline{T}$ such that for all $T \geq \overline{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in (\overline{\delta}, 1]$, for all $i \in I$, $x_{-i} \in X_{-i}$ with $x_{i-1} = B$, $\tilde{x}_{-i} \in X_{-i}$ with $\tilde{x}_{i-1} = G$, $\sigma_{i,T_b} \in \Sigma_{i,T_b}$ and $\tilde{\sigma}_{i,T_b} \in \mathcal{S}_{i,T_b}$, regardless of the specification of the monitoring rounds,*

$$\max_{\sigma_{i,T_b} \in \Sigma_{i,T_b}} w_i^P\left(\sigma_{i,T_b}, \sigma_{-i}^{x_{-i}} \mid \delta\right) < \underline{w}_i < \overline{w}_i < \min_{\sigma_{i,T_b} \in \mathcal{S}_{i,T_b}} w_i^P\left(\tilde{\sigma}_{i,T_b}, \sigma_{-i}^{\tilde{x}_{-i}} \mid \delta\right).$$

Proof. Analogous to **Lemma 4** Yamamoto (2008b). The difference is the existence of the monitoring rounds. However, the stage game payoff during the monitoring rounds is normalized to 0 and the length of each monitoring round is $\tilde{S}(T)(\lfloor \log_2(KT) \rfloor + 1)/KT = o(T)$, the result holds with sufficiently large $\overline{T}$ and $\overline{\delta}$. ∎

For the notational convenience, define $\min_{\sigma_{-i}^{x_{-i}} \mid x_{i-1}=B} \max_{\sigma_{i,T_b} \in \Sigma_{i,T_b}} w_i^P\left(\sigma_{i,T_b}, \sigma_{-i}^{x_{-i}} \mid \delta\right) \equiv \underline{w}_i^P(\delta)$ and $\max_{\sigma_{-i}^{x_{-i}} \mid x_{i-1}=G} \min_{\sigma_{i,T_b} \in \mathcal{S}_{i,T_b}} w_i^P\left(\tilde{\sigma}_{i,T_b}, \sigma_{-i}^{\tilde{x}_{-i}} \mid \delta\right) \equiv \overline{w}_i^P(\delta)$.

[5] If $\delta = 1$, the average payoff is defined as the time-average payoff.

10.2 Monitoring

Proof of Lemma 6. From the $(N-2)$-identifiability condition, for some $0 < q_1 < q_2 < q_3 < 1$,

$$Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = q_{3,2},$$

$$Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = q_{2,1},$$

$$Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right) = q_{3,2},$$

and

$$Q_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right) = q_{2,1}$$

have a solution with $\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right) : \Omega_i \times \Omega_l \to [0,1]$, $\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right) : \Omega_i \times \Omega_l \to [0,1]$, $\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) : \Omega_i \times \Omega_l \to [0,1]$, and $\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) : \Omega_i \times \Omega_l \to [0,1]$ with $\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = \psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ and $\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = \psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ for all t. The existence of the solution is guaranteed by the $(N-2)$ identifiability for any $0 < q_1 < q_2 < q_3 < 1$. If any element is negative, add a constant L to each element of ψ and q_1, q_2, q_3. If any element is larger than 1, divide each element of ψ and q_1, q_2, q_3 by the largest element. Note that since $\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = \psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$ and $\psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i, t\right) = \psi^{2,1}_{-(i,l)}\left(\mathcal{A}_j \mid a_i\right)$, this operation is independent of t.

Since $\Pr(\omega'_{-(i,j,l)} \mid \omega_{-(i,j,l)}) \to 1$ if $\omega'_{-(i,j,l)} = \omega_{-(i,j,l)}$ and $\Pr(\omega'_{-(i,j,l)} \mid \omega_{-(i,j,l)}) \to 0$ otherwise as $S \to \infty$ and $\left|\Omega_{-(j,l)}\right|$ is bounded, for all $\eta > 0$, there exists S such that

$$\sum_{\omega_{-(i,j,l)} \neq \omega'_{-(i,j,l)}} \Pr(\omega_{-(i,j,l)} \mid \omega'_{-(i,j,l)}) < \eta \tag{14}$$

for all $\omega_{-(i,j,l)}$ and $-(i,j,l)$.

Note that for all $L \geq 1$, from the argument analogous to **Claim 6** in Yamamoto (2008b), if player i sends the message a_i^G when player i sends the message about T_ε, the probability of player n to infer that player i sends a_i^B is $o(\tilde{S}^L)$. The symmetric argument holds if player

i sends the message a_i^B. Since $|a_i(t)| = \lfloor \log_2 KT \rfloor + 1$, if $o(\tilde{S}^L)(\lfloor \log_2 KT \rfloor + 1) = o(T^2)$,

$$\sum_{\mathbf{t} \neq t\mathbf{1}} \Pr(\mathbf{t} \mid t) = o\left(T^2\right). \tag{15}$$

Therefore, from (14) and (15), for any $\eta > 0$, there exists S such that for sufficiently large T,

$$\begin{aligned} \left\| \hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i, t) - Q_{-(i,l)}(\mathcal{A}_j \mid a_i) \right\|_\infty &< \eta \\ \left\| \hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i) - Q_{-(i,l)}(\mathcal{A}_j \mid a_i) \right\|_\infty &< \eta \end{aligned} \tag{16}$$

for all i, j, l, a_i, $\mathcal{A}_j$, and t, which means for sufficiently large T,

$$\hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i, t)\, \psi_{-(i,l)}^{3,2}(\mathcal{A}_j \mid a_i, t) = q_{3,2},$$

$$\hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i, t)\, \psi_{-(i,l)}^{2,1}(\mathcal{A}_j \mid a_i, t) = q_{2,1},$$

$$\hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i)\, \psi_{-(i,l)}^{3,2}(\mathcal{A}_j \mid a_i) = q_{3,2},$$

$$\hat{Q}_{-(i,l)}(\mathcal{A}_j \mid a_i)\, \psi_{-(i,l)}^{2,1}(\mathcal{A}_j \mid a_i) = q_{2,1},$$

have a solution with $\psi_{-(i,l)}^{3,2}(\mathcal{A}_j \mid a_i) : \Omega_i \times \Omega_l \to [0,1]$, $\psi_{-(i,l)}^{2,1}(\mathcal{A}_j \mid a_i) : \Omega_i \times \Omega_l \to [0,1]$, $\psi_{-(i,l)}^{3,2}(\mathcal{A}_j \mid a_i, t) : \Omega_i \times \Omega_l \to [0,1]$, and $\psi_{-(i,l)}^{2,1}(\mathcal{A}_j \mid a_i, t) : \Omega_i \times \Omega_l \to [0,1]$. ■

From **Lemma 6**, the following lemma holds.

Lemma 13 *Under the assumption of **Lemma 6**, for any a^{KT}, ω_j^{KT} and ω_l^{KT},*

$$\begin{aligned} \left| \Pr\left(\psi_{-(i,l)}^{3,2}(\mathcal{A}_j, \varepsilon) = 1 \mid a^{KT}, \omega_j^{KT}\right) - \Pr\left(\psi_{-(i,l)}^{3,2}(\mathcal{A}_j, t, \varepsilon) = 1 \mid a^{KT}, \omega_j^{KT}\right) \right| &= o\left(T^2\right) \\ \left| \Pr\left(\psi_{-(i,l)}^{3,2}(\mathcal{A}_j, \varepsilon) = 1 \mid a^{KT}, \omega_l^{KT}\right) - \Pr\left(\psi_{-(i,l)}^{3,2}(\mathcal{A}_j, t, \varepsilon) = 1 \mid a^{KT}, \omega_l^{KT}\right) \right| &= o\left(T^2\right) \\ \left| \Pr\left(\psi_{-(i,l)}^{2,1}(\mathcal{A}_j, \varepsilon) = 1 \mid a^{KT}, \omega_j^{KT}\right) - \Pr\left(\psi_{-(i,l)}^{2,1}(\mathcal{A}_j, t, \varepsilon) = 1 \mid a^{KT}, \omega_j^{KT}\right) \right| &= o\left(T^2\right) \\ \left| \Pr\left(\psi_{-(i,l)}^{2,1}(\mathcal{A}_j, \varepsilon) = 1 \mid a^{KT}, \omega_l^{KT}\right) - \Pr\left(\psi_{-(i,l)}^{2,1}(\mathcal{A}_j, t, \varepsilon) = 1 \mid a^{KT}, \omega_l^{KT}\right) \right| &= o\left(T^2\right) \end{aligned}$$

Proof. For any ω_j^{KT} and a^{KT},

$$\begin{aligned}
& \left|\Pr\left(\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j,\varepsilon\right)=1 \mid a^{KT},\omega_j^{KT}\right)-\Pr\left(\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j,t,\varepsilon\right)=1 \mid a^{KT},\omega_j^{KT}\right)\right| \\
= & \left|\Pr\left(\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j,\varepsilon\right)=1 \mid a_t,\omega_{j,t}\right)-\Pr\left(\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j,t,\varepsilon\right)=1 \mid a^{KT},\omega_j^{KT}\right)\right| \\
\leq & \sum_{\mathbf{t}\neq t\mathbf{1}} \Pr\left(\mathbf{t} \mid t\right) = o(\tilde{S}\left(T\right)^L)(\lfloor \log_2 KT \rfloor + 1) = o\left(T^2\right).
\end{aligned}$$

The equality follows from the common knowledge about T_ε for $\psi^{3,2}_{-(i,l)}\left(\mathcal{A}_j,\varepsilon\right)$ The other cases are completely symmetric. ■

10.3 Determination of K and ε

Fix $\overline{u}$ such that for all i,

$$-\overline{u} < \underline{w}_i < \overline{w}_i < \overline{u},$$
$$\max_i \max_a \left|\pi_i\left(a\right)\right| < \overline{u},$$

and there exists $u_i : A_{i-1} \times \Omega_{i-1} \to \mathbb{R}$ satisfying

$$\pi_i\left(a\right) + E\left[u_i\left(a_{i-1},\omega_{i-1}\right) \mid a\right] = 0 \text{ for all } a$$

and

$$-\overline{u} < u_i\left(a_{t,i-1},\omega_{t,i-1}\right) < \overline{u} \text{ for all } a_{i-1},\omega_{i-1}.$$

Note that the identifiability condition guarantees the existence.

We will define K and ε so that the payoffs during the main round dominates the payoffs during the other rounds. Firstly, we pin down the function $\tilde{S}\left(T\right)$.

Lemma 14 *There exists L such that there exists $\tilde{S}\left(T\right)$ such that $o(\tilde{S}\left(T\right)^L)(\lfloor \log_2 KT \rfloor + 1) = o\left(T^2\right)$ and $\left(\lfloor \log_2 KT \rfloor + 1\right)\tilde{S}\left(T\right)/T = o\left(T\right)$.*

Proof. $L = 6$ and $\tilde{S}\left(T\right) = T^{\frac{1}{3}}$. ■

Therefore, we can fix $\tilde{S}(T) = T^{\frac{1}{3}}$ and S so that **Lemma 6** holds. Then, the following lemma assures that there exists ε and K such that for sufficiently large T, the length of the main rounds is sufficiently large compared to that of the other rounds.

Lemma 15 *For any e, there exists a natural number K and $\varepsilon > 0$ such that for sufficiently large T,*

$$\left(1 - \frac{K^2 T}{T_b}\right) \overline{u} < e.$$

Proof. Note that

$$\begin{aligned} & \frac{K^2 T}{T_b} \\ = & \frac{K^2}{\Lambda + \sum_k N(N-1)(N-2)\left(\left(\lfloor \log_2 KT \rfloor + 1\right) \tilde{S}(T)/T + (N-3) ST_{\varepsilon} \varepsilon K\right) + K^2} \\ = & \frac{K^2}{o(K^2) + Ko(T) + N(N-1)(N-2) ST_{\omega} K^2 \varepsilon + K^2}. \end{aligned}$$

Thus,

$$\lim_{K \to \infty | \varepsilon = o(K^2)} \lim_{T \to \infty} \frac{K^2 T}{T_b} = 1.$$

■

Lemma 16 *There exists K, $\varepsilon > 0$, and $\eta > 0$ such that there exists $\overline{T}$ such that for all $T \geq \overline{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in (\overline{\delta}, 1]$, for all i and x_{-i},*

$$\left(1 - \frac{K^2 T}{T_b}\right) 3\overline{u} < 3\eta < \underline{w}_i - \underline{w}_i^P(\delta) \text{ if } x_{i-1} = B \tag{17}$$

$$\left(1 - \frac{K^2 T}{T_b}\right) (|A_i| + 2)\overline{u} < (|A_i| + 2)\eta < \overline{w}_i - \underline{w}_i^P(\delta) \text{ if } x_{i-1} = G \tag{18}$$

From now on, we fix K, $\varepsilon > 0$, and $\eta > 0$ such that the above lemma holds. Without loss of generality, we assume ε is a rational number. We require T satisfies $\varepsilon KT \in \mathbb{N}$.

10.4 Reward Functions for the Bad Strategy

Throughout this subsection, for each $k \in \{1, \ldots, K\}$, let $h_{i,[k]}$ denote player i's private history up to the end of the kth supplemental round. Also, let $h_{i,[k,m]}$ be player i's history up to the end of the kth main round, $h_{i,[0]}$ be player i's history up to the end of the confirmation round, $h_{i,[-1]}$ be player i's history up to the end of the monitoring round for the coordination round, and $h_{i,[-1,m]}$ be player i's history up to the end of the coordination round. For each $k \in \{-1, \ldots, K\}$, let $H_{i,[k]}$ represent the set of all $h_{i,[k]}$ and $H_{i,[k,m]}$ represent the set of all $h_{i,[k,m]}$.

Given $h_{i-1,T_b+1} \in H_{i-1,T_b+1}$, let $I_{-i} \in (M_j^0 \times M_j^1 \times \cdots \times M_j^K)_{j \neq i}$ represent player $(i-1)$'s inference on the messages from players $-i$ in the report round. Let $I_{-i,k}$ be the projection of I_{-i} onto $(M_j^0 \times M_j^1 \times \cdots \times M_j^k)_{j \neq i}$ for $k \in \{0, \ldots, K\}$.

Without loss of generality, consider a particular $i \in I$. Suppose that U_i^B is decomposable into real-valued functions $\overline{U}_i^B(h_{i-1,T_b+1})$ and $(\theta^{-1}, \ldots, \theta^{K+1})$ such that

$$\begin{aligned} & U_i^B(h_{i-1,T_b+1}) \\ = \; & \frac{1}{\delta^{T_b}} \left[\begin{array}{c} \delta^{T_{-1}} \theta^{-1}\left(h_{i-1,[-1,m]}\right) + \delta^{T_{-1}+T_0} \theta^0\left(h_{i-1,[0]}\right) \\ + \sum_{k=1}^{K} \delta^{T_{-1}+T_0+\sum_{\tilde{k} \leq k} T_{\tilde{k}}} \theta^k\left(h_{i-1,[k]}, I_{-i,k}\right) + \delta^{T_b} \theta^{K+1}(h_{i-1,T_b+1}) \end{array} \right]. \end{aligned}$$

Construction of θ^{K+1} Let

$$\theta^{K+1}(h_{i-1,T_b+1}) = \sum_{t=1}^{T_r} \frac{u_i(a_{t,i-1}, \omega_{t,i-1})}{\delta^{T_r+1-t}},$$

where $(a_{t,i-1}, \omega_{t,i-1})$ is player $(i-1)$'s action and signal in the tth period of the report round. Remember $u_i(a_{t,i-1}, \omega_{t,i-1})$ satisfies

$$\pi_i(a) + E[u_i(a_{-i}, \omega_{i-1}) \mid a] = 0 \text{ for all } a, \tag{19}$$

and

$$-\overline{u} < u_i(a_{i-1}, \omega_{i-1}) \text{ for all } a_{i-1}, \omega_{i-1}. \tag{20}$$

Lemma 17 *1. Under U_i^B, (7) holds.*

2. For all K and T, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(0, \overline{\delta}\right)$,

$$-T_r \overline{u} < \theta^{K+1}\left(h_{i-1,T_b+1}\right).$$

Proof.

1. The first part follows from the following observations.
   (a) From (19), all the actions are indifferent in terms of instantaneous utility.
   (b) As we will show, $\theta^{-1}, \theta^0, \ldots, \theta^K$ only depends on I_{-i}, which is independent of player i's message in the report round. Therefore, truth-telling is optimal in the report round.
   (c) From **Lemma 5**, player i's action cannot affect the messages of the other players.
2. The second part follows from (20).

■

Construction of θ^k with $k \in \{1, \ldots, K\}$ The following notation is useful. For each $x \in X$, let $H_{-i,[0]}(x)$ be the set of $h_{-i,[0]} \in H_{-i,[0]}$ such that for each $j \neq i$, regardless of player i's message in the monitoring period for the coordination period and regardless of player i's message in the confirmation round,

$$m_j^0 \in M_j^0(x),$$

that is, players $-i$ confirm x and play $a_{-i}^{x,1}$ in the first main round. Define

$$\overline{H}_{-i,[0]} \equiv \bigcup_{x \in X^B} H_{-i,[0]}(x).$$

Notice that if all the players play the block strategy σ^x in the coordination round and the players $-i$ play the block strategy profile σ^x up to the end of the confirmation round, then it is likely that the resulting history profile $h_{-i,[0]}$ is an element of $H_{-i,[0]}(x)$.

Likewise, for each $k \in \{1, \ldots, K-1\}$ and $x \in X$, let $H_{-i,[k]}(x)$ be the set of $h_{-i,[k]} \in H_{-i,[k]}$ such that for each $j \neq i$, regardless of player i's message in the kth monitoring period and regardless of player i's message in the kth supplemental round,

$$(m_j^0 \times m_j^1 \times \cdots \times m_j^k) \in M_j^0(x) \times M_j^1(0) \times \cdots \times M_j^k(0),$$

that is, regardless of player i's action in the kth monitoring round and the kth supplemental round, no one's deviation is inferred and player $-i$ play $a_{-i}^{x,1}$ in the $(k+1)$ main round. Define

$$\overline{H}_{-i,[k]} \equiv \bigcup_{x \in X^B} H_{-i,[k]}(x).$$

Note that, as before, if all the players perform the block strategy profile σ^x up to the kth main round and if the other players than player i perform σ^x up to the kth supplemental round, it is likely that the resulting history profile $h_{-i,[k]}$ is an element of $H_{-i,[k]}(x)$.

Also, for each $k \in \{1, \ldots, K\}$ and $x \in X$, let $H_{-i,[k]}(x, i)$ be the set of $h_{-i,[k]} \in H_{-i,[k]}$ such that there exists $\tilde{k} \in \{1, \ldots, k\}$ such that for each $j \neq i$, regardless of player i's message in the $\tilde{k}$th monitoring round and regardless of player i's message in the $\tilde{k}$th supplemental round,

$$(m_j^0 \times m_j^1 \times \cdots \times m_j^{\tilde{k}}) \in M_j^0(x) \times M_j^1(0) \times \cdots \times M_j^{\tilde{k}-1}(0) \times (M^{\tilde{k}}(i) \setminus \bigcup_{l \neq i} M^{\tilde{k}}(l)).$$

Note that $h_{-i,[k]}$ is likely to be an element of $H_{-i,[k]}(x, i)$ if the other players than player i play the block strategy profile σ^x but player i deviates from $a^{x,\tilde{k}}$ in the $\tilde{k}$th main round for some $\tilde{k} \in \{1, \ldots, k\}$.

Define

$$\overline{H}_{-i,[k]} \equiv \bigcup_{x \in X^B} \left(H_{-i,[k]}(x) \cup H_{-i,[k]}(x, i) \right).$$

For each $k \in \{1, \ldots, K\}$, for $h_{-i,[k]} \in \overline{H}_{-i,[k]}$, the value $V_i\left(h_{-i,[k]}\right)$ is defined to be the maximum of player i's actual (that is, non-averaged) continuation payoff after history $h_{-i,[k]}$ subject to the constraint that the monitoring is perfect, the payoffs in the communication and monitoring rounds are replaced with 0. Likewise, for each $h_{-i,[0]} \in \overline{H}_{-i,[0]}$, the value $V_i\left(h_{-i,[0]}\right)$ is defined to be the maximum of player i's actual continuation payoff after history $\tilde{h}_{-i,[0]}$ over all $\tilde{h}_{-i,[0]} \in \overline{H}_{-i,[0]}$ subject to the constraint that the monitoring is perfect and the payoffs in the communication and monitoring rounds are replaced with 0. For another history, that is, for $h_{-i,[k]} \notin \overline{H}_{-i,[k]}$ with $k \in \{0, 1, \ldots, K\}$, the value $V_i\left(h_{-i,[k]}\right)$ is define to player i's continuation payoff when she earns $\overline{u}$ in periods of the main rounds and zero in the other periods.

The following notation is helpful. Let $J_{-i,K}$ denote the message that players $-i$ send in the report round about $(M_j^0 \times M_j^1 \times \cdots \times M_j^K)_{j \neq i}$. Let $J_{-i,k}$ denote the projection of $J_{-i,K}$message on $(M_j^0 \times M_j^1 \times \cdots \times M_j^k)_{j \neq i}$Note that $V_i\left(h_{-i,[k]}\right)$ only depends on $J_{-i,k}$. Recall that everyone tells the truth in the report round from **Lemma 17** and that $I_{-i,k}$ denotes the realized message corresponding to $J_{-i,k}$ in the report round.

The purpose of this step is to specify $\left(\theta^1, \ldots, \theta^K\right)$ by backward induction so that player i's continuation payoff after $h_{-i,[k-1]} \in H_{-i,[k-1]}$ is equal to $V_i\left(h_{-i,[k-1]}\right)$. For $K+1$, we are done by **Lemma 17**.

Suppose we are done after $\tilde{k} \geq k+1$. Consider k. Suppose that θ^k is decomposable as

$$\theta^k\left(h_{i-1,[k]}, I_{-i,k}\right) = \tilde{\theta}^k\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}, I_{-i,k-1}\right) + \sum_{t=1}^{T_k - KT} \frac{u_i\left(a_{i-1,t}, \omega_{i-1,t}\right)}{\delta^{T_k - KT + 1 - t}}, \tag{21}$$

where $\hat{\Omega}_{-(i-1,i,l)}$ denotes the realized message from players $-(i-1, i, l)$ to player $i-1$ during the kth monitoring round with some fixed $l \neq i-1, i$. Here, $(a_{i-1,t}, \omega_{i-1,t})$ is player $(i-1)$'s action and signal in the tth period of the kth monitoring round and the kth supplemental round.

Lemma 18 *Under U_i^B, (7) holds for the kth monitoring and supplemental rounds.*

Proof. The following observations establish the result.

1. From the second term of (21), all the actions are indifferent in terms of instantaneous utility.

2. From **Lemma 5**, player i's action cannot affect player $-i$'s message.

3. From **Lemma 5**, $\hat{\Omega}_{-(i-1,i,l)}$ is independent of player i's action in the monitoring round. Thus, $\tilde{\theta}^k\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}, I_{-i,k-1}\right)$ does not depend on player i's action in the kth monitoring round.

4. $\tilde{\theta}^k\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}, I_{-i,k-1}\right)$ does not depend on the outcome in the kth supplemental round.

5. For $\theta^{k+1}, \ldots, \theta^{K+1}$,

   (a) If player i's message in kth monitoring round or the kth supplemental round does not have any impact on the inference of the deviation of the players, her action cannot affect her continuation payoff.

   (b) If it does, her action does not affect her continuation payoff since it is constant.

■

Next, we specify a real valued function $\tilde{\theta}^k$. For each $h_{-i,[k-1]} \in H_{-i,[k-1]}$ and $a_i \in A_i$, let $\tilde{W}_i\left(h_{-i,[k-1]}, a_i\right)$ denote player i's continuation payoff from the kth main round, augmented as the payoffs during the kth monitoring round and the kth supplemental round are equal to 0, when player i plays a_i constantly in the kth main round and plays a best reply thereafter. That is,

$$\begin{aligned} & \tilde{W}_i\left(h_{-i,[k-1]}, a_i\right) \\ \equiv\ & \sum_{t=1}^{KT} \delta^{t-1}\pi_i\left(a_i, \sigma_{-i}^{x_{-i}}\left(h_{-i,[k-1]}\right)\right) \\ & +\delta^{T_m} \sum_{h_{-i,[k]} \in H_{-i,[k]}} \Pr\left(h_{-i,[k]} | h_{-i,[k-1]}, a_i\right) V_i\left(h_{-i,[k]}\right). \end{aligned}$$

Note that $\sigma_{-i}^{x_{-i}}\left(h_{-i,[k-1]}\right)$, $V_i\left(h_{-i,[k]}\right)$, and $\Pr\left(J_{-i,k}|h_{-i,[k-1]},a_i\right)$ only depend on $J_{-i,k-1}$. Thus,

$$\begin{aligned}&\tilde{W}_i\left(J_{-i,k-1},a_i\right)\\=&\sum_{t=1}^{KT}\delta^{t-1}\pi_i\left(a_i,\sigma_{-i}^{x_{-i}}\left(J_{-i,k-1}\right)\right)+\delta^{T_m}\sum_{J_{-i,k}}\Pr\left(J_{-i,k}|J_{-i,k-1},a_i\right)V_i\left(J_{-i,k}\right).\end{aligned}$$

For each $J_{-i,k-1}$, let $\{a_i^1\left(J_{-i,k-1}\right),\dots,a_i^{|A_i|}\left(J_{-i,k-1}\right)\}$ be a sequence of all the elements of A_i such that

$$\lim_{T\to\infty}\lim_{\delta\to1}\frac{\tilde{W}_i\left(J_{-i,k-1},a_i^1\left(J_{-i,k-1}\right)\right)}{T}\geq\cdots\geq\lim_{T\to\infty}\lim_{\delta\to1}\frac{\tilde{W}_i(J_{-i,k-1},a_i^{|A_i|}\left(J_{-i,k-1}\right))}{T}.$$

Then define $\tilde{\theta}^k$ to be

$$\begin{aligned}&\tilde{\theta}^k\left(h_{i-1,[k,m]},I_{-i,k-1},\hat{\Omega}_{-(i-1,i,l)}\right)\\=&\sum_{a_i\in A_i}1_{a_i}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)KT\eta+\sum_{n=1}^{|A_i|}1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)\lambda^k\left(I_{-i,k-1},n\right),\end{aligned}$$

where for each $a_i\in A_i$, let $1_{a_i}:H_{i-1,[k,m]}\times\hat{\Omega}_{l(i-1,i)}\to\{0,1\}$ is the indicator function such that $1_{a_i}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)=1$ if the random event $\sum_{t=1}^{KT}\psi_{-(i-1,l)}^{3,2}\left(\{a_i\},t\right)>Z_{KT}$ in the kth monitoring round and $1_{a_i}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)=0$ otherwise. Likewise, $1_{\left[I_{-i,k-1},n\right]}:H_{i-1,[k,m]}\times\hat{\Omega}_{-(i-1,i,l)}\to\{0,1\}$ is the indicator function such that $1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)=1$ if the random event $\sum_{t=1}^{KT}\psi_{-(i-1,l)}^{3,2}(\{a_i^n\left(I_{-i,k-1}\right),\dots,a_i^{|A_i|}\left(I_{-i,k-1}\right)\},t)>Z_{KT}$ in the kth monitoring round and $1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]},\hat{\Omega}_{-(i-1,i,l)}\right)=0$ otherwise.

In addition, $\lambda^k(I_{-i,k-1}, n)$ solves

$$\begin{aligned} & V_i(J_{-i,k-1}) \quad (22) \\ = \; & \tilde{W}_i(J_{-i,k-1}, a_i) + \delta^{KT+T_m} \sum_{\tilde{a}_i \in A_i} \Pr(1_{\tilde{a}_i} \mid J_{-i,k-1}, a_i) KT\eta \\ & + \delta^{KT+T_m} \sum_{I_{-i,k-1}} \Pr(I_{-i,k-1} \mid J_{-i,k-1}) \sum_{n=1}^{|A_i|} \Pr\left(1_{[I_{-i,k-1},n]} \mid J_{-i,k-1}, a_i\right) \lambda^k(I_{-i,k-1}, n) \end{aligned}$$

for all $J_{-i,k-1}$ and $a_i \in A_i$. Here, $\Pr\left(1_{[\cdot]} \mid J_{-i,k-1}, a_i\right)$ denotes the probability that the indicator function takes 1 conditional on the event that player i chooses the constant action a_i while player $-i$ play the action $\sigma_{-i}^{x_{-i}}(J_{-i,k-1})$.

Lemma 19 *For all K, there exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\bar{\delta} \in (0,1)$ such that (22) has a unique solution and for all $\left(h_{i-1,[k]}, I_{-i,k-1}\right)$,*

$$-(T_k - KT)\overline{u} - 2KT\eta < \theta^k\left(h_{i-1,[k]}, I_{-i,k-1}\right).$$

Proof. Analogous to **Lemma 10** in Yamamoto (2008b). From **Lemma 13**, the probability of the miscommunication about t does not matter in the limit. The remaining difference is the existence of the monitoring rounds. However, since the length of each monitoring round satisfies $T_m/KT = o(T)$, the result holds with sufficiently large $\overline{T}$. ■

Player i's continuation payoff from the kth main round is given by

$$\begin{aligned} & \tilde{W}_i\left(h_{-i,[k-1]}, a_i\right) \\ \equiv \; & \sum_{t=1}^{KT} \delta^{t-1}\pi_i\left(a_i, \sigma_{-i}^{x_{-i}}\left(h_{-i,[k-1]}\right)\right) \\ & + \delta^{KT+T_m} \sum_{h_{-i,[k]} \in H_{-i,[k]}} \Pr\left(h_{-i,[k]} | h_{-i,[k-1]}, a_i\right) V_i\left(h_{-i,[k]}\right) \\ & + \delta^{KT+T_m} \left[\begin{array}{c} \sum_{a_i \in A_i} 1_{a_i}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) KT\eta \\ + \sum_{I_{-i,k-1}} \Pr(I_{-i,k-1} \mid J_{-i,k-1}) \sum_{n=1}^{|A_i|} 1_{[I_{-i,k-1},n]}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) \lambda^k(I_{-i,k-1}, n) \end{array} \right] \\ = \; & V_i(J_{-i,k-1}). \end{aligned}$$

We will show that a constant action is optimal in the kth main round. Suppose $\delta = 1$. Without loss of generality, consider a particular $h_{-i,[k-1]} \in H_{-i,[k-1]}$, $J_{-i,k-1}$, and $\sigma_{-i}^{x_{-i}}$. Pick any arbitrary actions $a_i^* \in A_i$ and $a_i^{**} \neq a_i^*$. For each $\tau \in \{0, \ldots, KT\}$, let $W_i\left(a_i^{KT}(\tau)\right)$ denote that value $W_i\left(h_{-i,[k-1]}, a_i^{KT}(\tau)\right)$ with

$$a_i^{KT}(\tau) = (a_{i,1}, \ldots, a_{i,KT}) \text{ with } \#\{t \mid a_{i,t} = a_i^*\} = \tau \text{ and } \#\{t \mid a_{i,t} = a_i^{**}\} = KT - \tau.$$

$$\begin{aligned}
& W_i\left(a_i^{KT}(\tau)\right) - W_i\left(a_i^{KT}(\tau-1)\right) \\
= \; & \pi_i\left(a_i^*, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right) - \pi_i\left(a_i^{**}, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right) \\
& + \sum_{a_i \in A_i} \left[\sum_{r > Z_{KT}} F_{-(i-1,i,l)}^{3,2}\left(\{a_i\}, KT, r \mid a_i^{KT}(\tau)\right) - \sum_{r > Z_{KT}} F_{-(i-1,i,l)}^{3,2}\left(\{a_i\}, KT, r \mid a_i^{KT}(\tau-1)\right) \right] KT\eta \\
& + \sum_{I_{-i,k-1}} \Pr\left(I_{-i,k-1} \mid J_{-i,k-1}\right) \\
& \times \sum_{n=1}^{|A_i|} \left[\begin{array}{c} \sum_{r > Z_{KT}} F_{-(i-1,i,l)}^{3,2}\left(\{a_i^n(I_{-i,k-1}), \ldots, a_i^{|A_i|}(I_{-i,k-1})\}, KT, r \mid a_i^{KT}(\tau)\right) \\ -\sum_{r > Z_{KT}} F_{-(i-1,i,l)}^{3,2}\left(\{a_i^n(I_{-i,k-1}), \ldots, a_i^{|A_i|}(I_{-i,k-1})\}, KT, r \mid a_i^{KT}(\tau-1)\right) \end{array} \right] \lambda^k(I_{-i,k-1}, n) \\
& + \sum_{J_{-i,k}} \left[\Pr\left(J_{-i,k} | J_{-i,k-1}, a_i^{KT}(\tau)\right) - \Pr\left(J_{-i,k} | J_{-i,k-1}, a_i^{KT}(\tau-1)\right)\right] V_i(J_{-i,k}).
\end{aligned}$$

Let $\Delta_1\left(a_i^{KT}(\tau)\right)$ be the terms in the last line of the right hand side and $\Delta_2\left(a_i^{KT}(\tau)\right)$ be the terms in the second, third and fourth lines. In addition, let $\Delta_1\left(a_i^{KT}(\tau), \omega_i^{KT}\right)$ and $\Delta_2\left(a_i^{KT}(\tau), \omega_i^{KT}\right)$ be the value of $\Delta_1\left(a_i^{KT}(\tau)\right)$ and $\Delta_2\left(a_i^{KT}(\tau)\right)$ conditional on ω_i^{KT}.

Let $n^*(I_{-i,k-1})$ denote the integer n with $a_i^n(I_{-i,k-1}) = a_i^*$ and $n^{**}(I_{-i,k-1})$ denote the integer n with $a_i^n(I_{-i,k-1}) = a_i^{**}$. Let $\mathcal{I}_{-i,k-1}$ denote the set of all $I_{-i,k-1}$ satisfying

$n^*(I_{-i,k-1}) > n^{**}(I_{-i,k-1})$. Then, let us define

$$
\begin{aligned}
\Delta_1(\tau) &= \sum_{J_{-i,k}} \left(\widetilde{\Pr}(J_{-i,k}|J_{-i,k-1},\tau) - \widetilde{\Pr}(J_{-i,k}|J_{-i,k-1},\tau-1) \right) V_i(J_{-i,k}), \\
\Delta_2(\tau) &= -(\varepsilon q_3 - \varepsilon q_2) \tilde{F}^{3,2}_{-(i-1,i,l)}(\tau-1) KT\eta + (\varepsilon q_3 - \varepsilon q_2) \tilde{F}^{3,2}_{-(i-1,i,l)}(KT-\tau) KT\eta \\
&\quad - \sum_{I_{-i,k-1}\in\mathcal{I}_{-i,k-1}} \Pr(I_{-i,k-1} \mid J_{-i,k-1}) \sum_{n=1+n^{**}(I_{-i,k-1})}^{n^*(I_{-i,k-1})} (\varepsilon q_3 - \varepsilon q_2) \tilde{F}^{3,2}_{-(i-1,i,l)}(\tau-1) \lambda^k \left(I^{k-1}_{-i}, n\right) \\
&\quad + \sum_{I_{-i,k-1}\notin\mathcal{I}_{-i,k-1}} \Pr(I_{-i,k-1} \mid J_{-i,k-1}) \sum_{n=1+n^{*}(I_{-i,k-1})}^{n^{**}(I_{-i,k-1})} (\varepsilon q_3 - \varepsilon q_2) \tilde{F}^{3,2}_{-(i-1,i,l)}(KT-\tau) \lambda^k \left(I^{k-1}_{-i}, n\right),
\end{aligned}
$$

where $\tilde{F}^{3,2}_{-(i-1,i,l)}(\tau) = \tilde{F}^{3,2}_{-(i-1,i,l)}(\tau, KT-1, Z_{KT})$ and $\widetilde{\Pr}(J_{-i,k}|J_{-i,k-1},\tau)$ denote the probability of $J_{-i,k}$ occurs when player i plays $a_i^{KT} = \underbrace{a_i^*, \ldots, a_i^*}_{\tau}, \underbrace{a_i^{**}, \ldots, a_i^{**}}_{KT-\tau}$ and the transition probability is determined by $(\psi_{-(j,l)}(\mathcal{A}_i, \varepsilon))_{j,l\neq i}$ and $(\psi_{-(j,i)}(\mathcal{A}_l, \varepsilon))_{j,l\neq i}$ instead of $(\psi_{-(j,l)}(\mathcal{A}_i, \varepsilon, t))_{j,l\neq i}$ and $(\psi_{-(j,i)}(\mathcal{A}_l, \varepsilon, t))_{j,l\neq i}$. In addition, let $\Delta_1\left(\tau, \omega_i^{KT}\right)$ and $\Delta_2\left(\tau, \omega_i^{KT}\right)$ be the value of $\Delta_1(\tau)$ and $\Delta_2(\tau)$ conditional on ω_i^{KT}.

Lemma 20 *$\max_{\tau,\omega_i^{KT}} \left|\Delta_1(\tau) - \Delta_1\left(a_i^{KT}, \omega_i^{KT}\right)\right| = o(T)$ and $\max_{\tau,\omega_i^{KT}} \left|\Delta_2(\tau) - \Delta_2\left(a_i^{KT}, \omega_i^{KT}\right)\right| = o(T)$.*

Proof. From **Lemma 13**, since $V_i(J_{-i,k})$ and $\lambda^k\left(I^{k-1}_{-i}, n\right)$ is $O(T)$, for any ω_i^{KT} and a_i^{KT}, $\max_{\tau,\omega_i^{KT}} \left|\Delta_1\left(a_i^{KT}, \omega_i^{KT}\right) - \Delta_1\left(\tau, \omega_i^{KT}\right)\right| = o(T)$ and $\max_{\tau,\omega_i^{KT}} \left|\Delta_2\left(a_i^{KT}, \omega_i^{KT}\right) - \Delta_2\left(\tau, \omega_i^{KT}\right)\right| = o(T)$. Thus, it suffices to show that $\Delta_1(\tau) = \Delta_1\left(\tau, \omega_i^{KT}\right)$ and $\Delta_2(\tau) = \Delta_2\left(\tau, \omega_i^{KT}\right)$. Note that each probability is conditional on $J_{-i,k-1}$.

1. Proof of $\Delta_1(\tau) = \hat{\Delta}_1\left(\tau, \omega_i^{KT}\right)$. In this proof, the transition probability is determined by $(\psi_{-(j,l)}(\mathcal{A}_i, \varepsilon))_{j,l\neq i}$ and $(\psi_{-(j,i)}(\mathcal{A}_l, \varepsilon))_{j,l\neq i}$ instead of $(\psi_{-(j,l)}(\mathcal{A}_i, t, \varepsilon))_{j,l\neq i}$ and $(\psi_{-(j,i)}(\mathcal{A}_l, t, \varepsilon))_{j,l\neq i}$.

    (a) $V_i(J_{-i,k})$ takes the lowest value when $J_{-i,k} \in H_{-i,[k]}(x, i)$.

    (b) $V_i(J_{-i,k})$ takes the second highest value when $J_{-i,k} \in H_{-i,[k]}(x)$.

(c) $V_i(J_{-i,k})$ takes the highest value when $J_{-i,k} \notin \overline{H}_{-i,[k]}$.

(d) The probability of $J_{-i,k} \notin \overline{H}_{-i,[k]}$ is independent of a_i^{KT} and ω_i^{KT} from **Lemma 6** since whether $J_{-i,k} \notin \overline{H}_{-i,[k]}$ is determined by $(\hat{x}^0(n, j, -(n, j, i))_{j \neq n})_{n \neq i}$ and $((\hat{x}^{\tilde{k}}(n, j, -(n, j, i))_{j \neq n,i})_{n \neq i})_{\tilde{k} \leq k}$.

(e) Conditional on $J_{-i,k} \in \overline{H}_{-i,[k]}$, the probability of $J_{-i,k} \in H_{-i,[k]}(x, i)$ is equal to the probability of $J_{-i,k} \in M_{-i}^k(i)$.

(f) The probability of $J_{-i,k} \in M_{-i}^k(i)$ conditional on $J_{-i,k} \in \overline{H}_{-i,[k]}$ is equal to the probability of $J_{-i,k} \in M_{-i}^k(i)$ unconditional on $J_{-i,k} \in \overline{H}_{-i,[k]}$ since the probability of $J_{-i,k} \notin \overline{H}_{-i,[k]}$ is independent of a_i^{KT} and ω_i^{KT}.

(g) The probability of $J_{-i,k} \in M_{-i}^k(i)$ is independent of ω_i^{KT} from **Lemma 6**.

2. $\Delta_2(\tau) = \Delta_2\left(\tau, \omega_i^{KT}\right)$ directly follows from **Lemma 6.**

■

Note that $\Delta_2(\tau) + \pi_i\left(a_i^*, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right) - \pi_i\left(a_i^{**}, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right)$ is the same as $\Delta_2(\tau)$ in Yamamoto (2008b). Therefore, **Lemmas 13, 16, 17** of Yamamoto (2008b) establish the following results about $\Delta_2(\tau)$.

Lemma 21 $\sum_{\tau=1}^{t}[\Delta_2(\tau) + \pi_i\left(a_i^*, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right) - \pi_i\left(a_i^{**}, \sigma_{-i}^{x_{-i}}(J_{-i,k-1})\right)]$ *is a convex function and* $\tilde{\Delta}_2(1) = -\infty$ *and* $\tilde{\Delta}_2(KT-1) = +\infty$.

Next, we derive the results about $\Delta_1(\tau)$.

Lemma 22 $\Delta_1(\tau) \geq 0$ *for* $a_i^* = a_i^{x,k}$. $\Delta_1(\tau) \leq 0$ *for* $a_i^* \neq a_i^{x,k}$.

Proof. Analogous to **Lemma 15** in Yamamoto (2008b). ■

Lemma 23 *For any* $\rho \in (0, 1)$ *and* $n \geq 1$, $\sum_{\tau=1}^{\lfloor \rho KT \rfloor} \Delta_1(\tau) = o(T^{-n})$.

Proof. From the proof of **Lemma 20**,

$$\Delta_1(\tau) \leq \Delta V_i \times \sum_{J_{-i,k} \in M_{-i}^k(i)} \left(\Pr(J_{-i,k} | J_{-i,k-1}, \tau) - \Pr(J_{-i,k} | J_{-i,k-1}, \tau - 1)\right).$$

τ matters only if it affects the probability of the event that for some player n with $n \neq i$ and $-(n,i,l)$ with $l \neq j, n$, $x^k(n,i,-(n,i,l))$ changes from 0 to 1. Therefore,

$$\begin{aligned} & \sum_{J_{-i,k} \in M_{-i}^k(i)} \left(\Pr\left(J_{-i,k} | J_{-i,k-1}, \tau\right) - \Pr\left(J_{-i,k} | J_{-i,k-1}, \tau - 1\right)\right) \\ \leq \; & (N-1)(N-2)(\varepsilon q_2 - \varepsilon q_1) \tilde{F}^{2,1}\left(\tau, KT-1, Z_T''\right). \end{aligned}$$

Then, from **Lemma 15** in Yamamoto (2008b),

$$\sum_{\tau=1}^{\lfloor \rho KT \rfloor} \Delta_1(\tau) = o\left(T^{-n}\right).$$

■

Therefore, from (22) and **Lemmas 20**, **21**, **22**, and **23**, we have shown that a constant action is better than any strategy that mixes two actions after some history for sufficiently large δ. The similar argument establishes a constant action is better than any strategy that mixes more than two actions after some history. This establishes (6) for the kth main round.

Construction of θ^0 Let

$$\theta^0\left(h_{i-1,[0]}\right) = \sum_{t=1}^{T_0} \frac{u_i\left(a_{i-1,t}, \omega_{i-1,t}\right)}{\delta^{T_0+1-t}}$$

where $(a_{i-1,t}, \omega_{i-1,t})$ is player $(i-1)$'s action and signal in the tth period of the confirmation round.

Lemma 24 *1. Under U_i^B, (7) holds for the confirmation round for any history.*

2. For all T, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(0, \overline{\delta}\right)$,

$$-T_0 \overline{u} < \theta^0\left(h_{i-1,[0]}\right).$$

Proof.

1. The first part follows from the following observations.

   (a) All the actions are indifferent in terms of instantaneous utility.

   (b) From **Lemma 5**, player i's action cannot affect player $-i$'s message.

   (c) For $\theta^1, \ldots, \theta^{K+1}$,

      i. If player i's message in the confirmation round does not have any impact on the inference of the deviation of the players, her action cannot affect her continuation payoff.

      ii. If it does, her action does not affect her continuation payoff since it is constant.

2. The second part follows from (20).

■

Construction of θ^{-1} Suppose that θ^{-1} is decomposable as

$$\theta^{-1}\left(h_{i-1,[-1]}\right) = \tilde{\theta}^{-1}\left(h_{i-1,[-1]}\right) + \sum_{t=1}^{T_{-1}} \frac{u_i\left(a_{i-1,t}, \omega_{i-1,t}\right)}{\delta^{T_{-1}+1-t}}, \tag{23}$$

where $(a_{i-1,t}, \omega_{i-1,t})$ is player $(i-1)$'s action and signal in the tth period of the block game.

For each $x \in X$, let $H_{i-1,[-1]}(x)$ denote the set of all $h_{i-1,[-1]} \in H_{i-1,[-1]}$ such that for $h_{i-1,[-1]}$, $x(i-1, j, -(i-1, j, i)) = x_j$ for any $j \neq i-1$ and $\varepsilon\psi^{3,2}_{-(i-1,l)}(\{a_i^{x_i}\})$ occurs more than Z_T times during the monitoring round for the coordination round for some fixed l. Then, for each $x \in X$, let $1_x : H_{i-1,[-1]} \to \{0,1\}$ denote the indicator function of $H_{i-1,[-1]}(x)$, that is, $1_x\left(h_{i-1,[-1]}\right) = 1$ if and only if $h_{i-1,[-1]} \in H_{i-1,[-1]}(x)$.

Let

$$\tilde{\theta}^{-1}\left(h_{i-1,[-1]}\right) = \sum_{x \in X^B} 1_x\left(h_{i-1,[-1]}\right) \lambda^{-1}(x),$$

where $\left(\lambda^{-1}(x)\right)_{x\in X_B}$ to solve

$$\begin{aligned}\sum_{t=1}^{T_b}\delta^{t-1}\underline{w}_i &= 3T_b\eta+\delta^{T_{-1}}\sum_{\tilde{x}\in X^B}\sum_{h_{i-1,[-1]}\in H_{i-1,[-1]}(\tilde{x})}\Pr\left(h_{i-1,[-1]}|\sigma^x\right)\lambda^{-1}(\tilde{x})\\ &\quad+\delta^{T_{-1}+T_0}\sum_{h_{-i,[0]}\in H_{-i,[0]}}\Pr\left(h_{-i,[0]}|\sigma^x\right)V_i\left(h_{-i,[0]}\right). \qquad (24)\end{aligned}$$

Here, $\Pr\left(h_{i-1,[-1]}\mid\sigma^x\right)$ and $\Pr\left(h_{-i,[0]}\mid\sigma^x\right)$ denotes the occurrence probability of $h_{i-1,[-1]}$ and $h_{-i,[0]}$ given that players perform the block strategy σ^x.

Lemma 25 *1. Under U_i^B, (7) holds for the monitoring round for the coordination round.*

2. There exists $\overline{T}$ such that for all $T>\overline{T}$, there exists $\bar{\delta}\in(0,1)$ such that (24) has a unique solution and

$$3T_b\eta-T_{-1}\overline{u}<\theta^{-1}\left(h_{i-1,T_b}\right).$$

Proof.

1. The first part follows from the following observations.
   (a) From the second term of (23), all the actions are indifferent in terms of instantaneous utility.
   (b) From **Lemma 5**, player i's action cannot affect player $-i$'s message.
   (c) $\tilde{\theta}^{-1}\left(h_{i-1,[-1]}\right)$ is independent of player i's action in the monitoring round.
   (d) For $\theta^0,\ldots,\theta^{K+1}$,
      i. If player i's message in the monitoring round for the coordination round does not have any impact on the inference of the deviation of the players, her action cannot affect her continuation payoff.
      ii. If it does, her action does not affect her continuation payoff since it is constant.
2. The first part follows from (20).

■

We will show that playing $\sigma_i^{x_i}$ is a best reply against $\sigma_{-i}^{x_{-i}}$.

Lemma 26 *In the jth coordination round with $j \neq i$, player i is indifferent among all actions for any history.*

Proof. Analogous to **Lemma 18** in Yamamoto (2008b). ■

Lemma 27 *In the ith coordination round, for any $a_i \in A_i \setminus \{a_i^G, a_i^B\}$, player i is indifferent between a_i and a_i^B for any history.*

Proof. Analogous to **Lemma 18** in Yamamoto (2008b). ■

Without loss of generality, consider a particular $x_{-i} \in X_{-i}^B$. Let $W_i\left(\sigma_{-i}^{x_{-i}}, \tau\right)$ denote player i's unnormalized payoff in the auxiliary scenario against $\sigma_{-i}^{x_{-i}}$ when player i follow a sequence

$$(\underbrace{a_i^B, \ldots, a_i^B}_{\tau}, \underbrace{a_i^G, \ldots, a_i^G}_{T-\tau})$$

in the ith coordination round and chooses a best reply in the other periods. Specifically,

$$\begin{aligned} & W_i\left(\sigma_{-i}^{x_{-i}}, \tau\right) \\ = \; & \delta^{T_{-1}+T_0} \sum_{h_{-i,[0]} \in H_{-i,[0]}} \Pr\left(h_{-i,[0]} | \sigma_{-i}^{x_{-i}}, \tau\right) V_i\left(h_{-i,[0]}\right) \\ & + \delta^{T_{-1}+T_0} \sum_{\tilde{x} \in X^B} \sum_{h_{i-1,[-1]} \in H_{i-1,[-1]}(\tilde{x})} \Pr\left(h_{i-1,[-1]} | \sigma_{-i}^{x_{-i}}, \tau\right) \lambda^{-1}(\tilde{x}), \end{aligned}$$

where $\Pr\left(h_{-i,[0]} | \sigma_{-i}^{x_{-i}}, \tau\right)$ and $\Pr\left(h_{i-1,[-1]} | \sigma_{-i}^{x_{-i}}, \tau\right)$ denote the occurrence probability of $h_{-i,[0]}$ and $h_{i-1,[-1]}$. Since $V_i\left(h_{-i,[0]}\right)$ only depends on $J_{-i,0}$,

$$\begin{aligned} & W_i\left(\sigma_{-i}^{x_{-i}}, \tau\right) \\ = \; & \delta^{T_{-1}+T_0} \sum_{J_{-i,0}} \Pr\left(J_{-i,0} | \sigma_{-i}^{x_{-i}}, \tau\right) V_i\left(J_{-i,0}\right) \\ & + \delta^{T_{-1}+T_0} \sum_{\tilde{x} \in X^B} \sum_{h_{i-1,[-1]} \in H_{i-1,[-1]}(\tilde{x})} \Pr\left(h_{i-1,[-1]} | \sigma_{-i}^{x_{-i}}, \tau\right) \lambda^{-1}(\tilde{x}), \end{aligned}$$

Consider

$$\begin{aligned} & W_i\left(\sigma_{-i}^{x_{-i}}, \tau\right) - W_i\left(\sigma_{-i}^{x_{-i}}, \tau - 1\right) \\ = \; & \delta^{T_{-1}+T_0} \sum_{J_{-i,0}} \left(\Pr\left(J_{-i,0} | \sigma_{-i}^{x_{-i}}, \tau\right) - \Pr\left(J_{-i,0} | \sigma_{-i}^{x_{-i}}, \tau - 1\right)\right) V_i\left(J_{-i,0}\right) \\ & + \delta^{T_{-1}} \sum_{\tilde{x} \in X^B} \sum_{h_{i-1,[-1]} \in H_{i-1,[-1]}(\tilde{x})} \left(\Pr\left(h_{i-1,[-1]} | \sigma_{-i}^{x_{-i}}, \tau\right) - \Pr\left(h_{i-1,[-1]} | \sigma_{-i}^{x_{-i}}, \tau - 1\right)\right) \lambda^{-1}\left(\tilde{x}\right). \end{aligned}$$

Let $\Delta_3(\tau)$ be the terms in the first line of the right hand side and $\Delta_4(\tau)$ be the remaining term. In addition, let $\Delta_3\left(\tau, \omega_i^{NT}\right)$ and $\Delta_4\left(\tau, \omega_i^{NT}\right)$ be the value of $\Delta_3(\tau)$ and $\Delta_4(\tau)$ conditional on ω_i^{NT}.

Lemma 28 *$\Delta_3(\tau) = \Delta_3\left(\tau, \omega_i^{NT}\right)$ and $\Delta_4(\tau) = \Delta_4\left(\tau, \omega_i^{NT}\right)$ for all ω_i^{NT}.*

Proof. Note that each probability is conditional on $\sigma_{-i}^{x_{-i}}$.

1. Proof of $\Delta_3(\tau) = \Delta_3\left(\tau, \omega_i^{NT}\right)$
   (a) $\Delta_3\left(\tau, \omega_i^{NT}\right)$ only depends on $((x^0(n, j, -(n, j, i)))_{j \neq i,n})_{n \neq i}$.
   (b) From **Lemma 6,** $((x^0(n, j, -(n, j, i)))_{j \neq i,n})_{n \neq i}$ is independent of ω_i^{NT}.
2. Proof of $\Delta_4(\tau) = \Delta_4\left(\tau, \omega_i^{NT}\right)$
   (a) $\Delta_4\left(\tau, \omega_i^{NT}\right)$ only depends on $(x^0(i-1, j, -(i-1, j, i)))_{j \neq i}$.
   (b) From **Lemma 6,** $(x^0(i-1, j, -(i-1, j, i)))_{j \neq i}$ is independent of ω_i^{NT}.

■

Therefore, from **Lemmas 26**, **27**, and **28**, it suffices to show that $W_i\left(\sigma_{-i}^{x_{-i}}, \tau\right)$ takes its maximum at $\tau = 0$ and T. The following lemmas establish the result.

Lemma 29 *There exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\overline{\tau}$ such that $\Delta_4(\tau)$ is negative for all $\tau \leq \overline{\tau}$ and is positive for all $\tau > \overline{\tau}$.*

Proof. Analogous to **Lemma 22** in Yamamoto (2008b). ■

Lemma 30 *$\lim_{T\to\infty}\Delta_4(1) = -\infty$ and $\lim_{T\to\infty}\Delta_4(NT) = \infty$.*

Proof. Analogous to **Lemma 23** in Yamamoto (2008b). ■

Lemma 31 *For any n, $\max_{\tau\in\{1,\dots,T\}}|\Delta_3(\tau)| = o(T^{-n})$.*

Proof. Since $V_i(J_{-i,0})$ is constant for all $J_{-i,[0]} \in \overline{H}_{-i,[0]}$ and $V_i(J_{i,0}) = O(T)$, it is sufficient to show that

$$\max_{\tau\in\{1,\dots,T\}} \sum_{J_{-i,[0]}\notin\overline{H}_{-i,[0]}} \left|\Pr\left(J_{-i,[0]} \mid \tau\right) - \Pr\left(J_{-i,[0]} \mid \tau-1\right)\right| = o\left(T^{-n}\right).$$

$J_{-i,[0]} \notin \overline{H}_{-i,[0]}$ only if

1. There exist n, l, n', and l' such that $x^0(n,j,-(n,j,l)) \neq x^0(n',j,-(n,j,l'))$ with $j \neq i$.
2. There exist n, l, n', and l' such that $x^0(n,i,-(n,j,l)) \neq x^0(n',i,-(n,j,l'))$.
3. $x^0(n,j,-(n,j,l)) = x^0(n',j,-(n,j,l'))$ for all n, l, n', l', and j but there exist n, l, n', and l' such that $\hat{x}^0(n,j,-(n,j,l)) \neq \hat{x}^0(n',j,-(n,j,l'))$ for some j.

From **Claim 6** in Yamamoto (2008b), the probability of the first case is $o(T^{-n})$. From **Claim 7** in Yamamoto (2008b), the probability of the second case is $o(T^{-n})$. From **Claim 8** in Yamamoto (2008b), the probability of the third case is $o(T^{-n})$. ■

Determination of $\overline{\delta}$ From the above argument, there exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(\overline{\delta}, 1\right)$ and for all $i \in I$, for all $l \geq 0$, $h_{t_l} \in H_{t_l}$, $h_{i-1,T_b+1} \in H_{i-1,T_b+1}$, and $x \in X$ with $x_{i-1} = B$, (6), (7), and (8) holds. In addition, from (17),

$$U_i\left(h_{i-1,T_b+1}\right) \geq 0 \text{ for all } h_{i-1,T_b+1}.$$

Fix some $T > \overline{T}$. Then, since $U_i\left(h_{i-1,T_b+1}\right) \leq 2T_b\overline{u}$ for all h_{i-1,T_b+1}, for sufficiently large δ,

$$U_i\left(h_{i-1,T_b+1}\right) < \frac{\overline{w}_i - \underline{w}_i}{1-\delta} \text{ for all } h_{i-1,T_b+1} \in H_{i-1,T_b+1},$$

which means (9) holds.

10.5 Reward Functions for the Good Strategy

To simplify the notation, let X^G be the set of all $x \in X$ satisfying $x_{i-1} = G$ and X_{-i}^G be the set of all $x_{-i} \in X_{-i}$ satisfying $x_{i-1} = G$. Let $\overline{H}_{-i,[0]} \equiv \bigcup_{x\in X^G} H_{-i,[0]}(x)$ and $\overline{H}_{-i,[k]} \equiv \bigcup_{x\in X^G} \left(H_{-i,[k]}(x) \cup H_{-i,[k]}(x,i)\right)$ for $k \in \{1,\ldots,K\}$. See Subsection 10.4 for the definition of $H_{-i,[0]}(x)$ and $H_{-i,[k]}(x) \cup H_{-i,[k]}(x,i)$ for $k \in \{1,\ldots,K\}$.

Without loss of generality, consider a particular $i \in I$. Suppose that U_i^G is decomposable into real-valued functions $\left(\theta^{-1},\ldots,\theta^{K+1}\right)$ as in Subsection 10.4. We specify $\left(\theta^{-1},\ldots,\theta^{K+1}\right)$ so that **Lemma 9** holds.

Let θ^0 and θ^{K+1} be as in the proof of **Lemma 8**, that is, these transfers are the discounted sums of u_i. Then, (11) holds for the confirmation round and the report round.

The sequence of transfers $\left(\theta^1,\ldots,\theta^K\right)$ is specified by backward induction. Note that θ^{K+1} is determined so that **Lemma 9** holds. To define θ^k, assume that the sequence $\left(\theta^{k+1},\ldots,\theta^{K+1}\right)$ is determined so that player i's continuation payoff after history $h_{i,[k]}$, augmented by $\left(\theta^{k+1},\ldots,\theta^{K+1}\right)$, is equal to $V_i\left(h_{-i,[k]}\right)$. Here, for each $k \in \{1,\ldots,K\}$ and $h_{-i,[k]} \in \overline{H}_{-i,[k]}$, the value $V_i\left(h_{-i,[k]}\right)$ is defined to be the minimum of player i's continuation payoff after history $h_{-i,[k]}$ over all her continuation strategies in the set $\{\sigma_{i,T_b} \mid h_{i,[k]} \mid \forall h_{i,[k]} \in H_{i,[k]}, \forall \sigma_{i,T_b} \in \mathcal{S}_{i,T_b}\}$, subject to the constraints that monitoring is perfect and that payoffs in the communication rounds and monitoring rounds are replaced with 0. For each $h_{-i,[0]} \in \overline{H}_{-i,[0]}$, the value $V_i\left(h_{-i,[0]}\right)$ is defined to be the minimum of player i's continuation payoff after history $\tilde{h}_{-i,[0]}$ over all $\tilde{h}_{-i,[0]} \in \overline{H}_{-i,[0]}$ and over all her continuation strategies in the set $\{\sigma_{i,T_b} \mid h_{i,[0]} \mid \forall h_{i,[0]} \in H_{i,[0]}, \forall \sigma_{i,T_b} \in \mathcal{S}_{i,T_b}\}$, subject to the constraints that monitoring is perfect and that payoffs in the communication rounds and monitoring rounds are replaced with 0. For each each $k \in \{0, \ldots, K\}$ and $h_{-i,[k]} \notin \overline{H}_{-i,[k]}$, the value $V_i\left(h_{-i,[k]}\right)$ is defined to be player i's continuation payoff when she earns $-\overline{u}$ in periods of the main rounds and zero in the other periods.

Suppose that θ^k is decomposed as (21). Then, from the analogous argument in **Lemma 18**, (11) holds for the kth monitoring round and kth supplemental round for all history.

To specify $\tilde{\theta}^k$, the following notion is useful. For each $h_{-i,[k-1]} \in H_{-i,[k-1]}$ and $a_i \in A_i$, let $\tilde{W}_i\left(h_{-i,[k-1]}, a_i\right)$ denote player i's continuation payoff from the kth main round, augmented by $\left(\theta^{k+1}, \ldots, \theta^{K+1}\right)$ and by the second term of (21) when player i plays a_i constantly in the kth main round and plays a best reply thereafter. As before, we can write $\tilde{W}_i\left(J_{-i,k-1}, a_i\right)$ instead of $\tilde{W}_i\left(h_{-i,[k-1]}, a_i\right)$.

For each $J_{-i,k-1}$, let $\{a_i^1\left(J_{-i,k-1}\right), \ldots, a_i^{\left|\mathcal{A}_i^k\right|}\left(J_{-i,k-1}\right)\}$ be a sequence of all the elements of $\mathcal{A}_i^k$ such that

$$\lim_{T\to\infty}\lim_{\delta\to 1}\frac{\tilde{W}_i\left(J_{-i,k-1}, a_i^1\left(J_{-i,k-1}\right)\right)}{T} \geq \cdots \geq \lim_{T\to\infty}\lim_{\delta\to 1}\frac{\tilde{W}_i(J_{-i,k-1}, a_i^{\left|\mathcal{A}_i^k\right|}\left(J_{-i,k-1}\right))}{T}.$$

Then define $\tilde{\theta}^k$ to be

$$\begin{aligned}
&\tilde{\theta}^k\left(h_{i-1,[k,m]}, I_{-i,k-1}, \hat{\Omega}_{-(i-1,i,l)}\right) \\
= \quad & -T_b C + \sum_{a_i \in \mathcal{A}_i^k} 1_{a_i}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) KT\eta + \sum_{n=1}^{\left|A_i^k\right|} 1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right)\lambda^k\left(I_{-i,k-1}, n\right),
\end{aligned}$$

where for each $a_i \in A_i$, let $1_{a_i} : H_{i-1,[k,m]} \times \hat{\Omega}_{l(i-1,i)} \to \{0,1\}$ is the indicator function such that $1_{a_i}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) = 1$ if the random event $\sum_{t=1}^{KT}\psi_{-(i-1,l)}^{3,2}\left(\{a_i\}, t\right) > Z_{KT}$ in the kth monitoring round and $1_{a_i}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) = 0$ otherwise. Likewise, $1_{\left[I_{-i,k-1},n\right]} : H_{i-1,[k,m]} \times \hat{\Omega}_{-(i-1,i,l)} \to \{0,1\}$ is the indicator function such that $1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) = 1$ if the random event $\sum_{t=1}^{KT}\psi_{-(i-1,l)}^{3,2}(\{a_i^n\left(I_{-i,k-1}\right), \ldots, a_i^{|A_i|}\left(I_{-i,k-1}\right)\}, t) > Z_{KT}$ in the kth monitoring round and $1_{\left[I_{-i,k-1},n\right]}\left(h_{i-1,[k,m]}, \hat{\Omega}_{-(i-1,i,l)}\right) = 0$ otherwise. Here, C is a real number satisfying $C > 2\overline{u}$ and $\lambda^k(I_{-i,k-1}, n)$ solves

$$\begin{aligned} & V_i(J_{-i,k-1}) \\ = \quad & \tilde{W}_i(J_{-i,k-1}, a_i) - \delta^{KT+T_m} CT_b + \delta^{KT+T_m} \sum_{\tilde{a}_i \in A_i^k} \Pr(1_{\tilde{a}_i} \mid J_{-i,k-1}, a_i) KT\eta \\ & + \delta^{KT+T_m} \sum_{I_{-i,k-1}} \Pr(I_{-i,k-1} \mid J_{-i,k-1}) \sum_{n=1}^{|A_i^k|} \Pr\left(1_{[I_{-i,k-1}, n]} \mid J_{-i,k-1}, a_i\right) \lambda^k(I_{-i,k-1}, n) \end{aligned} \tag{25}$$

for all $J_{-i,k-1}$ and $a_i \in A_i$.

Lemma 32 *For all K, there exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\bar{\delta} \in (0,1)$ such that (25) has a unique solution and for all $\left(h_{i-1,[k]}, I_{-i,k-1}\right)$,*

$$\theta^k\left(h_{i-1,[k]}, I_{-i,k-1}\right) < (T_k - KT)\,\overline{u} + \left|\mathcal{A}_i^k\right| KT\eta.$$

In addition, the constant action is optimal if that action is included in $\mathcal{A}_i^k$.

Proof. Analogous to **Lemmas 19**, **20**, **21**, **22**, and **23** and **Lemma 24** in Yamamoto (2008b). ■

Then, suppose that θ^{-1} is decomposable as

$$\theta^{-1}\left(h_{i-1,[-1]}\right) = \sum_{x \in X^G} 1_x\left(h_{i-1,[-1]}\right) \lambda^{-1}(x) + \sum_{t=1}^{T_{-1}} \frac{u_i(a_{i-1,t}, \omega_{i-1,t})}{\delta^{T_{-1}+1-t}},$$

where $(a_{i-1,t}, \omega_{i-1,t})$ is player $(i-1)$'s action and signal in the tth period of the block game and $\left(\lambda^{-1}(x)\right)_{x \in X_B}$ to solve

$$\begin{aligned} & \sum_{t=1}^{T_b} \delta^{t-1} \overline{w}_i = -T_b\left(\eta + \lim_{\delta \to 1}\left(\overline{w}_i^P(\delta) - \overline{w}_i\right)\right) + \delta^{T_{-1}} \sum_{\tilde{x} \in X^B} \sum_{h_{i-1,[-1]} \in H_{i-1,[-1]}(\tilde{x})} \Pr\left(h_{i-1,[-1]} | \sigma^x\right) \lambda^{-1}(\tilde{x}) \\ & + \delta^{T_{-1}+T_0} \sum_{h_{-i,[0]} \in H_{-i,[0]}} \Pr\left(h_{-i,[0]} | \sigma^x\right) V_i\left(h_{-i,[0]}\right) \end{aligned} \tag{26}$$

for all $x_{-i} \in X_{-i}^G$. Then, the analogous argument to **Lemmas 25**, **26**, **27**, and **28** and **Lemma 25** in Yamamoto (2008b) establishes the following lemma.

Lemma 33 *1. Under U_i^G, (11) holds for the monitoring round for the coordination round.*

2. There exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\bar{\delta} \in (0, 1)$ such that (26) has a unique solution and

$$\theta^{-1}\left(h_{i-1,[-1]}\right) < T_{-1}\overline{u} - (|A_i| + 1)\, T_b \eta$$

3. Constant action $a_i^{x_i}$ is optimal for all $x_{-i} \in X_{-i}$.

From the above argument, there exists $\overline{T}$ such that for all $T > \overline{T}$, there exists $\overline{\delta} < 1$ such that for all $\delta \in \left(\overline{\delta}, 1\right)$ and for all $i \in I$, for all $l \geq 0$, $h_{t_l} \in H_{t_l}$, $h_{i-1,T_b+1} \in H_{i-1,T_b+1}$, and $x \in X$ with $x_{i-1} = B$, (10), (11), and (12) hold. In addition, from (18),

$$U_i^B\left(h_{i-1,T_b+1}\right) \leq 0 \text{ for all } h_{i-1,T_b}.$$

Fix some $T > \overline{T}$. Then, since $U_i\left(h_{i-1,T_b+1}\right) \geq -2T_b\overline{u}$ for all h_{i-1,T_b+1}, for sufficiently large δ,

$$U_i\left(h_{i-1,T_b+1}\right) > -\frac{\overline{w}_i - \underline{w}_i}{1-\delta} \text{ for all } h_{i-1,T_b} \in H_{i-1,T_b},$$

which means (13) holds.

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