Optimality of FDMA in Gaussian Multiple-Access Channels with Non-zero SNR Margin and Gap

Sumanth Jagannathan, John M. Cioffi
Department of Electrical Engineering, Stanford University, Stanford, CA 94305
{sumanthj, cioffi}@stanford.edu

***Abstract*— This paper investigates the effect of non-zero SNR margin and coding gap on the maximum sum rate and achievable rate region of the Gaussian Multiple-Access Channel (MAC). Frequency-division multiple-access (FDMA) is shown to be the only sum-rate-maximizing scheme for a scalar Gaussian MAC when the margin/gap is not zero. Furthermore, successive decoding is shown to be strictly sub-optimal from a sum rate perspective. With fixed probability-of-error, single-user codes with finite gap are sufficient to achieve the maximum sum rate. The achievable rate region of the scalar Gaussian MAC with non-zero margin/gap is also characterized. Unlike the capacity region of a scalar MAC, the optimal achievable rate region with non-zero margin/gap is not a pentagon. Finally it is shown that when there is non-zero margin/gap, the optimality of the FDMA scheme can be generalized to a Gaussian MAC with inter-symbol interference (ISI).**

I. Introduction

The area of multi-user communications is assuming great importance not only on the academic front but also in the industry (e.g. next generation digital subscriber loop (DSL) systems employing dynamic spectrum management techniques [1]). Specifically, significant progress has been made in characterizing the capacity region of the Gaussian multiple-access and broadcast channels. This includes capacity-achieving precoding/decoding techniques [2] as well as algorithms to determine the optimal spectrum allocation in order to maximize the weighted sum rate under energy constraints or to minimize the power while achieving any point in the capacity region ([3], [4], [5], etc.). These algorithms are predominantly designed for the case when capacity-achieving codes are used.

However, codes that are used in practice have a finite gap to capacity. For a variety of uncoded and coded modulations, this gap to capacity can be approximated by a scaling factor (Γ) applied to the signal to noise ratio (SNR) when the probability-of-error is fixed [6]. Moreover, for a number of uncoded modulation schemes such as PAM and QAM, the gap depends only on the probability-of-error (P_e) and is independent of the SNR. For example, Γ=9.5 dB at P_e=10^{-7} for uncoded PAM/QAM. For many coded modulations, Γ is a function of SNR, however, it can be approximated to be constant over a useful range of SNRs. The gap concept allows a designer to optimize the system under the constraint of a desired maximum probability-of-error (for example, many DSL systems require P_e=10^{-7}). By replacing the SNR in the rate equations by $\frac{SNR}{\Gamma}$, the maximum probability-of-error requirement can be easily incorporated. The new rate equations with modified SNR can now be used in any optimization setting in the same way as would be done for the capacity achieving code ($\Gamma = 0$). Optimal encoding and decoding schemes can now be designed while also satisfying the maximum P_e requirement.

Another concept that is used in practical wireline communication system design is the SNR margin. The SNR margin is defined as the amount by which the SNR can be reduced while still maintaining the maximum probability-of-error. Typical DSL systems operate at a SNR margin of 6 dB. This can be thought of as a safety margin in order to maintain the probability-of-error while taking into account the variability in the noise level. The margin and coding gap both appear as scaling factors by which the SNR is reduced and the product of the two can be treated as the effective gap in the system.

In order to optimize multi-user communication systems with a target P_e, the gap concept can be applied to each of the individual user's SNRs. However, such a formulation cannot be directly applied to the joint mutual information since the gap concept has been validated for only for single user modulation and coding. In a scalar Gaussian multiple-access channel, the gap concept can be incorporated by assuming successive decoding (which is optimal in terms of sum rate) and introducing the gap into each of the users's signal-to-interference-and-noise ratios (SINRs). Fung *et. al.* [7] have recently applied the gap concept in a similar manner to Gaussian broadcast channels by assuming an encoding order and Tomlinson-Harashima precoding. However, such a decomposition may not be optimal and also leads to a non-convexity in the sum rate optimization problem. This paper investigates the optimality of such a decomposition for the scalar Gaussian MAC with non-zero margin/gap.

The remainder of the paper is organized as follows. The system model is described in Section II, and the capacity region of the scalar Gaussian MAC is described in section III. The optimal sum rate of the MAC with non-zero gap is presented in section IV. The achievable rate region for the MAC with non-zero gap is characterized in section V. The optimal sum rate of the ISI MAC with non-zero margin/gap is characterized in section VI. Finally, some concluding remarks are given in Section VII. In this paper, the proofs are presented for the two-user case. The results can be easily extended to the case where there are more than two users by induction.

II. The system model

Consider a two-user scalar Gaussian multiple-access channel as shown in Fig. 1. The channels for users 1 and 2 are denoted by h_1 and h_2 respectively, and the transmit powers

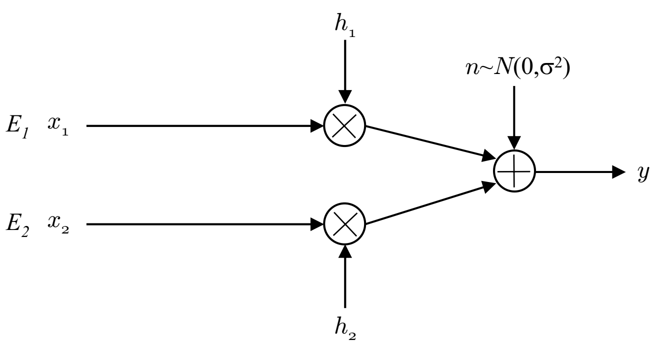

Fig. 1. A 2-user scalar Gaussian Multiple-Access Channel

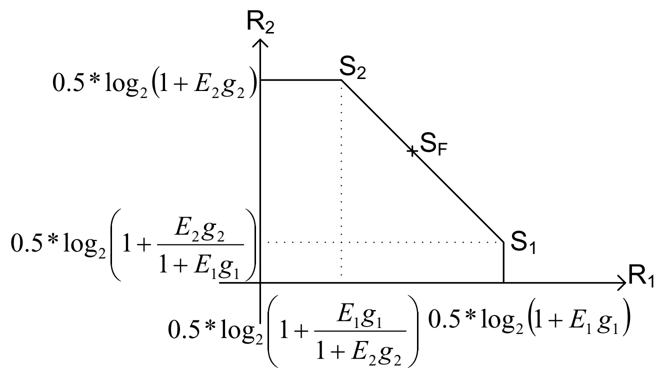

Fig. 2. Capacity region of a 2-user scalar MAC ($\gamma\Gamma = 0$ dB)

for the two users are $\mathcal{E}_1$ and $\mathcal{E}_2$. Without loss of generality, the bandwidth is normalized to unity. The noise at the receiver is an additive white Gaussian noise (AWGN) with zero mean and variance σ^2. The channel signal to noise ratios (CSNRs) for the two users are defined as $g_1 = \frac{|h_1|^2}{\sigma^2}$ and $g_2 = \frac{|h_2|^2}{\sigma^2}$. The SNR margin is denoted by γ and the gap of the codes used by the two users is denoted by Γ.

III. Capacity Region of a Scalar Gaussian MAC

The capacity region (corresponding to $\gamma\Gamma = 0$ dB) of a scalar Gaussian MAC is a pentagon [8] (Fig. 2), the corner points of which are determined by successive decoding using the two decoding orders $(1, 2)$ (point S_2) and $(2, 1)$ (point S_1). For decoding order (i, j), the receiver first decodes user i treating user j as interference. The decoded message of user i is then subtracted from the received signal, and user j is then decoded in the presence of the remaining AWGN noise. Furthermore, each user transmits at full power in order to achieve the sum capacity (also equal to the joint mutual information), which is $\frac{1}{2}\log_2(1 + \mathcal{E}_1 g_1 + \mathcal{E}_2 g_2)$. The maximum sum rate is achieved by the successive decoding points S_1 and S_2 as well as all points on the 45^o line joining these two corner points. In particular, a FDMA scheme achieves one of the points (S_F) on this maximum sum rate line. The other rate points on the 45^o line can be achieved by time sharing between the two successive decoding points or by rate splitting [9].

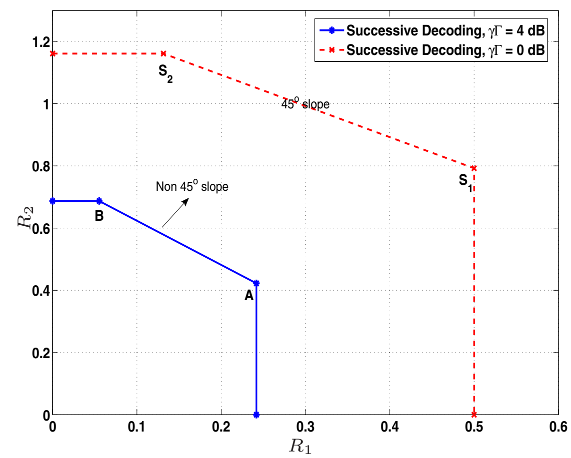

Fig. 3. Successive decoding (with time sharing) based achievable rate region of a 2-user scalar MAC with $\mathcal{E}_1 g_1 = 1$ and $\mathcal{E}_2 g_2 = 4$

IV. Optimal Sum Rate of a Scalar Gaussian MAC with Non-zero Gap

When there is a non-zero margin or gap (i.e. $\gamma\Gamma > 1$ in linear units), the rate pairs achieved by successive decoding (with each user utilizing its full power) are given by

$$(R_1^A, R_2^A) = \left(\frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_1 g_1}{\gamma\Gamma}\right), \frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_2 g_2}{\gamma\Gamma(\mathcal{E}_1 g_1 + 1)}\right)\right) \tag{1}$$

$$(R_1^B, R_2^B) = \left(\frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_1 g_1}{\gamma\Gamma(\mathcal{E}_2 g_2 + 1)}\right), \frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_2 g_2}{\gamma\Gamma}\right)\right) \tag{2}$$

where point A corresponds to the decoding order $(2, 1)$ and point B corresponds to the decoding order $(1, 2)$. Fig. 3 shows the successive decoding corner points when there is a non-zero margin/gap. The line joining the two corner points (A and B) no longer has a slope of 45^o.

Consider an FDMA scheme where a fraction $0 \le \alpha \le 1$ of the bandwidth is allocated to user 1 and the remaining bandwidth is allocated to user 2. The rate pairs achieved by such a scheme are given by

$$(R_1^{FDMA}(\alpha), R_2^{FDMA}(\alpha)) = \left(\alpha\frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_1 g_1}{\alpha\gamma\Gamma}\right), (1-\alpha)\frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_2 g_2}{(1-\alpha)\gamma\Gamma}\right)\right) \tag{3}$$

The reduction in bandwidth for each user is represented by the scale factor outside the log term. Each user transmits its full power within their respective allocated frequency bands.

Theorem 1: The optimal sum rate for a scalar MAC channel with non-zero margin/gap is given by

$$R_1 + R_2 = \frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_1 g_1 + \mathcal{E}_2 g_2}{\gamma\Gamma}\right) \tag{4}$$

and is achieved by FDMA. Moreover, successive decoding leads to a strictly sub-optimal sum rate when $\gamma\Gamma > 0$ dB.

Proof: The sum rates obtained by successive decoding at the corner points A and B are

$$R_1^A + R_2^A = \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma}\right)+\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{\gamma\Gamma(\mathcal{E}_1 g_1+1)}\right) \tag{5}$$

$$R_1^B + R_2^B = \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma(\mathcal{E}_2 g_2+1)}\right)+\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{\gamma\Gamma}\right) \tag{6}$$

while the sum rate achieved by FDMA as a function of α is given by

$$\begin{aligned} R_{\text{sum}}^{FDMA}(\alpha) &= R_1^{FDMA}(\alpha)+R_2^{FDMA}(\alpha)\\ &= \alpha\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\alpha\gamma\Gamma}\right)+\\ &\quad (1-\alpha)\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{(1-\alpha)\gamma\Gamma}\right)\end{aligned} \tag{7}$$

Each term in (7) has the same form as the sum rate obtained by FDMA in a scalar MAC with zero margin and gap and with modified power constraints $\frac{\mathcal{E}_1}{\gamma\Gamma}$ and $\frac{\mathcal{E}_2}{\gamma\Gamma}$ for the two users. Therefore, as in the zero gap case, the maximum sum rate is obtained by choosing $\alpha = \frac{\mathcal{E}_1 g_1}{\mathcal{E}_1 g_1+\mathcal{E}_2 g_2}$ [8], [2] and the maximum sum rate achieved by FDMA is

$$R_{\text{sum}}^{FDMA}(\max) = \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1+\mathcal{E}_2 g_2}{\gamma\Gamma}\right) \tag{8}$$

Note that the sum rate achieved by FDMA is same as the rate obtained by applied the margin/gap directly to the joint mutual information. Thus, the FDMA scheme along with single user codes seems to act like an effective multi-user code for the scalar Gaussian MAC with a multi-user gap (=$\gamma\Gamma$).

Now, for the successive decoding point A (decoding order $(2,1)$), the effective noise seen by user 2 is $\gamma\Gamma(\mathcal{E}_1 g_1+1)$ which is strictly greater than $(\gamma\Gamma+\mathcal{E}_1 g_1)$ for $\gamma\Gamma > 0$ dB i.e $\gamma\Gamma > 1$ on a linear scale. Therefore, the sum rate (5) obtained by corner point A is strictly upper bounded by

$$\begin{aligned} R_1^A + R_2^A &< \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma}\right)+\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{\gamma\Gamma+\mathcal{E}_1 g_1}\right)\\ &= \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1+\mathcal{E}_2 g_2}{\gamma\Gamma}\right)\\ &= R_{\text{sum}}^{FDMA}(\max)\end{aligned}$$

Similarly, for the corner point B (i.e decoding order $(1,2)$), we obtain $R_1^B + R_2^B < R_{\text{sum}}^{FDMA}(\max)$. Thus with non-zero magin/gap, both the successive decoding points A and B result in a lower sum rate than the maximum FDMA sum rate.

Till now, we have assumed that each user transmits its full power to obtain the successive decoding points A and B. Consider the decoding order $(1,2)$. If user 1 reduces its power, then the resulting rate pair B' is within the pentagon defined by the points A and B (Fig. 4) since user 1 is decoded first anyway and does not affect user 2. That is $R_1^{B'} < R_1^B$ as user 1 sees the same interference but has lesser power while the rate for user 2 remains the same i.e. $R_2^{B'} = R_2^B$.

On the other hand, if the power of user 2 is reduced by a factor $\delta < 1$, then the following rate pair is obtained (B'').

$$\begin{aligned}&(R_1^{B''}, R_2^{B''}) =\\ &\left(\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma(\delta\mathcal{E}_2 g_2+1)}\right), \frac{1}{2}\log_2\left(1+\frac{\delta\mathcal{E}_2 g_2}{\gamma\Gamma}\right)\right)\end{aligned} \tag{9}$$

Observe that while the rate of user 2 decreases as $\delta < 1$, user 1's rate increases and it is not clear that the sum rate decreases compared to points A and B. Therefore, we need to prove that the sum rate achieved by successive decoding with the power scaling is still less than the optimal FDMA sum rate. We will prove this for the decoding order $(1,2)$ and the same proof will hold for the decoding order $(2,1)$ as well.

Lemma 1: For $0 \le \delta \le 1$, the rate pair $(R_1^{B''}, R_2^{B''})$ is strictly within the pentagon defined by the points A and B as corner points.

Proof: From the equation (9), we express $\delta\mathcal{E}_2 g_2$ in terms of $R_2^{B''}$ as

$$\delta\mathcal{E}_2 g_2 = \gamma\Gamma\left(2^{2R_2^{B''}}-1\right) \tag{10}$$

Substituting this expression into the equation for $R_1^{B''}$, we obtain

$$R_1^{B''} = \frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma\left(\gamma\Gamma\left(2^{2R_2^{B''}}-1\right)+1\right)}\right) \tag{11}$$

$$= \frac{1}{2}\log_2\left(1+\frac{1}{a2^x-b}\right) \tag{12}$$

where

$$a = \frac{(\gamma\Gamma)^2}{\mathcal{E}_1 g_1}, \qquad b = \frac{\gamma\Gamma(\gamma\Gamma-1)}{\mathcal{E}_1 g_1},$$
$$x = 2R_2^{B''} > 0 \qquad \text{and} \qquad a2^x - b > 0\ \forall x > 0$$

Lemma 2: The function

$$r(x) = \ln\left(1+\frac{1}{ae^x-b}\right) \tag{13}$$

with $a > 0$, $x > 0$, $b > 0$ and $ae^x - b > 0$ is a convex function in x.

Proof: See appendix. ■

Using lemma 2 by replacing ln with $\log_2$ and e^x with 2^x, we find that $R_1^{B''}$ is strictly a convex function of $R_2^{B''}$. As we sweep δ from 0 to 1, we obtain the values of $R_2^{B''}$ between 0 and R_2^B. When $R_2^{B''} = 0$, the rate pair achieved by successive decoding $\left((R_1^{B''}, R_2^{B''})\right)$ is equal to $(R_1^A, 0)$ (denoted by C in Fig. 4) and when $R_2^{B''} = R_2^B$, the rate pair achieved is (R_1^B, R_2^B) (which is point B).

Both these points lie within the pentagon defined by the points A and B. For any rate $R_2^{B''}$ between 0 and R_2^B (i.e. for any convex combination of the 2 extremes of $R_2^{B''}$), the rate pair $(R_1^{B''}, R_2^{B''})$ is below the line BC. This is because $R_1^{B''}$ is a convex function of $R_2^{B''}$ and for any convex function $f(x)$, $f(\beta x_1+(1-\beta)x_2) \le \beta f(x_1)+(1-\beta)f(x_2)$ for $0 \le \beta \le 1$.

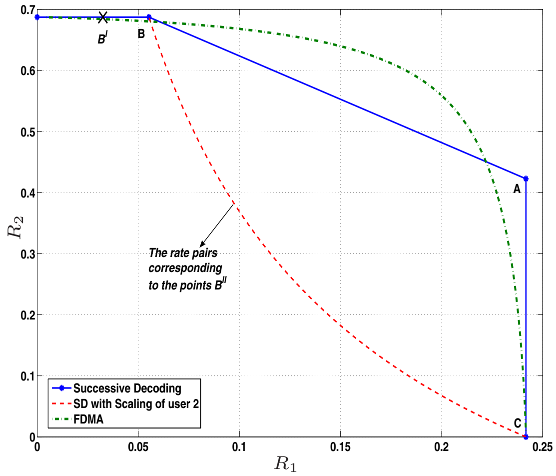

Fig. 4. Successive decoding rate pairs (**A** and **B**) compared with energy scaling for user 2 with $\mathcal{E}_1 g_1$ = 1, $\mathcal{E}_2 g_2$ = 4, $\gamma\Gamma$ = 4 dB

Therefore, the rate pairs $(R_1^{B''}, R_2^{B''})$ lie within the pentagon defined by A and B for $0 \leq \delta \leq 1$ (refer Fig. 4). ∎

Therefore, successive decoding is strictly sub-optimal than FDMA in terms of sum rate. Since rate splitting also involves a sequence of successive decoding steps (after splitting the messages of the users), by applying FDMA at each of those steps, we can obtain a better sum rate. Now, for any other allocation of spectrums to users 1 and 2, if any frequency band is shared by the two users, then successive decoding or rate splitting needs to be used in that band. However, we can improve the total sum rate by using FDMA in this band with the same energies. Continuing with this procedure from any arbitrary spectrums for users 1 and 2, we obtain the result that FDMA is the optimal sum rate achieving scheme. ∎

V. Achievable Rate Region of a Scalar Gaussian MAC with Non-zero Gap

Having characterized the optimal sum rate for the scalar Gaussian MAC with non-zero margin/gap, we now try to characterize the optimal achievable rate region. Fig. 4 compares the FDMA rate region with the successive decoding region when the margin/gap is non-zero. Clearly, the nice pentagon (with the SD points as corner points) that was obtained in the zero margin/gap case is no longer optimal since there are a number of rate pairs achieved by FDMA which are outside the successive decoding pentagon. Therefore, the optimal achievable rate region of a scalar Gaussian MAC with non-zero margin/gap is not a pentagon.

For any spectrum allocation for users 1 and 2, a receiver may successively decode on the frequency bands where the spectra of the two users overlap and use FDMA (decoding of one user) in the remaining bands. Given a flat noise power spectral density over the entire frequency band, since the channels of the two users are flat (scalar channels), we can group the bands that are shared by the 2 users into one band, and the remaining bandwidth will be used for FDMA. All the possible spectra for the 2 users can be characterized in this manner by dividing the bandwidth between successive decoding and FDMA. Now, for any division of the bandwidth (say a fraction β is used for successive decoding and the remaining is used for FDMA), we assume that the energies of the two users are divided into these two bands in proportion to the bandwidth. In other words, users 1 and 2 will use energies $\beta\mathcal{E}_1$ and $\beta\mathcal{E}_2$ respectively for successive decoding and will use energies $(1-\beta)\mathcal{E}_1$ and $(1-\beta)\mathcal{E}_2$ respectively for FDMA. Splitting the energies proportional to the bandwidth is intuitively a reasonable assumption as this analysis considers only scalar channels and a flat noise spectrum.

The rate pairs achieved for any such spectrum allocation are the sum of the rate pairs achieved by successive decoding over the fraction of bandwidth β and the FDMA rate pairs over the remaining bandwidth. The resultant rate pair achieved by successive decoding (for the order $(1,2)$) is

$$\begin{aligned}&(R_1^{SD}, R_2^{SD}) = \\ &\left(\beta\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\gamma\Gamma(\mathcal{E}_2 g_2+1)}\right), \beta\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{\gamma\Gamma}\right)\right) \\ &= \beta\cdot(R_1^B, R_2^B)\end{aligned} \tag{14}$$

Essentially, the factor β cancels inside the log terms as the energies of both users are scaled by β and the noise is also reduced by β since the bandwidth is reduced to that fraction. Similarly, the FDMA rate pairs are

$$\begin{aligned}&(R_1^{FDMA}(\alpha,\beta), R_2^{FDMA}(\alpha,\beta)) = \\ &\left((1-\beta)\alpha\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_1 g_1}{\alpha\gamma\Gamma}\right),\right. \\ &\qquad \left.(1-\beta)(1-\alpha)\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_2 g_2}{(1-\alpha)\gamma\Gamma}\right)\right) \\ &\qquad = (1-\beta)\cdot(R_1^{FDMA}(\alpha), R_2^{FDMA}(\alpha))\end{aligned} \tag{15}$$

Therefore, the net rate pairs (R_1, R_2) are given by

$$(R_1, R_2) = \beta\cdot(R_1^B, R_2^B) + (1-\beta)\cdot(R_1^{FDMA}(\alpha), R_2^{FDMA}(\alpha)) \tag{16}$$

which are convex combinations of the FDMA rate pairs (using full bandwidth) with the successive decoding rate pair (point B). These rate pairs therefore lie on the line joining the successive decoding point (B) and the FDMA rate pairs. A similar analysis holds for the decoding order $(1,2)$ (point A). Hence the boundary of the rate region will be characterized by the tangents from the successive decoding points A and B to the FDMA rate region. Fig. 5 shows the achievable rate region for the scalar MAC with non-zero margin/gap.

VI. Optimal Sum Rate of a Gaussian MAC with ISI

For a Gaussian MAC with ISI with zero margin/gap, Cheng and Verdu proved that there exists an FDMA solution for the sum rate point [10]. Generalization of section IV's argument proves that there exists an FDMA scheme that achieves the sum rate point for a Gaussian MAC with ISI even when there is a non-zero margin/gap. The generalization first performs a tonal decomposition of the ISI channel such that each tone

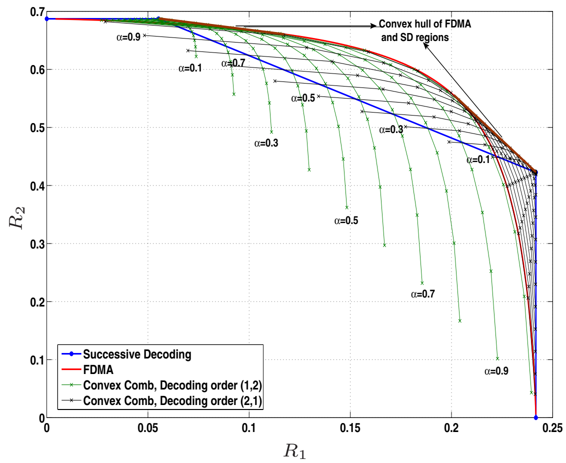

Fig. 5. Best achievable rate region for a scalar Gaussian MAC with $\mathcal{E}_1 g_1$ = 1, $\mathcal{E}_2 g_2$ = 4, $\gamma\Gamma$ = 4 dB

is approximately flat. For any allocation of spectra to users 1 and 2, if any tone is shared by the two users, successive decoding needs to be used for that tone. For such a SD tone, holding all other tones fixed, a design can improve the total sum rate by further dividing this tone and using FDMA in this band with the same energies that were used for successive decoding. Since each tone is considered to be flat, we have a scalar Gaussian MAC on each tone and using FDMA on this tone would lead to the best sum rate. We then move on to the next tone which is shared by the 2 users. Continuing with this procedure over all the tones, we will obtain a spectrum allocation that maximizes the sum rate starting from any arbitrary spectrums for users 1 and 2. Since each tone is used only by one of the users in this optimal spectrum, we obtain the result that FDMA is the optimal sum rate achieving scheme for the Gaussian ISI MAC.

VII. Conclusion

The optimal sum rate of a scalar Gaussian Multiple-Access Channel is characterized for the practical case of non-zero SNR margin and codes with non-zero gap. FDMA achieves this best sum rate and successive decoding is strictly sub-optimal from a sum rate perspective when the margin/gap is non-zero. Also, the achievable rate region for a scalar Gaussian MAC is characterized and is found not to be a pentagon. The FDMA scheme along with single user codes is shown generalize the gap concept to the scalar Gaussian MAC. The optimal sum rate for the Gaussian ISI MAC with non-zero margin/gap was also shown to be achieved using FDMA.

Appendix
Proof of Lemma 1

Let us define the following functions

$$f(x) = \frac{1}{ae^x - b},\ a > 0,\ x > 0,\ b > 0 \text{ and } ae^x - b > 0$$
$$p(x) = \ln(x),\ x > 0 \tag{17}$$

Therefore, $r(x) = p(1 + f(x))$. $r(x)$ is a continuous, differentiable function $\forall\ x > 0$. The first derivative of $r(x)$ is given by

$$\begin{aligned} r'(x) &= p'(1 + f(x)) \cdot f'(x) = \frac{1}{1 + f(x)} \cdot f'(x) \\ &= \frac{1}{1 + \frac{1}{ae^x - b}} \cdot \left(\frac{-ae^x}{(ae^x - b)^2} \right) \\ &= \frac{-ae^x}{(ae^x - b + 1)(ae^x - b)} \end{aligned} \tag{18}$$

Continuing to find the second derivative of $r(x)$, we obtain

$$\begin{aligned} r''(x) &= \frac{-ae^x}{(ae^x - b + 1)(ae^x - b)} + \frac{a^2 e^{2x}}{(ae^x - b + 1)(ae^x - b)^2} \\ &\quad + \frac{a^2 e^{2x}}{(ae^x - b + 1)^2 (ae^x - b)} \\ &= \frac{a^2 e^{2x} - b^2 + b}{(ae^x - b + 1)^2 (ae^x - b)^2} \end{aligned} \tag{19}$$

which is strictly greater than 0 since $a^2 e^{2x} - b^2 + b = (ae^x - b)(ae^x + b) + b > 0$. Therefore, $r(x)$ is strictly a convex function of x.

Acknowledgements

This work was supported by the NSF under contract CNS-0427711 and France Télécom R&D. The authors would like to thank David Yu for helpful discussions.

References

[1] J. M. Cioffi and M. Mohseni, Dynamic spectrum management A methodology for providing significantly higher broadband capacity to the users, *Telenor Telektronikk*, No. 4, pp. 126137, 2004.

[2] J. M. Cioffi, "Lecture Notes: EE479 Multiuser Digital Transmission systems," *Stanford University*, 2005.

[3] D. Tse, S. Hanly, "Multi-Access Fading Channels: Part I: Polymatroid Structure, Optimal Resource Allocation and Throughput Capacities", *IEEE Trans. Info. Theory*, Vol. 44, No. 7, pp. 2796-2815, Nov. 1998.

[4] N. Jindal, W. Rhee, S. Vishwanath, S.A. Jafar, and A. Goldsmith, "Sum Power Iterative Water-filling for Multi-Antenna Gaussian Broadcast Channels", *IEEE Trans. Info. Theory*, Vol. 51, No. 4, pp. 1570-1580, April 2005.

[5] M. Mohseni, R. Zhang, J. M. Cioffi, "Optimized Transmission for Fading Multiple-Access and Broadcast Channels with Multiple Antennas," *IEEE Journal on Selected Areas in Comm.*,Vol. 24, No. 8, pp. 1627-1639, Aug. 2006.

[6] J. M. Cioffi, "Lecture Notes: EE379A Digital Communication," *Stanford University*, 2005.

[7] C. F. Fung, W. Yu and T. J. Lim, "Precoding for the Multi-Antenna Downlink: Multi-user Gap Approximation and Optimal User Ordering," to appear in *IEEE Trans. Comm.*, 2006.

[8] T. M. Cover, J. A Thomas, "Elements of Information Theory," John Wiley & Sons, 1991.

[9] B. Rimoldi, R. Urbanke, "A Rate-Splitting Approach to the Gaussian Multiple-Access Channel ," *IEEE Trans. Info. Theory*, Vol. 42, No. 2, pp. 364–375, 1996.

[10] R. S. Cheng, S. Verdu, "Gaussian Multiaccess Channels with Intersymbol Interference: Capacity Region and Multiuser Water-Filling," *IEEE Trans. Info. Theory*, Vol. IT-39, No. 3, pp. 773-785, May 1993.