Iterative Water-filling for Optimal Resource Allocation in OFDM Multiple-Access and Broadcast Channels

David D. Yu and John M. Cioffi
Department of Electrical Engineering
Stanford University
Stanford, CA 94305 USA
Email: {dyu, cioffi}@stanford.edu

***Abstract*— A class of optimal resource allocation problems in linear Gaussian Multiple-access and Broadcast channels (MAC and BC) can be summarized as weighted sum power minimization problem. In this paper an iterative water-filling algorithm is proposed to solve this problem efficiently. It is shown that by formulating an explicit rate expression for MAC, though non-convex of power spectral densities, the KKT conditions demonstrate a strong water-filling flavor. By iteratively solving the KKT conditions, whereas in each iteration a slightly modified single-user Margin Adaptive Water-Filling(MAWF) algorithm is applied to update the dual variable in a greedy manner, the power spectral density of each user converges to the optimal solution very fast. Simulations verify zero duality gap and fast convergence. The problem in BC can be solved in its dual MAC.**

I. Introduction

The information theoretic characterization of the capacity region of linear Gaussian Multiple-Access and Broadcast Channels(MAC and BC)is very well understood (see [1] for MAC, [2] for BC, and [3] for a general treatment of MAC and BC). The extension of these results to general channel conditions with fading or InterSymbol Interference (ISI), given the assumption that all the transmitter(s) and reciever(s) have complete Channel State Information(CSI) instantaneously, became a problem of optimally allocating resources (power and rate) across a set of parallel and independent dimensions in frequency, time(fading state) and space(multiple antennas) [4][8][9].

Two classes of optimal power allocation problems are of fundamental interest: one is given power constraints, what is optimal power and rate allocation to support the boundary surface of the capacity region; the other is given a rate-tuple, what is the optimal power and rate allocation to support the boundary surface of the power region (see definitions in [7]). The first problem can be casted as a weighted sum rate maximization problem, and the second as a weighted sum power minimization problem. Efficient greedy algorithms were derived based on the polymatroid structure of the MAC capacity region and the contra-polymatroid structure of the MAC power region introduced by Tse and Hanley [4][7][9]. The resource allocation problem in BC can be solved in its dual MAC by the celebrated MAC-BC duality [12][13][9].

The focus of this paper is the weighted sum power minimization problem for SISO MAC and sum power minimization problem for SISO BC, both with ISI. This problem was first raised and answered by Tse and Hanley in [4], which is a natural and crucial question to be addressed under scheduling and Quality of Service (QoS) requirements in a cross-layer design environment. Special cases in which only minimum sum power is considered (equal power weight) have also been studied in other works. In [10], Oh, Kim and Cioffi solved it in two steps: first, assume the best decoding order, which is the same for all frequency tones, is known; then a convex transformation allows a "power adaptive iterative waterfilling" algorithm to apply. Michel and Wunder considered the same problem and used a per-tone based optimal ordering to transform the problem formulation into convex, and a similar Lagrange dual method as in [10] was then developed.

All algorithms above are based on convex optimization techniques, for they provide powerful analyzing and numerical tools. The drawbacks are twofold: as powerful as the algorithm is, it often requires powerful computing platform, which many practical systems can not afford, e.g. the Access Point(AP) of a wireless Lan only has embedded processor with limited computing power; more importantly, although transforming a non-convex problem into convex often relies on the structure of the problem, it may also lose insight to the structure of the solutions. Instead, in this paper we will utilize both the structure of the problem and the structure of the optimal solution to develop a simple and efficient algorithm to fully characterize the boundary surface of any given MAC power region. The algorithm can be easily extended to the sum power minimization problem in BC by using MAC-BC duality.

This work is also motivated by the fact that the single user Rate Adaptive Water-Filling algorithm (RAWF)[14], where given a power constraint maximize the rate, can be directly applied to iteratively solve the sum rate maximization problem in a Gaussian MAC with individual power constraints[6]. We ask the following question: can the single user Margin Adaptive Water-Filling algorithm (MAWF)(see thorough discussions in [14]), where given a target rate minimize the power, be applied to iteratively solve the sum power minimization problem in a Gaussain MAC? The answer is yes, but with slight modification of the single user MAWF algorithm. We will show in this paper that a non-convex formulation of the sum power minimization problem based on optimal decoding order in each subchannel made this extension possible, and intuition and insight can be drawn from the modified MAWF algorithm.

The remaining of this paper is organized as follows: Section II introduces the system model for OFDM MAC , and the sum power minization problem is formulated by optimal orders in each subchannel. The iterative water-filling algorithm based on single user MAWF is developed and verified in Section III. Section IV explains how to use this algorithm to fully characterize the power region and how to use MAC-BC duality to solve sum power minimization problem in BC. Conclusions are drawn in Section V.

II. System Model and Problem Formulation

A. System Model

The basic channel model considered in this paper is a SISO Gaussian multiple access linear channel with intersymbol interference. The channel response from each independent transmitter to a common receiver has finite length. The additive Gaussian noise process at the common receiver is stationary, has zero mean, and its autocorrelation function has finite support (two noise samples are independent if sample time is sufficiently apart). Then conventional OFDM scheme with sufficiently large block length decomposes the channel into a set of parallel independent subchannels. The OFDM scheme is asymptotically optimal as its block length goes to infinity. We define such a channel model as a OFDM MAC. Gaussian signaling is optimal for this channel model.

Consider a OFDM MAC with K transmitters(users), N parallel independent subchannels, and white Gaussian noise with variance σ_n^2, where n denotes subchannel index. In each subchannel n, the inputs and output are related as follows:

$$Y_n = \sum_{k=1}^{K} H_{k,n} X_{k,n} + N_n$$

where Y_n is the received signal, $H_{k,n}$ is user k's channel gain, $X_{k,n}$is user k's transmitted symbol, and N_n is the noise sample. Power constraint can be imposed on each user:

$$\frac{1}{N} \sum_{n=1}^{N} \mathbf{E}(|X_{k,n}|^2) \leq P_k, \forall k$$

Notations will be used in this model: for each user k, $r_{k,n}$ and $p_{k,n}$ denote its rate allocation and power allocation in subchannel n, respectively. $P_k = \sum_{n=1}^{N} p_{k,n}$ is user k's power constraint and $R_k = \sum_{n=1}^{N} r_{k,n}$ is its rate. Given power constraint $\mathbf{P} = [P_1, P_2, \cdots, P_K]$, any rate-tuple $\mathbf{R} = [R_1, R_2, \cdots, R_K]$ is achievable with arbitrarily low probability of error if $\mathbf{R} \in C_{MAC}(\mathbf{H}, \mathbf{P})$, the capacity region of the OFDM MAC, where $\mathbf{H}$ represents the channel model depicted above.

B. Problem Formulation

A general formulation of the weighted sum power minimization problem is as follows. Given a target rate-tuple $\mathbf{R}$ and a weighting vector $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_K]$,

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} \lambda_k P_k \\ \text{subject to} \quad & \mathbf{R} \in C_{MAC}(\mathbf{H}, \mathbf{P}) \end{aligned} \tag{1}$$

It is instructive to study the special case of equal weighting factors, i.e. the sum power minimization problem. The insight gained and numerical algorithm developed from this special case can be easily extended to the general problem (1). A restatement of the sum power minimization problem is:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} P_k \\ \text{subject to} \quad & \mathbf{R} \in C_{MAC}(\mathbf{H}, \mathbf{P}) \end{aligned} \tag{2}$$

Given the optimal solution $\mathbf{P}^*$, it is not hard to see that $\mathbf{R}$ lies on the boundary surface of $C_{MAC}(\mathbf{H}, \mathbf{P}^*)$ [4], implying an optimal global decoding order(though may not be unique) that is the same across all subchannels. Not knowing the optimal decoding order beforehand (there are $K!$ possible orders), it is difficult to form an explicit expression between R and P, an necessary step to solve (2). In [10], the optimal decoding order is assumed known presumably, while in [4] the contra-polymatroid structure of the power region allows searching for the optimal order in a greedy manner. The optimality of successive decoding can be seen as follows. Since $\mathbf{P}^*$ minimizes the sum power that achieves $\mathbf{R}$, $\mathbf{R}$ must also lie on the boundary surface of $C_{MAC}^{SUM}(\mathbf{H}, \sum_{k=1}^{K} P_k^*)$, the capacity region with sum power constraint. Thus $C_{MAC}(\mathbf{H}, \mathbf{P}^*)$ and $C_{MAC}^{SUM}(\mathbf{H}, \sum_{k=1}^{K} P_k^*)$ coincide at this target rate-tuple $\mathbf{R}$. It is well known that contrary to individual power constraints, with sum power constraint successive decoding always achieves every point on the boundary surface of a SISO MAC [12]. An easy way to see this is through the dual BC of each subchannel. Since the dual BC is degraded, superposition encoding and successive decoding with a fixed decoding order, which is solely determined by the channel gains, achieves all the boundary point. Thus by duality successive decoding with a fixed decoding order that is the reverse of the dual BC's decoding order in each subchannel is also optimal to achieve the boundary surface of MAC with sum power constraint.

The argument above is also a proof for that per-subchannel based ordering that is solely based on channel gains in each sub-channel is optimal. Since there is only one global optimal decoding order, these two seemingly contradicting decoding orders suggest that many subchannels will be assigned to less than K users. A thorough discussion on this interesting ordering issue can be found in [11].

The optimal decoding order in each subchannel allows explicit expressions to link power allocation to rate allocation. The following lemma, which was first stated in [5], is the foundation for many weighted sum power minimization algorithms that are developed based on the per-subchannel (called per-"tone" or per-"channel realization" in other works) optimal orders.

Lemma 1: For a scalar Gaussian MAC with channel gain $\mathbf{H} = [H_1, H_2, \cdots, H_K]$ and noise variance σ^2, the optimal decoding order to solve (1) is a permutation $\boldsymbol{\pi}$ such that:

$$\frac{H^2_{\boldsymbol{\pi}(1)}}{\lambda_{\boldsymbol{\pi}(1)}} \geq \frac{H^2_{\boldsymbol{\pi}(2)}}{\lambda_{\boldsymbol{\pi}(2)}} \geq \cdots \geq \frac{H^2_{\boldsymbol{\pi}(K)}}{\lambda_{\boldsymbol{\pi}(K)}}$$

and user $\boldsymbol{\pi}(1)$ decodes first while user $\boldsymbol{\pi}(K)$ decodes last. The users who decode earlier treat the un-decoded received signals as noise. Therefore for the scalar case(1) can be reformulated as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} \lambda_k P_k \\ \text{subject to} \quad & \log_2\left(1+\frac{|H_{\boldsymbol{\pi}(k)}|^2 P_{\boldsymbol{\pi}(k)}}{\sigma^2+\sum_{j=k+1}^{K}|H_{\boldsymbol{\pi}(j)}|^2 P_{\boldsymbol{\pi}(j)}}\right) \\ \geq \quad & R_{\boldsymbol{\pi}(k)}, \forall k; \end{aligned} \tag{3}$$

(3) in *Lemma 1* actually has close form solution because for the optimal solution the inequalities degenerate into equalities. However, for general OFDM MAC the rate allocation among subchannels is what to be solved and close form solution does not exist. Note that sum power minimization is a special case of *Lemma 1* when equal weights are applied.

With *Lemma 1*, (2) can be reformulated with explicit rate constraints. Let $\boldsymbol{\pi_n}$ be the optimal decoding order by *Lemma 1* in subchannel n, and $\boldsymbol{\pi_n^{-1}}(k)$ denote the location of user k in the permutation of $\boldsymbol{\pi_n}$. (2) can be recast as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K}\sum_{n=1}^{N} P_{k,n} \\ \text{subject to} \quad & \\ & \sum_{n=1}^{N} \log_2\left(1+\frac{|H_{k,n}|^2 P_{k,n}}{\sigma_n^2+\sigma_{k,n}^2}\right) \\ & \geq R_k, \forall k; \\ & P_{k,n} \geq 0, \forall k, n; \end{aligned} \tag{4}$$

where $\sigma_{k,n}^2 = \sum_{j=\boldsymbol{\pi_n^{-1}}(k)+1}^{K} |H_{\boldsymbol{\pi_n}(j),n}|^2 P_{\boldsymbol{\pi_n}(j),n}$ is the noise contributed by un-decoded signals in subchannel n.

The rate function of the inequality constraints in (4) are not convex. A key step in [10] and [11] is to convert the constraints to convex functions. Instead, in this paper we will directly exploit the structure of the optimal solution provided by this non-convex formulation.

III. Iterative Margin Adaptive Water-filling Algorithm

A. The Lagrangian

the Lagrangian associated with the optimization problem (4) is defined over the domain $\mathcal{D} = \{p_{k,n} : p_{k,n} \geq 0, \forall k, n\}$ as

$$\begin{aligned} \mathcal{L}(\{p_{k,n}\},\{\mu_k\},\{\boldsymbol{\pi_n}\}) = \sum_{k=1}^{K}\sum_{n=1}^{N} P_{k,n} - \sum_{k=1}^{K}\mu_k \cdot \\ \left(\sum_{n=1}^{N}\log_2\left(1+\frac{|H_{k,n}|^2 P_{k,n}}{\sigma_n^2+\sum_{j=k'}^{K}|H_{\boldsymbol{\pi_n}(j),n}|^2 P_{\boldsymbol{\pi_n}(j),n}}\right)\right) \\ +\sum_{k=1}^{K}\mu_k R_k. \end{aligned} \tag{5}$$

where μ_k's are Lagrange multipliers with non-negative values, and $k' = \boldsymbol{\pi_n^{-1}}(k)+1$.

Assuming zero duality, which is verified later in this section, the Karush-Kuhn-Tucker (KKT) conditions[15] are summarized by the following theorem:

Theorem 1: The KKT conditions of (5) are

$$\begin{aligned} P_{k,n} + \frac{\sigma_n^2+\sum_{j=\boldsymbol{\pi_n^{-1}}(k)+1}^{K}|H_{\boldsymbol{\pi_n}(j),n}|^2 P_{\boldsymbol{\pi_n}(j),n}}{H_{k,n}^2} \\ = \mu_k - \sum_{j=1}^{\boldsymbol{\pi_n^{-1}}(k)-1} P_{\boldsymbol{\pi_n}(j),n}, \forall k, n; \end{aligned} \tag{6}$$

Again $\boldsymbol{\pi_n^{-1}}(k)$ represents the location of user k in the permutation of $\boldsymbol{\pi_n}$

Proof: Though tedious, the proof basically involves two steps. First, note that the Lagrangian can be decomposed to separate Lagrangian's for each subchannel n. For each Lagrangian, take derivative of each $P_{k,n}$ for all k, then term collection and rearrangement followed by simple induction lead to equation (6).□

Equation (6) reveals the structure of the optimal solutions to the optimization problem (4): for each user k, the optimal power and rate allocation reduces to single user margin adaptive water-filling with water level reduction in each subchannel. To see this, define $g_{k,n}$ as

$$\frac{1}{g_{k,n}} = \frac{\sigma_n^2+\sum_{j=\boldsymbol{\pi_n^{-1}}(k)+1}^{K}|H_{\boldsymbol{\pi_n}(j),n}|^2 P_{\boldsymbol{\pi_n}(j),n}}{H_{k,n}^2}; \tag{7}$$

and define $\mu_{adj^{(k,n)}}$ as

$$\mu_{adj^{(k,n)}} = \sum_{j=1}^{\boldsymbol{\pi_n^{-1}}(k)-1} P_{\boldsymbol{\pi_n}(j),n}; \tag{8}$$

where adj denotes "water-level adjustment". From the successive decoding scheme user k's rate in subchannel n is

$$r_{k,n} = \log_2((\mu_k - \mu_{adj^{(k,n)}})g_{k,n}); \tag{9}$$

Thus the target rate constraint for user k is satisfied with

$$\prod_{n=1}^{N^*}(\mu_k - \widetilde{\mu}_{adj^{(k,n)}}) = \frac{2^{R_k}}{\prod_{n=1}^{N^*}\widetilde{g}_n} \tag{10}$$

where N^* denotes the number of subchannels that have rate allocation, and $\sim$ represents sorting operation over N^* (**??**). It is now obvious that equation (10) is a single user margin adaptive water-filling equation with water level reduction in each subchannel.

The intuition drawn from *Theorem 1* is also interesting: in each subchannel, a user treats the sum of received signal powers of weaker users as noise, while the sum power of stronger users serves as a water level reduction factor. This says that each user suffers from both the weaker and the stronger users' power in different ways and it really makes sense. Note also the equation of (6) actually provides a geometric view of how powers are related to minimize the sum power.

B. The IWF Algorithm

The optimal solution of (4) must satisfy the KKT conditions (6) when duality gap is zero. The structure of the optimal solution revealed in equation (9) and (10) allows us to solve the KKT conditions (6) by applying single user margin adaptive water-filling algorithm iteratively. The algorithm is proposed as the following:

Algorithm 1:

```
Initialize the K × N power allocation matrix P = 0
Sort H_{k,n} in each tone n; store index in a K × N matrix;
    Iterate until stopping criterion satisfied:
        Inner Loop
            for i=1 to K (users)
                for j=1 to N (subchannels)
                    compute g_{i,j};
                    compute μ_{adj^{i,j}};
                end
                Call Margin Adaptive Subroutine
                to update P_{i,j} by computing μ_i
            End
        End Inner Loop
```

The stopping criterion is typically the vanishing difference between the sum powers in each iteration.

The margin adaptive waterfilling subroutine is summarized in Fig. 1, which is very similar with Fig.4.9 in [14].

In single user case, the computation of μ_k at each iteration in Fig. 1 is simple while in multiuser case, to compute μ_k involves solving the root of polynomial (10), thus it appears computation intensive. However, a close examination of (10) shows that the largest positive root of polynomial (10) is the right solution for μ_k, which enables a simple binary search method to compute μ_k.

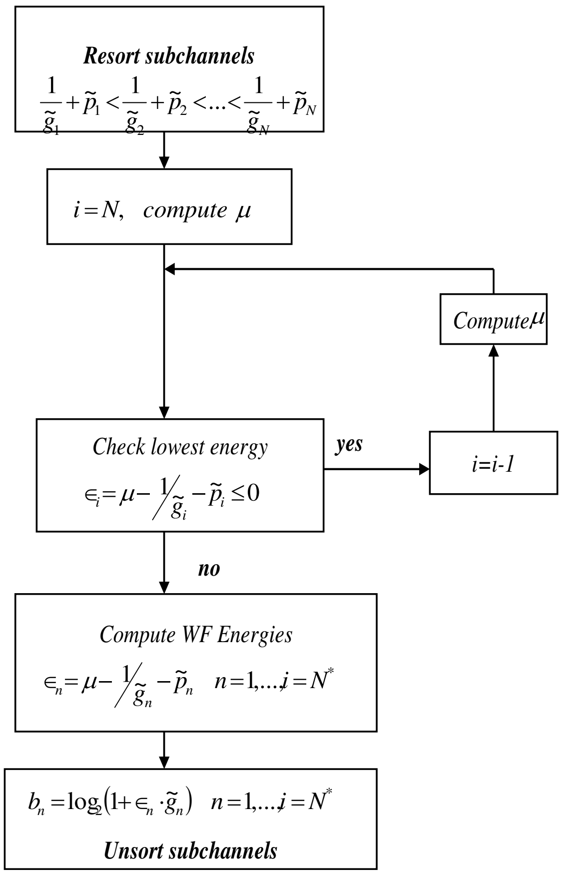

Fig. 1. Margin Adaptive Waterfilling Subroutine

C. Experimental Results

Extensive simulations are used to verify the convergence of *Algorithm 1* and the optimality of the solutions found by this algorithm. For every simulation result, the solution is compared against the optimal solution found by known algorithm based on convex optimization techniques which guarantees convergence and optimality. We choose the algorithm provided in [9] as reference. Simulation shows the duality gap of problem formulation (4) is indeed zero, and *Algorithm 1* exhibits fast convergence and converges to the optimal solution. Fig. 2 illustrates typical converging behavior by *Algorithm 1*.

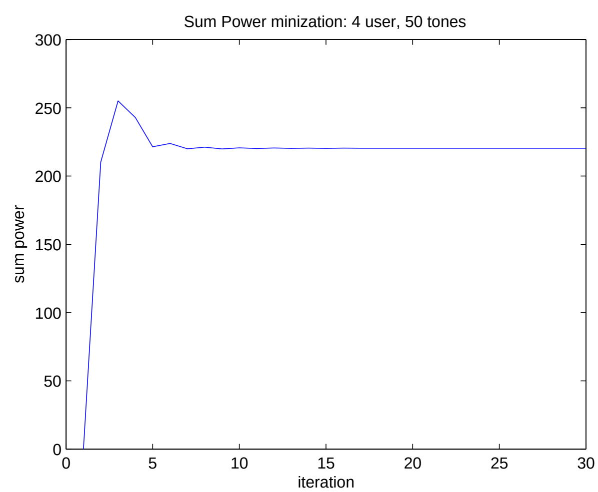

Fig. 2. Typical Convergence Behavior of *Algorithm 1*

IV. Characterization of Full Power Region

Algorithm 1 can be easily extended to characterize the full power region of a given target rate-tuple in an OFDM MAC. By *Lemma 1*, the general weighted sum power minimization problem can be formulated as:

$$
\begin{aligned}
&\text{minimize} \sum_{k=1}^{K} \lambda_k \sum_{n=1}^{N} P_{k,n} \\
&\text{subject to} \\
&\sum_{n=1}^{N} \log_2 \left(1 + \frac{|H_{k,n}|^2 P_{k,n}}{\sigma_n^2 + \sum_{j=\boldsymbol{\pi}_{\boldsymbol{n}}^{-1}(k)+1}^{K} |H_{\boldsymbol{\pi}_{\boldsymbol{n}}(j),n}|^2 P_{\boldsymbol{\pi}_{\boldsymbol{n}}(j),n}} \right) \\
&\geq R_k, \forall k; \\
&P_{k,n} \geq 0, \forall k, n;
\end{aligned} \tag{11}
$$

Now a simple ”change variable” trick can turn (11) into the form of (4) to which *Algorithm 1* is readily applied:

Let $P'_{k,n} = \lambda_k P_{k,n}$ and $H'_{k,n} = \frac{H_{k,n}}{\sqrt{\lambda_k}}$, (11) becomes

$$
\begin{aligned}
&\text{minimize} \sum_{k=1}^{K} \sum_{n=1}^{N} P'_{k,n} \\
&\text{subject to} \\
&\sum_{n=1}^{N} \log_2 \left(1 + \frac{|H'_{k,n}|^2 P'_{k,n}}{\sigma_n^2 + \sum_{j=\boldsymbol{\pi}_{\boldsymbol{n}}^{-1}(k)+1}^{K} |H'_{\boldsymbol{\pi}_{\boldsymbol{n}}(j),n}|^2 P'_{\boldsymbol{\pi}_{\boldsymbol{n}}(j),n}} \right) \\
&\geq R_k, \forall k; \\
&P'_{k,n} \geq 0, \forall k, n;
\end{aligned} \tag{12}
$$

The solution of (12) can be scaled back to find the solution for (11).

The extension of *Algorithm 1* to solving sum power minimization problem in BC is also trivial by applying the MAC-BC duality [12]. The dual MAC of each subchannel of the BC is first formulated, and the minimum sum power of the same given rate-tuple as imposed to BC can be found by applying *Algorithm 1* to this equivalent OFDM MAC. This minimum sum power is the optimal value for the BC by duality. Also by duality, each individual $P_{k,n}$ can be mapped from the values found in its dual MAC. It is worth pointing out that the KKT conditions of a direct formulation in BC does not exhibit the elegance shown in (6).

V. Conclusion

It is shown in this paper that the single user margin adaptive water-filling algorithm can be extended to multiuser environment by introducing a water level reduction term to each user's water level in each subchannel. This extended margin adaptive water-filling algorithm not only exhibits fast convergence provided by the greedy method in updating the water-level of each user, but also converges to the optimal solution. The operations involved in this algorithm only includes basic arithmetics in addition to the $\log$ function, which can be implemented as a table lookup. The data structures are also as simple as maintaining two tables: a matrix to hold intermediate $P_{k,n}$'s as well as its final optimal value; a second matrix to hold the index of sorted channel gains. This data structure also allows simple tracking algorithm to address slow fading channels: keep updating the channel gain matrix while iterating the inner loop of *Algorithm 1*.

References

[1] R. Cheng and S. Verdú ”Gaussian multiaccess channels with ISI: Capacity region and multisuer water-filling,” *IEEE Trans. Inform. Theory*, Vol. 39, May 1993.

[2] H. Weingarten, Y. Steinberg and S. Shamai (Shitz) ”Capacity Region of the MIMO Broadcast Channel,” to appear in IEEE Trans. Information Theory.

[3] T.M. Cover and J.A. Thomas *Elements of Informatin Tehory*, New York: Wiely, 1991

[4] D. Tse and S. Hanly ”Multi-access fading channels-Part I: polymatroid structure, optimal resouce allocation and throuput capacities”, *IEEE Trans. Inform. Theory*, vol.44, no.7, pp.2796-2815, Nov. 1998.

[5] D. Tse and S. Hanly ”Multi-access fading channels-Part II: Delay-Limited capacities”, *IEEE Trans. Inform. Theory*, vol.44, no.7, pp.2816-2831, Nov. 1998.

[6] W. Yu, W. Rhee, S. Boyd and J. Cioffi ”Iterative Water-filling for Gaussian Vector Multiple Access Channels”, *IEEE Trans. Inform. Theory*, vol. 50, no. 1, pp.145-151, Jan 2004.

[7] L. Li, N. Jindal and A. Goldsmith ”Outage Capacityes and Optimal Power Allocation for Fading Multiple-Accss Channels,” *IEEE Trans. Inform. Theory*, vol.51, no.4, pp.1326-1347, Apr. 2005.

[8] D. Tse ”Optimal power allocation over parallel Gaussian broadcast channels,” *Proc. of International Symposium on Information*, Ulm Germany,June 1997, p.27.

[9] M. Mohseni, R. Zhang and John Cioffi ”Optimized transmission of fading multiple-access and broadcast channels with multiple antennas,” accepted by *IEEE JSAC, special issue on nonlinear optimization of communication systems*.

[10] J. OH, S.J. Kim and J. Cioffi ”Optimum power allocation and control for OFDM in multiple access channels,” *Proc. IEEE Vehicular Techn. Conf.(VTC)*, Los Angeles, Sep. 2004.

[11] T. Michel and G. Wunder ”Solution to the Sum Power Minimization Problem under Given Rate Requirements for the OFDM Multiple Access Channel”, Alleton Conf. Sep. 2004.

[12] N. Jindal, S. Vishwanath, and A. Goldsmith, “On the duality of Gaussian multiple-access and broadcast Channels,” *IEEE Trans. Inform. Theory*, vol.50, no.5, pp768-783, May 2004.

[13] S. Vishwanath, N. Jindal and A. Goldsmith, ”Duality, Achievable Rates, and Sum-Rate Capacity of Gaussian MIMO Broadcast Channels”, *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp2658-2668, Oct. 2003.

[14] J. Cioffi ”EE379C Course Reader, Chapter 4”, *http://www.stanford.edu/class/ee379c*, 2005

[15] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2003.