Spatially-Correlated Jamming in Gaussian Multiple Access and Broadcast Channels

Mark H. Brady, Mehdi Mohseni, and John M. Cioffi
Department of Electrical Engineering
Stanford University
Stanford, CA 94305
Email: {mhbrady,mmohseni,cioffi}@stanford.edu

***Abstract*— The effect of jamming on multiuser transmission in MIMO Gaussian multiple access and broadcast channels is analyzed. A malicious jammer seeks to impair communication in a multiuser channel by injecting a spatially-correlated Gaussian interference signal. Full channel state information (CSI) is assumed, but the jammer has no knowledge of the users' signals.**

In the broadcast channel, a worst-case throughput (sum rate) is obtained as the Nash equilibrium of a certain strictly competitive game; in the multiple access channel, a general weighted sum-rate is similarly considered. Certain properties of the Nash equilibria of each game are developed.

***Keywords*—** Jamming, MIMO Broadcast Channel, MIMO Multiple Access Channel, Zero-sum Games.

I. Introduction

MIMO transmission techniques continue to mature both theoretically and in practice. This paper considers interference in MIMO Gaussian multiple access and MIMO Gaussian broadcast channels under a jamming framework. For each channel, a jammer seeking to impair transmission injects a spatially-correlated Gaussian interference signal, while the users attempt to best mitigate the jammer's interference.

The game theoretic analysis of a single user channel with a single jammer has been examined under various conditions [1] [2] [3] [4]. In such analyses, the objective is commonly channel mutual information [5], or a squared-difference error criterion [6]. The terminology of "correlated jamming" has been used to denote correlation between a legitimate user's signal and the jammer's signal. Recently, multiple-access channels under a throughput objective have been studied [7].

Potential applications of this analysis are MIMO wireless systems and next-generation "vectored" DSL systems [8]; for specificity, terminology from the former example is adopted in the sequel.

The paper is organized as follows. Section II considers the Gaussian multiple access channel model of upstream transmission. The performance criterion in this setting is an arbitrary non-negative weighting of the users' rates. Section III considers downstream transmission in a Gaussian BC, with the performance criterion of channel throughput. Concluding remarks are made in Section V.

A word on notation: for the matrix X, $X \succeq 0$ denotes that X is positive semidefinite, while for the vector $\mathbf{x}$, $\mathbf{x}_k$ denotes the kth element and $\mathbf{x} \succeq 0$ designates that each element is nonnegative. All capacities are given in nats.

II. MIMO Multiple Access Channels

A. Channel and Coding Model

A total of K independent users, each equipped with N_T antennas, transmit to a common base station with N_R antennas. A malicious jammer has a total of N_J transmit antennas. The linear discrete-time channel is governed by

$$\mathbf{y} = \sum_{k=1}^{K} H_k \mathbf{x}_k + G\mathbf{z} + \mathbf{n}, \tag{1}$$

where $H_k \in \mathbb{R}^{N_R \times N_T}$ is the gain between the kth user and the receiver, $G \in \mathbb{R}^{N_R \times N_J}$ is the gain between the jammer and the receiver, $\mathbf{y} \in \mathbb{R}^{N_R}$ is the channel output, and $\mathbf{x}_k \in \mathbb{R}^{N_T}$ is kth user's signal. A Gaussian jammer emits signal $\mathbf{z}$, where $\mathbf{z} \sim \mathcal{N}(0, Z)$. $\mathbf{n} \in \mathbb{R}^{N_R}$ denotes AWGN independent of the users and jammer, distributed as $\mathbf{n} \sim \mathcal{N}(0, \Lambda)$ where $\Lambda \succ 0$. Each legitimate transmitter has a power constraint: $\mathbf{E}[\mathbf{x}_k^T \mathbf{x}_k] \leq P_k^x$, for all $k = 1, \ldots, K$, as does the jammer: $\mathbf{E}[\mathbf{z}^T \mathbf{z}] \leq P^j$.

All of the transmitters and the jammer are assumed to have perfect CSI. Although the jammer may correlate its interference in space (Z), eavesdropping by the jammer is not allowed *i.e.* the jammer has no knowledge of the users' signals ($\{\mathbf{x}_k\}$) and thus may not perform "correlated jamming".

B. Gaussian MAC Capacity

First consider the case when the jammer is removed ($N_J = 0$ or $P^j = 0$). In this case, the channel (1) corresponds to a Gaussian Multiple Access Channel. The (Shannon) capacity of this channel is well-known [9], and is given by the following expression

$$\mathcal{C}_{\text{MAC}}(\Lambda) = \bigcup_{\substack{\mathbf{Tr}(X_k) \leq P_k, \\ X_k \succeq 0, \\ k=1,\ldots,K}} \left\{ R : \; 2\sum_{s \in S} R_s \leq \log \frac{|\sum_{m \in S} H_m X_m H_m^T + \Lambda|}{|\Lambda|} \; \forall S \subset E \right\}, \tag{2}$$

where $E = \{1, \ldots, K\}$ is an index set, $R \in \mathbb{R}_+^K$ is a vector of user rates wherein R_k is the rate of user k, and $X_k \in$

Player	*Objective*	*Strategy Set*
1 - Users	$\max J$	$\{X_1, \ldots, X_K : \ X_k \succeq 0, \ \ \mathbf{Tr}(X_k) \leq P_k^x\}$
2 - Jammer	$\min J$	$\{Z : \ Z \succeq 0, \ \ \mathbf{Tr}(Z) \leq P_k^j, \}$

TABLE I

SUMMARY OF MULTIPLE ACCESS CHANNEL GAME $\mathcal{M}$.

$\mathbb{S}_+^{N_T}$ denotes the transmit covariance of user k. The notation $\mathcal{C}_{\text{MAC}}(\Lambda)$ is employed to denote explicitly the dependence of the capacity region on Λ.

Turning to the more general case with the jammer present ($N_J > 0$), denote the net (Gaussian) interference covariance at the receiver by Ψ

$$\mathbf{Cov}[\mathbf{n} + G\mathbf{z}] = \Lambda + GZG^T \quad (3)$$

$$\triangleq \Psi(Z). \quad (4)$$

We consider the problem of computing

$$\begin{array}{ll} \max & \mu^T R \\ \text{subject to} & R \in \mathcal{C}_{\text{MAC}}(\Psi(Z)), \end{array} \quad (5)$$

where μ is a fixed non-negative weighting $\mu \in \mathbb{R}_+^K$ and $Z \succ 0$ is fixed. Note that the maximum is attained because $\mathcal{C}_{\text{MAC}}(\Psi(Z))$ is closed and bounded. This optimization is illustrated in Figure 1 as the intersection of a tangent plane having normal vector μ with $\mathcal{C}_{\text{MAC}}(\Lambda)$. It can be shown that (5) is a *convex optimization problem* in the sense of [10]; without loss of generality, by assuming $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_K$ one has the equivalent problem [11]

$$\begin{array}{ll} \max & \frac{1}{2}\sum_{k=1}^K (\mu_k - \mu_{k-1}) \log \frac{\left|\Psi(Z) + \sum_{m=k}^K H_m X_m H_m^T\right|}{|\Psi(Z)|} \\ \text{subject to} & \mathbf{Tr}(X_k) \leq P_k, \\ & X_k \succeq 0, \end{array} \quad (6)$$

where in a slight abuse of notation, we define $\mu_0 = 0$. Note that the formulation (6) has only K total $\log|\cdot|$ expressions, which is a substantial simplification of the naive formulation of (5) based on the $2^K - 1$ convex inequalities defining $\mathcal{C}_{\text{MAC}}(\Lambda)$ (2).

C. Game-Theoretic Formulation

Suppose now that the jammer wishes to choose a fixed transmit covariance Z so as to minimize the weighted rate achievable by the transmitters. The jammer's optimization is

$$\begin{array}{ll} \inf_Z \max_{\{X_k\}} & \frac{1}{2}\sum_{k=1}^K (\mu_k - \mu_{k-1}) \\ & \cdot \log \frac{\left|\Psi(Z) + \sum_{m=k}^K H_m X_m H_m^T\right|}{|\Psi(Z)|} \\ \text{subject to} & (X_1, X_2, \ldots, X_K) \in \mathcal{S}_1, \\ & Z \in \mathcal{S}_2, \end{array} \quad (7)$$

where we define $\mathcal{S}_1 = \{(X_1, \ldots, X_K) : X_k \succeq 0, \ \ \mathbf{Tr}(X_k) \leq P_k^x, \ k = 1, \ldots, K\}$ and $\mathcal{S}_2 = \{Z : Z \succeq 0, \ \ \mathbf{Tr}(Z) \leq P^j\}$.

It is possible to obtain additional insight to the problem (7) by appealing to game-theoretic results. Consider the following

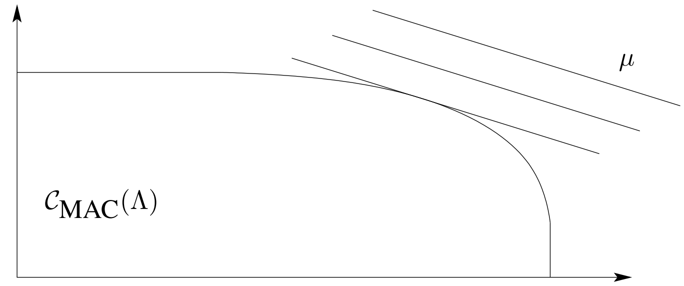

Fig. 1. Illustration of μ-weighted rate maximization over $\mathcal{C}_{\text{MAC}}(\Lambda)$ considered in multiple access game $\mathcal{M}$.

two-player strictly-competitive game, as summarized in Table I. The objective function of the game is

$$J(X_1, \ldots, X_K, Z) = \frac{1}{2}\sum_{k=1}^K (\mu_k - \mu_{k-1}) \log \frac{\left|\Psi(Z) + \sum_{m=k}^K H_m X_m H_m^T\right|}{|\Psi(Z)|}. \quad (8)$$

The non-negative weighting $\mu \in \mathbb{R}_+^K$ is fixed. Player 1 (corresponding to the legitimate users) chooses transmit covariances $(X_1, \ldots, X_K)$ from the set $\mathcal{S}_1$ to maximize the objective function (corresponding to their weighted rate). Player 2 (corresponding to the jammer) chooses a covariance (Z) to minimize the objective function. The game $\mathcal{M} = (J, \mathcal{S}_1, \mathcal{S}_2)$ is defined as the Multiple Access Channel Worst-Case Weighted Rate game.

This game-theoretic formulation of (7) admits the application of Nash equilibrium results from game theory. A Nash equilibrium (in pure strategies) of the game $\mathcal{M}$, denoted $(X_1^\star, \ldots, X_K^\star, Z^\star)$, has the physical interpretation of a worst interference covariance ($Z^\star$) by the jammer, and the optimal transmit covariance for the users $(S_1^\star, \ldots, S_k^\star)$ under such interference.

The following two lemmata are useful in the subsequent analysis:

Lemma 1 ([4]): The function $f : \mathbb{S}_+^n \mapsto \overline{\mathbb{R}}$ defined as

$$f(K_z) = \log\left(|K_x + K_z|/|K_z|\right), \quad (9)$$

is convex in K_z, where $0 \preceq K_x \in \mathbb{S}^n$. Furthermore, the convexity is strict if $K_x \succ 0$.

Lemma 2 ([4] [12]): The function $g : \mathbb{S}_+^n \mapsto \mathbb{R}$ defined as

$$g(K_x) = \log\left(|K_x + K_z|/|K_z|\right), \quad (10)$$

is strictly concave in K_x, where $0 \prec K_z \in \mathbb{S}^n$.

Theorem 1: A pure-strategy Nash equilibrium of the game $\mathcal{M}$, denoted $(X_1^\star, \ldots, X_K^\star, Z^\star)$, always exists. Also, the game has a value.

Proof: The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are both compact and convex. The function J is continuous, and each $\log|\cdot|$ term in the summation (8) is convex in $Z \in \mathcal{S}_2$ for any fixed $(X_1, \ldots, X_K) \in \mathcal{S}_1$ due to Lemma 1 and the affine composition property. The differences $\mu_k - \mu_{k-1}$ are all nonnegative, and therefore J is the nonnegative weighted sum of terms convex in Z; hence J is convex in Z (for any fixed $(X_1,\ldots,X_K)\in\mathcal{S}_1$). Also, for any fixed $Z\in\mathcal{S}_2$, each $\log|\cdot|$ term in the sum (8) is a concave function of $(X_1,\ldots,X_K)$ on $\mathcal{S}_1$ due to Lemma 2 and the affine composition property. Therefore J is the nonnegative weighted sum of terms concave in $(X_1,\ldots,X_K)$ (for any fixed $\overline{Z}\in\mathcal{S}_2$). The necessary conditions of [13, Thm. 4.4] are thereby satisfied and the result follows. ■

As a corollary, the infimum in (7) is achieved.

It can be shown that, in general, the Nash equilibrium of the game $\mathcal{M}$ is not unique. However, the following result gives conditions under which a certain "partial" uniqueness holds.

Theorem 2: Let $(X_1^\star,\ldots,X_K^\star,Z^\star)$ be any Nash equilibrium of $\mathcal{M}$. If the matrices $H_1,\ldots,H_K$ are each full column rank and $0<\lambda_1<\ldots<\lambda_K$, then for all $(\widehat{X}_1,\ldots,\widehat{X}_K,\widehat{Z})$ that are Nash equilibria of $\mathcal{M}$ it holds $\widehat{X}_k=X_k^\star$ for every $k=1,\ldots,K$.

Proof: By the interchangeability property, $(\widehat{X}_1,\ldots,\widehat{X}_K,Z^\star)$ is also a Nash equilibrium. Define $(\widetilde{X}_1,\ldots,\widetilde{X}_K)=\frac{1}{2}(\widehat{X}_1,\ldots,\widehat{X}_K)+\frac{1}{2}(X_1^\star,\ldots,X_K^\star)$. Because J is concave in $(X_1,\ldots,X_k)$ with $Z^\star$ fixed and the game has a value, it holds

$$J\left(\widetilde{X}_1,\ldots,\widetilde{X}_K,Z^\star\right)=\frac{1}{2}J\left(X_1^\star,\ldots,X_K^\star,Z^\star\right)+\frac{1}{2}J\left(\widehat{X}_1,\ldots,\widehat{X}_K^\star,Z^\star\right). \quad (11)$$

And because each term in the summation (8) similarly is concave in $(X_1,\ldots,X_k)$, it holds

$$(\mu_k-\mu_{k-1})\log\frac{\left|\Psi(Z^\star)+\sum_{m=k}^K H_m\widetilde{X}_mH_m^T\right|}{|\Psi(Z^\star)|}\geq\frac{1}{2}(\mu_k-\mu_{k-1})\log\frac{\left|\Psi(Z^\star)+\sum_{m=k}^K H_mX_m^\star H_m^T\right|}{|\Psi(Z^\star)|}+\frac{1}{2}(\mu_k-\mu_{k-1})\log\frac{\left|\Psi(Z^\star)+\sum_{m=k}^K H_m\widehat{X}_mH_m^T\right|}{|\Psi(Z^\star)|}, \quad (12)$$

for each $k=1,\ldots,K$. Together (11) and (12) imply that the inequality in (12) holds with equality. Because $\mu_k-\mu_{k-1}>0$ and $\Psi(Z^\star)\succ0$, Lemma 2 implies that

$$\sum_{m=k}^K H_m\widehat{X}_mH_m^T=\sum_{m=k}^K H_mX_m^\star H_m^T, \quad (13)$$

for each $k=1,\ldots,K$. Consider now the case of $k=K$. Because H_K has full column rank, it has a left inverse $H_K^\dagger$ such that $H_K^\dagger H_K=I$. Then by (13), $H_K^\dagger H_K\widehat{X}_KH_K^T(H_K^\dagger)^T=H_K^\dagger H_KH_KX_K^\star H_K^T(H_K^\dagger)^T$ and hence $\widehat{X}_K=X_K^\star$.

Now consider $k=K-1$. By (13) it holds

$$\sum_{m=K-1}^K H_m\widehat{X}_mH_m^T=\sum_{m=K-1}^K H_mX_m^\star H_m^T. \quad (14)$$

Because $\widehat{X}_K=X_K^\star$ this implies that $H_{K-1}\widehat{X}_{K-1}H_{K-1}^T=H_{K-1}X_{K-1}^\star H_{K-1}^T$. H_{K-1} also has full column rank and a left inverse, whence $\widehat{X}_{K-1}=X_{K-1}^\star$. By an identical induction argument, $\widehat{X}_k=X_k^\star$ for each $k=1,\ldots,K$, implying the result. ■

In practical DSL systems, the full column rank condition can be shown to hold in deployed loop channels [8] [14].

III. MIMO Broadcast Channels

This section considers a "dual" configuration whereby a single transmitter (*e.g.* base station) wishes to communicate with several independent receivers in the presence of hostile Gaussian jamming.

A. Channel and Coding Model

A single transmitter, equipped with N_T antennas, transmits to K independent users; for clarity of exposition, it is assumed that each user is equipped with N_R antennas. The jammer has a total of N_J transmit antennas.

The discrete-time channel is governed by the following linear model:

$$\mathbf{y}_k=H_k\mathbf{x}+G_k\mathbf{z}+\mathbf{n}_k, \quad (15)$$

where $H_k\in\mathbb{R}^{N_R\times N_T}$ is the gain between the transmitter and the kth user, $G_k\in\mathbb{R}^{N_R\times N_J}$ is the gain between the jammer and user k, $\mathbf{y}_k\in\mathbb{R}^{N_R}$ is the channel output observed by user k, $\mathbf{x}\in\mathbb{R}^{N_T}$ is the transmitter's signal, and $\mathbf{z}_k\in\mathbb{R}^{N_R}$ is AWGN observed by user k. The distribution of the AWGN, which is independent of the jammer, is $\mathbf{n}_k\sim\mathcal{N}(0,\Lambda_k)$ where $\Lambda_k\succ0$ for all $k=1,\ldots,K$. The transmitter and jammer's power are upper-bounded: $\mathbf{E}[\mathbf{x}^T\mathbf{x}]\leq P^x$, $\mathbf{E}[\mathbf{z}^T\mathbf{z}]\leq P^j$. Both the users and the jammer are assumed to have perfect CSI, but again the jammer has no knowledge of $\mathbf{x}$.

B. Sum Capacity

It has been shown [15] (see also [16] [17]) that without the jammer ($N_J=0$) in the channel (15), the sum capacity $\mathcal{C}_{\text{sum}}$ is given by:

$$\begin{array}{ll}\max_X\min_\Upsilon & \frac{1}{2}\log|HXH^T+\Upsilon|-\frac{1}{2}\log|\Upsilon|\\ \text{subject to} & \Upsilon^{[k]}=\Lambda_k\quad k=1,\ldots,K,\\ & \mathbf{Tr}(X)\leq P^x,\\ & \Upsilon\succeq0,\\ & X\succeq0.\end{array} \quad (16)$$

where the notation $H^T=[H_1^T,\ldots,H_k^T]$ and $\Upsilon^{[k]}$ denotes the kth $N_R\times N_R$ block-diagonal of the matrix $\Upsilon\in\mathbb{R}^{KN_r\times KN_r}$. An interpretation [15, Thm. 3] of the formulation (16) is a strictly-competitive two-player game between the transmitter (X) choosing an optimal transmit covariance and a "malicious nature" (Υ) choosing a worst joint distribution for the broadcast channel; furthermore, solutions of (16) correspond to Nash equilibria of the game. The sum capacity may be achieved by *e.g.* dirty-paper coding [9], or trellis and convolutional precoding [18].

Considering now the interference experienced from the Gaussian jammer, the noise covariance seen by user k is given by

$$\mathbf{Cov}[\mathbf{n}_k+G_k\mathbf{z}]=\Lambda_k+G_kZG_k^T \quad (17)$$

Player	*Objective*	*Strategy Set*
1 - Users	$\max K$	$\{X : X \succeq 0,\ \mathbf{Tr}(X) \leq P^x\}$
2 - Jammer, "Nature"	$\min K$	$\{S, Z : Z \succeq 0,\ \mathbf{Tr}(Z) \leq P^j,$ $\Psi(Z) + S \succeq 0,\ S^{[k]} = 0\}$

TABLE II

SUMMARY OF BROADCAST CHANNEL GAME $\mathcal{B}$.

For notational convenience, define

$$\Psi(Z) \triangleq \begin{bmatrix} \Lambda_1 + G_1 Z G_1^T & & \\ & \ddots & \\ & & \Lambda_K + G_K Z G_K^T \end{bmatrix}, \quad (18)$$

where all the off block-diagonal terms are 0. For any fixed $Z \succeq 0$, the sum capacity $\mathcal{C}_{\text{sum}}(Z)$ of the channel may be written

$$\begin{array}{ll} \min_S \max_X & \frac{1}{2}\log\left|HXH^T + S + \Psi(Z)\right| \\ & -\frac{1}{2}\log\left|S + \Psi(Z)\right| \\ \text{subject to} & S^{[k]} = 0 \quad k = 1, \ldots, K, \\ & \mathbf{Tr}(X) \leq P^x, \\ & \mathbf{Tr}(Z) \leq P^j, \\ & S + \Psi(Z) \succeq 0, \\ & X \succeq 0, \\ & Z \succeq 0. \end{array} \quad (19)$$

C. Game-Theoretic Formulation

Suppose that the jammer wishes to choose its covariance Z so as to minimize the channel sum capacity $\mathcal{C}_{\text{sum}}(Z)$. The jammer's optimization is

$$\begin{array}{ll} \inf_Z \min_S \max_X & \frac{1}{2}\log\left|HXH^T + S + \Psi(Z)\right| \\ & -\frac{1}{2}\log\left|S + \Psi(Z)\right| \\ \text{subject to} & X \in \mathcal{S}_1, \\ & (S, Z) \in \mathcal{S}_2, \end{array} \quad (20)$$

where $\mathcal{S}_1 = \{X : X \succeq 0, \mathbf{Tr}(X) \leq P^x\}$ and $\mathcal{S}_2 = \{(S, Z) : S^{[k]} = 0, k = 1, \ldots, K, Z \succeq 0, \mathbf{Tr}(Z) \leq P^j, \Psi(Z) \succeq -S\}$.

The formulation (20) may be interpreted as a strictly-competitive game, and is summarized in Table II. In this two-player game, the objective function $K : \mathcal{S}_1 \times \mathcal{S}_2 \mapsto \overline{\mathbb{R}}$ is given by

$$K(X, S, Z) = \frac{1}{2}\log\left|HXH^T + S + \Psi(Z)\right| - \frac{1}{2}\log\left|S + \Psi(Z)\right|. \quad (21)$$

Player 1 chooses X from the set $\mathcal{S}_1$ to maximize the objective, while Player 2 chooses (S, Z) from the set $\mathcal{S}_2$ to minimize the objective. The game $\mathcal{B} = (J, \mathcal{S}_1, \mathcal{S}_2)$ is defined as the Broadcast Worst Throughput game. Observe that by setting $P^j = 0$, the Broadcast Worst Throughput game reduces to the game (16) defined for broadcast channel sum capacity; thus the former game generalizes the latter.

A pure-strategy Nash equilibrium represents a "worst" choice transmit covariance (Z) by the jammer and channel joint distribution (S) by "nature" [15], as well as the optimal response (X) to this interference by the transmitter such that neither player has a unilateral incentive to deviate its strategy. It turns out that such a Nash equilibrium always exists in $\mathcal{B}$.

Theorem 3: The game $\mathcal{B}$ has a Nash equilibrium in pure strategies and a value.

Proof: We give only a sketch of the lengthy proof; the crux of the approach is a result due to Diggavi and Cover [4] that has been used similarly in [15]. With the identification $N = S + \Psi(Z)$ it can be shown that a Gaussian saddle point of the mutual information expression $I(X; HX + N)$ exists over a certain feasible set of distributions. Evaluating this mutual information expression for the Gaussian case, one obtains the $\log|\cdot|$ expression of (21) and

$$\min_{(S,Z)\in\mathcal{S}_2} \max_{X\in\mathcal{S}_1} K(X, S, Z) = \max_{X\in\mathcal{S}_1} \min_{(S,Z)\in\mathcal{S}_2} K(X, S, Z). \quad (22)$$

■

Note therefore that the infimum in (20) is achieved.

In general, there need not exist a unique Nash equilibrium of $\mathcal{B}$. However, sufficient conditions are given in the following Theorem for the uniqueness of $X^\star$.

Theorem 4: In the game $\mathcal{B}$, if H has full column rank and if there exists a Nash equilibrium $(X^\star, S^\star, Z^\star)$ such that $\Psi(Z^\star) \succ -S^\star$, then for every Nash equilibrium $(\widehat{X}, \widehat{S}, \widehat{Z})$, it holds $\widehat{X} = X^\star$.

Proof: The convex optimization problem

$$\begin{array}{ll} \max & \frac{1}{2}\log\left|Q + S + \Psi(Z)\right| - \frac{1}{2}\log\left|S + \Psi(Z)\right| \\ \text{subject to} & Q \in \mathcal{S}_1, \end{array} \quad (23)$$

has a unique solution [10], denoted $Q^\star$ because by Lemma 2, the objective is strictly convex in Q. Therefore $HX^\star H^T = Q^\star$. By the exchangeability property of Nash equilibria, $(\widehat{X}, S^\star, Z^\star)$ is also a Nash equilibrium. Therefore by the identical argument above, $H\widehat{X}H^T = Q^\star$ and hence $H\widehat{X}H^T = HX^\star H^T$. Because H is full column rank, it has a left inverse $H^\dagger$. Thus $H^\dagger H\widehat{X}H^T(H^\dagger)^T = H^\dagger HX^\star H^T(H^\dagger)^T$, which simplifies to $\widehat{X} = X^\star$. ■

Note that in contrast to Theorem 4, no conditions on the Nash equilibria themselves (*i.e.* $\Psi(Z^\star) \succ -S^\star$) are required by Theorem 2.

IV. NUMERICAL EXAMPLES

This section gives numerical examples arising from the games $\mathcal{M}$ and $\mathcal{B}$. Nash equilibria of these games may be efficiently computed by convex optimization techniques [10].

A. Gaussian Vector MAC Example

Let $K = 2$, $N_T = N_R = 2$, $P_1^x = P_2^x = 1$, $P^j = 1$, $\mu = [1\ 1]^T$ and

$$\begin{aligned} H_1 &= \begin{bmatrix} 0.2 & 0.1 \\ -0.2 & 1.0 \end{bmatrix}, \ H_2 = \begin{bmatrix} -0.3 & 0.0 \\ 0.5 & 0.6 \end{bmatrix}, \\ G &= \begin{bmatrix} 0.2 & 0.1 \\ -0.3 & 0.15 \end{bmatrix}, \\ \Lambda &= I. \end{aligned} \quad (24)$$

The following is a Nash equilibrium of $\mathcal{M}$ with these parameters

$$\begin{aligned} X_1^\star &= \begin{bmatrix} 0.017 & -0.130 \\ -0.130 & 0.983 \end{bmatrix}, \\ X_2^\star &= \begin{bmatrix} 0.570 & 0.495 \\ 0.495 & 0.429 \end{bmatrix}, \\ Z^\star &= \begin{bmatrix} 0.822 & -0.383 \\ -0.383 & 0.178 \end{bmatrix}, \\ J(X_1^\star, X_2^\star, Z^\star) &= 0.475. \end{aligned} \tag{25}$$

Note that the jammer uses full power ($\mathbf{Tr}(Z^\star) = P^j = 1$) as does Player 1 ($\mathbf{Tr}(X_1) = P_1^x$, $\mathbf{Tr}(X_2) = P_2^x$).

When the jammer is removed ($P^j = 0$), it may be verified (*e.g.* by a modified waterfilling algorithm [11]) that the sum capacity of the resulting MAC is 0.508; thus, the jammer has reduced throughput by a nominal amount. Because the jammer's interference power $GZ^\star G^T$ is not large compared to the AWGN ($\Lambda = I$), it is intuitive that the throughput loss is not substantial in this example.

B. Gaussian Vector BC Example

This example considers a multiple-input single-output (MISO) broadcast channel. Let $K = 3$, $N_T = 3$, $N_R = 1$, $N_J = 2$, $P^x = 5$, $P^j = 5$, and

$$\begin{aligned} H &= \begin{bmatrix} 1.0 & -0.3 & 0.2 \\ -0.4 & 2.0 & 0.5 \\ -0.1 & 0.2 & 3.0 \end{bmatrix}, \\ G &= \begin{bmatrix} 0.1 & -0.2 \\ 2.0 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}, \\ \Lambda_1 &= \Lambda_2 = \Lambda_3 = 1. \end{aligned} \tag{26}$$

The game $\mathcal{B}$ with these parameters has the following Nash equilibrium

$$\begin{aligned} X^\star &= \begin{bmatrix} 1.734 & -0.480 & 0.032 \\ -0.480 & 0.671 & 0.126 \\ 0.032 & 0.126 & 2.596 \end{bmatrix}, \\ Z^\star &= \begin{bmatrix} 0.873 & 1.898 \\ 1.898 & 4.127 \end{bmatrix}, \\ S^\star &= \begin{bmatrix} 0 & 0.034 & 0.469 \\ 0.034 & 0 & -2.979 \\ 0.469 & -2.979 & 0 \end{bmatrix}, \\ K(X^\star, (S^\star, Z^\star)) &= 1.765. \end{aligned} \tag{27}$$

It was shown in [15] that without the jammer ($P^j = 0$), the BC sum capacity is 2.895. Thus in this example, the jammer causes a significant reduction in rate. This example also shows that even when $\Psi(Z^\star) \succ -S^\star$, the jammer's Nash equilibrium strategy may be rank-deficient ($Z^\star \not\succ 0$) as

$$Z^\star = \begin{bmatrix} 0.873 & 1.898 \\ 1.898 & 4.127 \end{bmatrix} = 5 \begin{bmatrix} 0.418 \\ 0.906 \end{bmatrix} \begin{bmatrix} 0.418 & 0.906 \end{bmatrix}. \tag{28}$$

V. Conclusion

The properties of Gaussian jamming in MIMO multiple access and broadcast channels has been examined from the standpoint of game theory. Worst-case interferences and optimal responses thereto were obtained from Nash equilibria of suitably-defined strictly competitive games: in the multiple access channel, arbitrary weightings of users' rates were adopted as the performance criterion. In the broadcast channel, throughput is considered, in a generalization of a game previously formulated for Gaussian sum capacity.

Partial uniqueness properties of these Nash equilibria have been developed. Under appropriate conditions, the Nash equilibrium strategies of the legitimate users were shown to be unique; such strategies therefore may be interpreted as "robust" transmit covariances that afford protection against a hostile Gaussian jammer.

Acknowledgment

This work was supported by NSF contract CNS-0427677.

References

[1] M. Mèdard, "Capacity of correlated jamming channels," in *35th Annual Allerton Conference on Communications*, 1997, pp. 1043–1052.

[2] S. Shafiee and S. Ulukus, "Capacity of multiple access channels with correlated jamming," in *IEEE MILCOMM*, 2005.

[3] A. Bayesteh, M. Ansari, and A.K. Khandani, "Effect of jamming on the capacity of MIMO channels," in *22nd Biennial Symp. on Communications, Queen's University, Kingston, Ontario, Canada*, 2004.

[4] Suhas N. Diggavi and Thomas M. Cover, "The worst additive noise under a covariance constraint," *IEEE Trans. on IT*, vol. 47, no. 7, pp. 3072–3081, 2001.

[5] T. Cover, *Elements of Information Theory*, Wiley-Interscience, 1991.

[6] T. Basar and Y-W. Wu, "A complete characterization of minimax and maximin encoder-decoder policies for communication channels with incomplete statistical description," *IEEE Trans. on IT*, vol. 31, no. 4, pp. 482–489, 1985.

[7] S. Shafiee and S. Ulukus, "Correlated jamming in multiple access channels," in *Conf. on Information Sci. and Systems CISS*, 2005.

[8] George Ginis and John M. Cioffi, "Vectored transmission for digital subscriber line systems," *IEEE Journal on Sel. Areas in Comm.*, vol. 20, no. 5, pp. 1085–1104, June 2002.

[9] A. Goldsmith, S.A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE Journal on Sel. Areas in Comm.*, vol. 21, no. 5, pp. 684–702, 2003.

[10] S. Boyd, *Convex Optimization*, Cambridge University Press, 2004.

[11] Wei Yu, Wonjong Rhee, Stephen Boyd, and John M. Cioffi, "Iterative water-filling for Gaussian vector multiple-access channels," *IEEE Trans. on IT*, vol. 50, no. 1, pp. 145–152, 2004.

[12] T. Cover and J. Thomas, "Determinant inequalities via information theory," *SIAM Jnl. on Matrix Analysis and Applications*, vol. 9, no. 3, pp. 384–392, 1998.

[13] T. Basar and G.J. Olsder, *Dynamic Noncooperative Game Theory*, Academic Press, 1982.

[14] R. Cendrillion, G. Ginis, M. Moonen, and K. Van Acker, "Partial crosstalk precompensation in downstream VDSL," *Elsevier Signal Processing*, vol. 84, pp. 2005–2019, June 2004.

[15] Wei Yu and John M. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *IEEE Trans. on IT*, vol. 50, no. 9, pp. 1875–1892, 2004.

[16] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, achievable rates and sum-rate capacity of Gaussian MIMO broadcast channels," *IEEE Trans. on IT*, vol. 49, no. 10, pp. 2658–2668, 2003.

[17] P. Viswanath and D. N. C. Tse, "Sum capacity of the vector Gaussian broadcast channel and uplink-downlink capacity," *IEEE Trans. on IT*, vol. 49, pp. 1912–1921, 2003.

[18] Wei Yu, David Varodayan, and John Cioffi, "Trellis and convolutional precoding for transmitter-based interference presubtraction," *IEEE Trans. on Comm.*, vol. 53, no. 7, pp. 1220–1230, 2005.