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A comparison of bounded diffusion models for choice in time controlled tasks

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ABSTRACT

The Wiener diffusion model (WDM) for 2-alternative tasks assumes that sensory information is integrated over time. Recent neurophysiological studies have found neural correlates of this integration process in certain neuronal populations. This paper analyses the properties of the WDM with two different boundary conditions in decision making tasks in which the time of response is indicated by a cue. A dual reflecting boundary mechanism is proposed and its performance is compared with a well-established absorbing boundary in the cases of the WDM, the WDM with extensions, and the WDM with prior probability. The two types of boundary influence the dynamics of the model and introduce differential weighting of evidence. Comparisons with Ornstein–Uhlenbeck models are also done, and it is shown that the WDM with both types of boundary achieves similar performance and produces similar fits to existing behavioural data. Further studies are proposed to distinguish which boundary mechanism is more consistent with experimental data.

1. Introduction

Making choices is a frequent and critical element of human and animal lives. This problem has been studied by psychologists under two types of 2-alternative-forced-choice (2AFC) paradigms, in which subjects must decide between two available alternatives. The *information controlled* (IC) paradigm allows subjects to respond whenever they feel confident (Luce, 1986). Alternatively, subjects can be required to report their choice immediately after a cue to respond (Swensson, 1972; Yellott, 1971), under the *time controlled* (TC) paradigm (sometimes referred to as the response signal paradigm; see Dosher (1984)).

Over the last half century, several sequential sampling models have been proposed to describe experimental results as well as underlying decision making mechanisms in 2AFC tasks (for reviews, see Luce (1986) and Townsend and Ashby (1983)). The Wiener diffusion model (WDM) – the focus of this paper – assumes that subjects integrate partial information representing the relative support for the two alternatives over time (Laming, 1968; Ratcliff, 1978; Stone, 1960), and it has been shown to be the statistically optimal method for choosing between two alternatives on the basis of noisy evidence. For a fixed set of stimulus conditions, the WDM minimizes the reaction time for given accuracy, or maximizes the accuracy for given reaction time (Edwards, 1965; Gold & Shadlen, 2001, 2002; Laming, 1968; Wald, 1947). The WDM successfully describes the reaction time distribution and accuracy in various cognitive decision tasks in the IC paradigm (Laming, 1968; Link, 1975; Link & Heath, 1975; Ratcliff, 1978; Ratcliff & Smith, 2004; Ratcliff, Van Zandt, & McKoon, 1999; Stone, 1960). However, in the TC paradigm, the earlier version of the model (Ratcliff, 1978) allows integrator states to take arbitrary values. This leads to the prediction that the accuracy always increases over time if the drift of the process is in the correct direction. This is contrary to the finding that accuracy in the paradigm grows over time to an asymptote. To avoid this shortcoming, the model needs to assume that drift values are normally distributed across trials (Ratcliff, 1978) or that the integration range is bounded (Ratcliff, 1988). With one of these elaborations, the model correctly predicts the time–accuracy curves found in the IC condition.

This paper introduces a *reflecting boundary* mechanism to model 2AFC tasks in the TC paradigm. We compare the dynamics and performance of the WDM with reflecting and absorbing boundaries (Feller, 1968) and assess their ability to account for published behavioural data from 2AFC experiments. We also compare them with an Ornstein–Uhlenbeck (O–U) model (Busemeyer & Townsend, 1992, 1993; Diederich, 1995, 1997; Ratcliff & Smith, 2004; Smith, 1995, 2000). Analytical and numerical results show that both boundary types lead to similar performance and produce similar fits, but we identify some differences between them. Further studies are proposed that might distinguish which type of boundary better describes the decision making process.

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The paper is organized as follows. Section 2 reviews neurophysiological evidence of decision making, and describes the WDM and alternative boundary mechanisms. More detailed reviews are available elsewhere (Bogacz, Brown, Moehlis, Holmes, & Cohen, 2006; Glimcher, 2001; Gold & Shadlen, 2007; Schall, 2001; Smith & Ratcliff, 2004). Section 3 compares the dynamics and the performance of models with the two types of boundary. Section 4 compares fits of bounded models with behavioural data. Finally, Section 5 discusses further experimental studies that could distinguish between the boundary mechanisms. Mathematical details are provided in Appendices A and B.

2. The biology of decision and the sequential sampling models

2.1. The neural basis of 2AFC tasks

Recently, neuronal activity from awake animals has been recorded in choice experiments. For example, in the motion discrimination task, visual stimuli comprise arrays of random moving dots, a proportion of which move coherently to the left or right. Subjects (monkeys) are required to indicate their decision regarding the coherent direction by making a saccade to a left or right target (Roitman & Shadlen, 2002; Shadlen & Newsome, 2001).

The activity of neurons in the middle temporal (MT) area has been shown to correlate with the motion coherence (Britten, Shadlen, Newsome, & Movshon, 1993). However, since these neurons are highly noisy and poorly correlated with choices, the MT area is less likely to be a "decision maker" than to provide temporal information for a further process. It has been suggested that the lateral interparietal (LIP) area may interpret raw information from MT neurons. Shadlen and Newsome (1996) reported that LIP neurons gradually build up or attenuate their activity within a trial, and exhibit persistent activity in the absence of stimuli (Shadlen & Newsome, 2001). The time course of LIP neuronal activity suggests that the LIP area integrates inputs from MT neurons (Hanks, Ditterich, & Shadlen, 2006; Huk & Shadlen, 2005; Roitman & Shadlen, 2002). Similar discharge patterns are also found in the frontal eye field (FEF) (Schall, 2002) and the superior colliculus (SC) (Basso & Wurtz, 1998).

These results indicate a general decision mechanism manifested in different brain regions in which certain neuronal populations integrate sensory information over time to increase the accuracy of selection between alternatives (Gold & Shadlen, 2007; Schall, 2001). Gold and Shadlen (2001, 2002) formalize the decision process in 2AFC tasks as following two processes: two populations of sensory neurons (e.g., in the MT area) generate continuous noisy information streams ($Y_1(t)$ and $Y_2(t)$) for each of two alternatives Y_1 and Y_2 at time t. For simplicity, we assume that $Y_1(t)$ and $Y_2(t)$ have constant means μ_1 and μ_2 during each trial, with the same constant standard deviation, σ. The goal of the second process (reflected in LIP activities) is to successfully identify which input population has higher mean based on sample sequences $Y_1(t)$ and $Y_2(t)$. This framework is the basis for several sequential sampling models in behavioural studies, including the WDM (Ratcliff, 1978), the O–U model (Busemeyer & Townsend, 1992), and the leaky-competing-accumulator (LCA) model (Usher & McClelland, 2001).

2.2. The Wiener diffusion model (WDM)

The Wiener diffusion or Brownian motion is a continuous limit of the random walk (Laming, 1968; Ratcliff, 1978; Stone, 1960). It implies a leak-free integrator that accumulates the *difference* $Y_1(t)-Y_2(t)$ between noisy evidence streams for the two alternatives. Let $X(t)$ denote the accumulated difference at time t: the value of the integrator state, with initial state $X_0 = X(0)$.

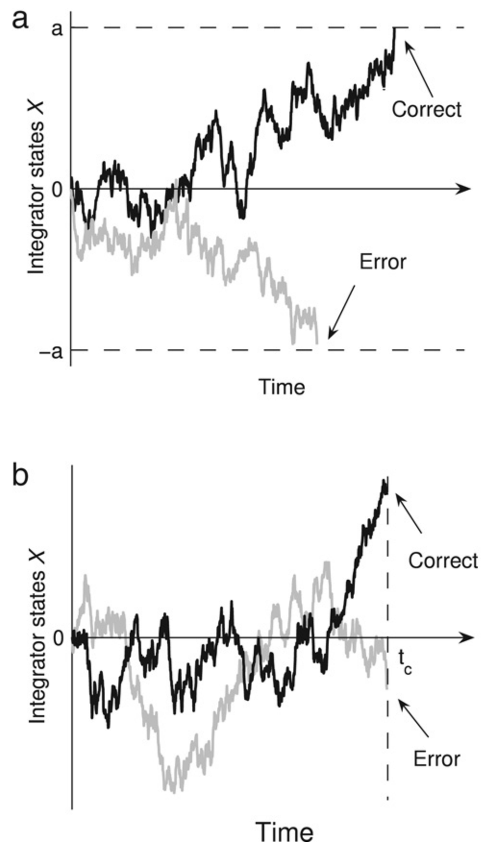

Fig. 1. Evolutions of integrator states after stimulus onset for the WDM, showing correct trials (black) and error trials (grey). The model was simulated with $\mu = \sigma = a = 1$ and time-step $dt = 0.01$ s, and hence $X(t) > 0$ corresponds to the correct alternative. (a) IC paradigm: choices are made on first reaching one of the two thresholds. (b) TC paradigm: choices are determined by the sign of the integrator state $X(t_c)$ at time t_c.

If there is no prior bias towards either choice, the process starts at baseline $X_0 = 0$, and is described by a stochastic differential equation:

$$dX(t) = \mu\, dt + \sigma\, dW(t), \quad \text{with } X_0 = 0, \tag{1}$$

where $dX(t)$ denotes the evidence obtained during time step dt. μ is a constant drift (the accumulation rate), representing the average of evidence difference $\mu_1 - \mu_2$. On a given trial, $\mu > 0$ ($\mu_1 > \mu_2$) implies that Y_1 is the correct choice, while $\mu < 0$ ($\mu_1 < \mu_2$) if Y_2 is correct. For consistency we hereafter set $\mu > 0$ unless indicated specifically, and hence assume that Y_1 is the correct choice. The magnitude of μ reflects the difficulty level of the task: for small μ ($\mu_1 \approx \mu_2$), it is difficult to distinguish which evidence samples have higher mean. The second term, $\sigma\, dW(t)$, denotes Gaussian noise with mean 0 and variance $\sigma^2 dt$. In the absence of noise ($\sigma = 0$), $X(t)$ changes at rate μ and always reaches a correct decision. Noisy inputs cause the $X(t)$ to fluctuate and hence induce incorrect choices on some trials.

Fig. 1 shows the growth of $X(t)$ in the two paradigms. For the IC paradigm, the decision time is unrestricted and two thresholds $\pm a$ are introduced to indicate termination states. Once $X(t)$ reaches a threshold, the corresponding alternative is chosen. For the TC paradigm, the decision process is interrupted by a response cue, and a response is immediately required. We hereafter denote the time delay from stimulus onset to response cue by t_c. The alternatives are selected by locating the final integrator state $X(t_c)$ and selecting Y_1 if $X(t_c) > 0$, and Y_2 if $X(t_c) < 0$.

Performance can be measured by the error rate: the probability of making an incorrect choice in a block of trials,[1] hereafter denoted by P. The error rate is a function of model parameters μ, σ, and threshold setting, a, or response signal, t_c, depending on the paradigm. For unbiased initial conditions $X_0 = 0$, the error rate of the WDM in the IC and TC paradigms is given by Eqs. (2) and (3), respectively (Bogacz, Usher, Zhang, and McClelland (2007); cf. Gardiner (1985) and Ratcliff (1978)):

$$P(a) = \frac{1}{1 + e^{\frac{2\mu a}{\sigma^2}}}, \quad \text{and} \tag{2}$$

$$P(t_c) = \int_{-\infty}^{-\frac{\mu}{\sigma}\sqrt{t_c}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \tag{3}$$

Two extensions have been proposed to improve fits to experimental data (Ratcliff et al., 1999). They allow certain parameters to vary randomly across trials. First, the drift rate $\tilde{\mu}$ is assumed to have a Gaussian distribution across trials with mean μ^* and variance σ_μ^2, which might reflect the variability of difficulty between trials, the subject's attention level, or other variable inter-trial factors. On each trial, $\tilde{\mu}$ can take either positive or negative values, promoting accumulation towards different alternatives. The correct choice is determined by the mean drift μ^*, even if the sampled value $\tilde{\mu}$ has opposite sign in some trials. This is motivated by the fact that, in difficult situations, stimulus distributions corresponding to the two alternatives often overlap (Ratcliff et al., 1999). Even after long training, perfect performance in such tasks is impossible. Drift variability is also necessary to ensure that the asymptotic accuracy of the WDM in the IC paradigm is not infinite in the absence of boundaries (Ratcliff, 1978). Second, the theory of premature sampling assumes that subjects start to integrate noise before sensory information is available. Hence the starting point is not at 0 when stimuli onset (Laming, 1968). Instead, on each trial X_0 is chosen from a uniform distribution [2] on the interval $[-\sigma_X, \sigma_X]$.

The extended WDM produces different reaction times on correct and error trials in the IC paradigm (Ratcliff et al., 1999). We assume that the same variability sources also operate in the TC paradigm; their effects on the decision process will be evaluated in the next section.

2.3. Boundary mechanisms

In the TC paradigm, the fact that integrator states $X(t)$ are unbounded implies that the error rate of the WDM with no variability in drift rate diminishes to zero for large t_c (cf. Eq. (3)). To eliminate this contradiction, absorbing boundaries can be introduced at $X = \pm b$. This mechanism was originally used to model tasks in which IC and TC paradigms were intermixed (Ratcliff, 1988). Such tasks promote subjects to respond as quickly as possible before a predetermined deadline. Under this condition the proposed model, also called the internal deadline model, assumes that the decision process is terminated when the accumulated evidence reaches one of the three possible boundaries, whichever comes first. The three boundaries are: top and bottom absorbing boundaries in the state domain, and a deadline boundary in the time domain (Diederich & Busemeyer, 2006; Ratcliff & Rouder, 2000).

In this work, we consider a pure TC paradigm in which subjects are only allowed to respond after the deadline t_c (Roitman & Shadlen, 2002). If the decision process reaches one of the absorbing boundaries before t_c, the accumulation process stops and the activity $X(t) = \pm b$ is maintained until the end of the trial. For sufficiently large t_c, $X(t)$ will almost surely reach one of the boundaries before t_c (cf. Section 3.1). Absorbing boundaries have the same effect as the decision threshold in the IC paradigm. Hence the error rate of the absorbing WDM with infinite t_c can be analytically obtained as (cf. Eq. (2))

$$\lim_{t_c \to \infty} P_{(abs)}(t_c) = \frac{1}{1 + e^{\frac{2\mu b}{\sigma^2}}}, \tag{4}$$

where the subscript *abs* stands for the WDM with absorbing boundary. For $b < \infty$, the error rate does not decrease to zero as t_c increases, which is consistent with experimental observations (Meyer, Irwin, Osman, & Kounios, 1988; Usher & McClelland, 2001).

In contrast with the absorbing boundary mechanism, since no time pressure exists in the pure TC paradigm, subjects may use the deadline alone to terminate the decision process. In this case two reflecting boundaries may be more suitable to constrain the accumulation process, because they allow the preferred choice to change even if $X(t)$ reaches a boundary. Here the boundaries restrict the amount of evidence that can be represented (much as a sigmoidal function provides cutoffs at high and low activation). Some previous studies (Diederich, 1995; Diederich & Busemeyer, 2003) use a lower reflecting and an upper absorbing boundary to model the simple reaction time task in which subjects respond immediately after a stimulus is detected. The reflecting boundary in their model and the one proposed here share similar motivations but there is a major difference. In the simple reaction time task, the accumulated sensory information directly represents the absolute evidence to make a response. Since the integrated information cannot drop below a certain baseline, one reflecting boundary is required to model the minimum level of absolute evidence. Single reflecting boundaries have also been used to represent a lower bound on the integration process (see Ratcliff and Smith (2004), Smith and Ratcliff (in press) and Usher and McClelland (2001)).

In the TC paradigm of 2AFC tasks, the integrator in the WDM represents the *relative* evidence supporting the alternatives. The preferred alternative at time t is determined by the sign of $X(t)$. A value of $X(t) > 0$ means that the first alternative is the provisional choice, whereas $X(t) < 0$ means that the second alternative is currently preferred. If we also assume that a minimum activity baseline exists and that decision preferences may switch during the trial, two reflecting boundaries are required to restrict the relative evidence of two alternatives within a certain range. Details of the model are given in Section 3.

2.4. The Ornstein–Uhlenbeck (O–U) model

Another widely applied sequential sampling method is the O–U model (Busemeyer & Diederich, 2002; Busemeyer & Townsend, 1992, 1993). It introduces a new parameter λ to the WDM to represent decay ($\lambda < 0$) or growth ($\lambda > 0$) of accumulated information, its evolution being described by

$$dX(t) = (\lambda X(t) + \mu)\, dt + \sigma dW, \quad X_0 = 0, \tag{5}$$

where the notations are as in Eq. (1). In the O–U model the accumulation rate depends not only on the drift μ but also on the

[1] Here we do not directly measure the probability of choosing certain alternatives (e.g., P_{Y_1} or P_{Y_2}) since in most experiments the correct choice is randomly assigned from the two alternatives across trials (e.g., Roitman and Shadlen (2002) and Shadlen and Newsome (2001)). Note that when we assume Y_1 is correct, then $P = P_{Y_2}$.

[2] The uniform distribution is assumed to prevent X_0 exceeding the thresholds $\pm a$ (by setting $\sigma_X < a$).