Equilibrium of Heterogeneous Congestion Control Protocols

Ao Tang*, Jiantao Wang*, Steven Low*, and Mung Chiang†
*Division of Engineering and Applied Science
California Institute of Technology, Pasadena, CA 91125, USA
†Department of Electrical Engineering
Princeton University, Princeton, NJ 08544, USA

***Abstract*— When heterogeneous congestion control protocols that react to different pricing signals share the same network, the resulting equilibrium may no longer be interpreted as a solution to the standard utility maximization problem. We prove the existence of equilibrium in general multi-protocol networks under mild assumptions. For almost all networks, the equilibria are locally unique, and finite and odd in number. They cannot all be locally stable unless it is globally unique. Finally, we show that if the price mapping functions that map link prices to effective prices observed by the sources are similar, then global uniqueness is guaranteed.**

I. Introduction

Congestion control protocols have been modeled as distributed algorithms for network utility maximization, e.g., [3], [7], [10], [15], [4], [6]. With the exception of a few limited analysis on very simple topologies [9], [5], [6], [1], existing literature generally assumes that all sources are homogeneous in that, even though they may control their rates using different algorithms, they all adapt to the same type of congestion signals, e.g., all react to loss probabilities, as in TCP Reno, or all to queueing delay, as in TCP Vegas or FAST [2]. When sources with *heterogeneous* protocols that react to different congestion signals share the same network, the current duality framework is no longer applicable. With more congestion control protocols being proposed and ideas of using congestion signals other than packet losses, including explicit feedbacks, being developed in the networking community, we need a mathematically rigorous framework to understand the behavior of large-scale networks with heterogeneous protocols. The purpose of this paper is to propose such a framework and some of the theoretical predictions in this paper have already been demonstrated experimentally in [14].

A congestion control protocol generally takes the form

$$\dot{p}_l = g_l\left(\sum_{j:l\in L(j)} x_j(t), p_l(t)\right) \qquad (1)$$

$$\dot{x}_j = f_j\left(x_j(t), \sum_{l\in L(j)} m_l^j(p_l(t))\right) \qquad (2)$$

Here, $L(j)$ denotes the set of links used by source j, and $g_l(\cdot)$ models a queue management algorithm that updates the price $p_l(t)$ at link l, often implicitly, based on its current value and the sum of source rates $x_j(t)$ that traverse link l. The prices may represent loss probabilities, queueing delays, or quantities explicitly calculated by the links and fed back to the sources. The function f_j models a TCP algorithm that adjusts the transmission rate $x_j(t)$ of source j based on its current value and the sum of "effective prices" $m_l^j(p_l(t))$ in its path. The effective prices $m_l^j(p_l(t))$ are functions of the link prices $p_l(t)$, and the functions m_l^j in general can depend on the links and sources.

When all algorithms use the same pricing signal, i.e., $m_l^j = m_l$ are the same for all sources j, the equilibrium properties of (1)–(2) turn out to be very simple. Indeed, under mild conditions on g_l and f_j, the equilibrium of (1)–(2) exists and is unique [6]. This is proved by identifying the equilibrium of (1)–(2) with the unique solution of the utility maximization problem defined in [3] and its Lagrange dual problem [7]. Here, the equilibrium prices p_l play the role of Lagrange multipliers, one at each link. This utility maximization problem thus provides a simple and complete characterization of the equilibrium of a single-protocol network and also leads to a relatively simple dynamic behavior.

When heterogeneous algorithms that use different pricing signals share the same network, i.e., m_l^j are different for different sources j, the situation is much more complicated. For instance, when TCP Reno and TCP Vegas or FAST share the same network, neither loss probability nor queueing delay can serve as the Lagrange multiplier at the link, and (1)–(2) can no longer be interpreted as solving the standard network utility maximization problem. Basic questions, such as the existence and uniqueness of equilibrium, its local and global stability, need to be re-examined. We focus in this paper on the existence and uniqueness of equilibrium. Due to page limitation, all proofs are omitted and they can be found in [13].

II. Model

A. Notation

A network consists of a set of L links, indexed by $l = 1, \ldots, L$, with finite capacities c_l. We often abuse notation and use L to denote both the number of links and the set $L = \{1, \ldots, L\}$ of links. Each link has a price p_l as its congestion measure. There are J different protocols indexed by superscript j, and N^j sources using protocol j, indexed by (j,i) where $j=1,\ldots,J$ and $i=1,\ldots,N^j$. The total number of sources is $N:=\sum_j N^j$.

The $L\times N^j$ routing matrix R^j for type j sources is defined by $R^j_{li}=1$ if source (j,i) uses link l, and 0 otherwise. The overall routing matrix is denoted by

$$R=\left[\begin{array}{cccc} R^1 & R^2 & \cdots & R^J \end{array}\right]$$

Even though different classes of sources react to different prices, e.g. Reno to packet loss probability and Vegas/FAST to queueing delay, the prices are related. We model this relationship through a price mapping function that maps a common price (e.g. queue length) at a link to different prices (e.g. loss probability and queueing delay) observed by different sources. Formally, every link l has a price p_l. A type j source reacts to the "effective price" $m^j_l(p_l)$ in its path, where m^j_l is a price mapping function, which can depend on both the link and the protocol type. The exact form of m^j_l depends on the AQM algorithm used at the link; see [14] for links with RED.[1] Let $m^j(p)=(m^j_l(p_l), l=1,\ldots L)$ and $m(p)=(m^j(p_l), j=1,\ldots J)$. The aggregate prices for source (j,i) is defined as

$$q^j_i \;=\; \sum_l R^j_{li} m^j_l(p_l) \tag{3}$$

Let $q^j=(q^j_i, i=1,\ldots,N^j)$ and $q=(q^j, j=1\ldots,J)$ be vectors of aggregate prices. Then $q^j=\left(R^j\right)^T m^j(p)$ and $q=R^T m(p)$.

Let x^j be a vector with the rate x^j_i of source (j,i) as its ith entry, and x be the vector of x^j

$$x \;=\; \left[\begin{array}{c} (x^1)^T, (x^2)^T, \ldots, (x^J)^T \end{array}\right]^T$$

Source (j,i) has a utility function $U^j_i(x^j_i)$ that is strictly concave increasing in its rate x^j_i. Let $U=(U^j_i, i=1,\ldots,N^j, j=1,\ldots,J)$.

In general, if z_k are defined, then z denotes the (column) vector $z=(z_k,\ \forall k)$. Other notations will be introduced later when they are encountered. We call (c,m,R,U) a *network*.

B. Network equilibrium

A network is in equilibrium, or the link prices p and source rates x are in equilibrium, when each source (j,i) maximizes its net benefit (utility minus bandwidth cost), and the demand for and supply of bandwidth at each bottleneck link are balanced. Formally, a network equilibrium is defined as follows.

Given any prices p, we assume in this paper that the source rates x^j_i are uniquely determined by

$$x^j_i\left(q^j_i\right) \;=\; \left[\left(U^j_i\right)^{\prime -1}\left(q^j_i\right)\right]^+$$

where $\left(U^j_i\right)^{\prime}$ is the derivative of U^j_i, and $\left(U^j_i\right)^{\prime -1}$ is its inverse which exists since U^j_i is strictly concave. Here $[z]^+=\max\{z,0\}$. This implies that the source rates x^j_i uniquely solve

$$\max_{z\geq 0} \;\; U^j_i(z) - z q^j_i$$

As we will see, under the assumptions in this paper, $\left(U^j_i\right)^{\prime -1}\left(q^j_i\right)>0$ for all the prices p that we consider, and hence we can ignore the projection $[\cdot]^+$ and assume without loss of generality that

$$x^j_i\left(q^j_i\right) \;=\; \left(U^j_i\right)^{\prime -1}\left(q^j_i\right) \tag{4}$$

As usual, we use $x^j\left(q^j\right)=\left(x^j_i\left(q^j_i\right), i=1,\ldots,N^j\right)$ and $x(q)=\left(x^j\left(q^j\right), j=1,\ldots,J\right)$ to denote the vector-valued functions composed of x^j_i. Since $q=R^T m(p)$, we often abuse notation and write $x^j_i(p), x^j(p), x(p)$. Define the aggregate source rates $y(p)=(y_l(p), l=1,\ldots,L)$ at links l as:

$$y^j(p) \;=\; R^j x^j(p), \qquad y(p) \;=\; Rx(p) \tag{5}$$

In equilibrium, the aggregate rate at each link is no more than the link capacity, and they are equal if the link price is strictly positive. Formally, we call p an *equilibrium price*, a *network equilibrium*, or just an *equilibrium* if it satisfies (from (3)–(5))

$$P(y(p)-c)=0, \quad y(p)\leq c, \quad p\geq 0 \tag{6}$$

where $P:=\text{diag}(p_l)$ is a diagonal matrix. The goal of this paper is to study the existence and uniqueness properties of network equilibrium specified by (3)–(6). Let E be the equilibrium set:

$$E \;=\; \{p\in\Re^L_+ \mid P(y(p)-c)=0,\ y(p)\leq c\} \tag{7}$$

For future use, we now define an active constraint set and the Jacobian for links that are actively constrained. Fix an equilibrium price $p^*\in E$. Let the *active constraint set* $\hat{L}=\hat{L}(p^*)\subseteq L$ (with respect to p^*) be the set of links l at which $p^*_l>0$. Consider the reduced system that consists only of links in $\hat{L}$, and denote all variables in the reduced system by $\hat{c}$, $\hat{p}$, $\hat{y}$, etc. Then, since $y_l(p)=c_l$ for every $l\in\hat{L}$, we have $\hat{y}(\hat{p})=\hat{c}$. Let the Jacobian for the reduced system be $\hat{\boldsymbol{J}}(\hat{p})=\partial\hat{y}(\hat{p})/\partial\hat{p}$. Then

$$\hat{\boldsymbol{J}}(\hat{p}) \;=\; \sum_j \hat{R}^j \frac{\partial x^j}{\partial \hat{q}^j}(\hat{p})\left(\hat{R}^j\right)^T \frac{\partial \hat{m}^j}{\partial \hat{p}}(\hat{p}) \tag{8}$$

where

$$\frac{\partial x^j}{\partial \hat{q}^j} \;=\; \text{diag}\left(\left(\frac{\partial^2 U^j_i}{\partial (x^j_i)^2}\right)^{-1}\right) \tag{9}$$

$$\frac{\partial \hat{m}^j}{\partial \hat{p}} \;=\; \text{diag}\left(\frac{\partial \hat{m}^j_l}{\partial \hat{p}_l}\right) \tag{10}$$

and all the partial derivatives are evaluated at the generic point $\hat{p}$.

[1] One can also take the price p^j_l used by one of the protocols, e.g. queueing delay, as the common price p_l. In this case the corresponding price mapping function is the identity function, $m^j_l(p_l)=p_l$.

C. Current theory: $J = 1$

In this subsection, we briefly review the current theory for the case where there is only one protocol, i.e., $J = 1$, and explain why it cannot be directly applied to the case of heterogeneous protocols.

When all sources react to the same price, then the equilibrium described by (3)–(6) is the unique solution of the following utility maximization problem defined in [3] and its Lagrange dual [7]:

$$\max_{x \geq 0} \quad \sum_i U_i(x_i) \tag{11}$$

$$\text{subject to} \quad Rx \ \leq \ c \tag{12}$$

where we have omitted the superscript $j = 1$. The strict concavity of U_i guarantees the existence and uniqueness of the optimal solution of (11)–(12). The basic idea to relate the utility maximization problem (11)–(12) to the equilibrium equations (3)–(6) is to examine the dual of the utility maximization problem, and interpret the effective price $m_l(p_l)$ as a Lagrange multiplier associated with each link capacity constraint (see, e.g., [7], [10], [6]). As long as $m_l(p_l) \geq 0$ and $m_l(0) = 0$, one can replace p_l in (6) by $m_l(p_l)$. The resulting equation together with (3)–(5) provides the necessary and sufficient condition for $x_i(p)$ and $m_l(p_l)$ to be primal and dual optimal respectively.

This approach breaks down when there are $J > 1$ types of prices because there cannot be more than one Lagrange multiplier at each link. In general, an equilibrium no longer maximizes aggregate utility, nor is it unique. However, as shown in Theorem 1, existence of equilibrium is still guaranteed under the following assumptions:

- A1: Utility functions U_i^j are strictly concave increasing, and twice continuously differentiable in their domains. Price mapping functions m_l^j are continuously differentiable in their domains and strictly increasing with $m_l^j(0) = 0$.
- A2: For any $\epsilon > 0$, there exists a number $p_{\max}$ such that if $p_l > p_{\max}$ for link l, then

$$x_i^j(p) < \epsilon \text{ for all } (j, i) \text{ with } R_{li}^j = 1$$

These are mild assumptions. Concavity and monotonicity of utility functions are often assumed in network pricing for elastic traffic. Moreover, most TCP algorithms proposed or deployed turn out to have strictly concave increasing utility functions; see e.g. [6]. The assumption on m_l^j preserves the relative order of prices and maps zero price to zero effective price. Assumption A2 says that when p_l is high enough, then every source going through link l has a rate less than ϵ. minutes for presentation.

Theorem 1. *Suppose A1 and A2 hold. There exists an equilibrium price p^* for any network (c, m, R, U).*

III. Regular networks

Theorem 1 guarantees the existence of network equilibrium. We now study its uniqueness properties.

A. Multiple equilibria: examples

In a single-protocol network, if the routing matrix R has full row rank, then there is a unique active constraint set $\hat{L}$ and a unique equilibrium price p associated with it. If R does not have full row rank, then equilibrium prices p may be non-unique but the equilibrium rates $x(p)$ are still unique since the utility functions are strictly concave.

In contrast, the active constraint set in a multi-protocol network can be non-unique even if R has full row rank (Example 2). Clearly, the equilibrium prices associated with different active constraint sets are different. Moreover, there can be multiple equilibrium prices associated with the same active constraint set (Example 1).

Example 1: unique active constraint set but uncountably many equilibria

In this example, we assume all the sources use the same utility function

$$U_i^j(x_i^j) \quad = \quad -\frac{1}{2}\left(1 - x_i^j\right)^2 \tag{13}$$

Then the equilibrium rates x^j of type j sources are determined by the equilibrium prices p as

$$x^j(p) \quad = \quad \mathbf{1} - (R^j)^T m^j(p)$$

where $\mathbf{1}$ is a vector of appropriate dimension whose entries are all 1s. We use linear price mapping functions:

$$m^j(p) \quad = \quad K^j p$$

where K^j are $L \times L$ diagonal matrices. Then the equilibrium rate vector of type j sources can be expressed as

$$x^j(p) \quad = \quad \mathbf{1} - (R^j)^T K^j p$$

When only links with strictly positive equilibrium prices are included in the model, we have

$$y(p) \ = \ \sum_{j=1}^{J} R^j x^j(p) \quad = \quad c$$

Substituting in $x^j(p)$ yields

$$\sum_{j=1}^{J} R^j (R^j)^T K^j p \quad = \quad \sum_{j=1}^{J} R^j \mathbf{1} - c$$

which is a linear equation in p for given R^j, K^j, and c. It has a unique solution if the determinant is nonzero, but has no or multiple solutions if

$$\det\left(\sum_{j=1}^{J} R^j (R^j)^T K^j\right) \quad = \quad 0$$

When $J = 1$, i.e., there is only one protocol, and R^1 has full row rank, $\det(R^1(R^1)^T K^1) > 0$ since both $R^1(R^1)^T$ and K^1 are positive definite. In this case, there is a unique equilibrium price vector. When $J = 2$, there are networks whose determinants are zero that have uncountably many equilibria. See [12] for an example where R does not have full row rank. We provide here an example with $J=3$ where R still has full row rank.

The network is shown in Figure 1 with three unit-capacity links, $c_l = 1$. There are three different protocols with the corresponding routing matrices

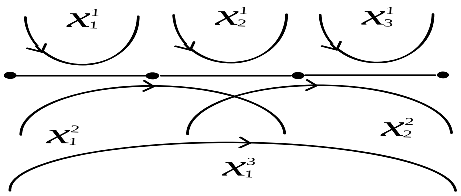

Fig. 1. Example 1: uncountably many equilibria.

$$R^1 = I, \quad R^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T, \quad R^3 = (1,1,1)^T$$

The linear mapping functions are given by

$$K^1 = I, \quad K^2 = \text{diag}(5,1,5), \quad K^3 = \text{diag}(1,3,1)$$

It is easy to calculate that

$$\sum_{i=1}^{3} R^i (R^i)^T K^i = \begin{bmatrix} 7 & 4 & 1 \\ 6 & 6 & 6 \\ 1 & 4 & 7 \end{bmatrix}$$

which has determinant 0. Using the utility function defined in (13), we can check that the following are equilibrium prices for all $\epsilon \in [0, 1/24]$:

$$p_1^1 = p_3^1 = 1/8 + \epsilon \qquad p_2^1 = 1/4 - 2\epsilon$$

The corresponding rates are

$$x_1^1 = x_3^1 = 7/8 - \epsilon \qquad x_2^1 = 3/4 + 2\epsilon$$
$$x_1^2 = x_2^2 = 1/8 - 3\epsilon \qquad x_1^3 = 4\epsilon$$

All capacity constraints are tight with these rates. Since there is an one-link flow at every link, the active constraint set is unique and contains every link. Yet there are uncountably many equilibria.

Example 2: multiple active constraint sets each with a unique equilibrium

Consider the symmetric network in Figure 2 with 3 flows. There are two protocols in the network with the following routing matrices

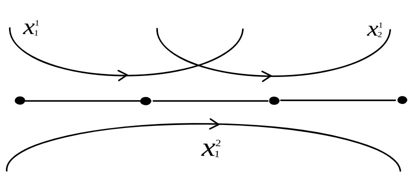

Fig. 2. Example 2: two active constraint sets.

$$R^1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R^2 = (1,1,1)^T$$

Flows $(1,1)$ and $(1,2)$ have identical utility function U^1 and source rate x^1, and flow $(2,1)$ has a utility function U^2 and source rate x^2.

Links 1 and 3 both have capacity c_1 and price mapping functions $m_1^1(p) = p$ and $m_1^2(p)$ for protocols 1 and 2 respectively. Link 2 has capacity c_2 and price mapping functions $m_2^1(p) = p$ and $m_2^2(p)$.

In [14], we prove that when assumption A1 holds, the network shown in Figure 2 has at least two equilibria provided:

1) $c_1 < c_2 < 2c_1$;
2) for $j = 1, 2$, $(U^j)'(x^j) \to \overline{p}^j$, possibly ∞, if and only if $x^j \to 0$.
3) for $l = 1, 2$, $m_l^2(p_l) \to \overline{p}^2$ as $p_l \to \overline{p}^1$, and satisfy

$$2m_1^2((U^1)'(c_2 - c_1)) < (U^2)'(2c_1 - c_2) < m_2^2((U^1)'(c_2 - c_1))$$

B. Regular networks

Examples 1 and 2 show that global uniqueness is generally not guaranteed in a multi-protocol network. We now show, however, that local uniqueness is basically a generic property of the equilibrium set. We present our main results on the structure of the equilibrium set here, providing conditions for the equilibrium points to be locally unique, finite and odd in number, and globally unique.

Consider an equilibrium price $p^* \in E$. Recall the active constraint set $\hat{L}$ defined by p^*. The equilibrium price $\hat{p}^*$ for the links in $\hat{L}$ is a solution of

$$\hat{y}(\hat{p}) = \hat{c} \tag{14}$$

By the inverse function theorem, the solution of (14), and hence the equilibrium price $\hat{p}^*$, is *locally unique* if the Jacobian matrix $\hat{\boldsymbol{J}}(\hat{p}^*) = \partial \hat{y} / \partial \hat{p}$ is nonsingular at $\hat{p}^*$. We call a network (c, m, R, U) *regular* if all its equilibrium prices are locally unique.

The next result shows that almost all networks are regular, and that regular networks have finitely many equilibrium prices. This justifies restricting our attention to regular networks.

Theorem 2. *Suppose assumptions A1 and A2 hold. Given any price mapping functions m, any routing matrix R and utility functions U,*

1) *the set of link capacities c for which not all equilibrium prices are locally unique has Lebesgue measure zero in $\Re_+^L$.*
2) *the number of equilibria for a regular network (c, m, R, U) is finite.*

For the rest of this subsection, we narrow our attention to networks that satisfy an additional assumption:

A3: Every link l has a single-link flow (j, i) with $\left(U_i^j\right)'(c_l) > 0$.

Assumption A3 says that when the price of link l is small enough, the aggregate rate through it will exceed its capacity. This ensures that the active constraint set contains all links and facilitates the application of Poincare-Hopf theorem by avoiding equilibrium on the boundary (some $p_l = 0$). [2]

Since all the equilibria of a regular network have nonsingular Jacobian matrices, we can define the *index* $I(p)$ of $p \in E$ as

$$I(p) \quad = \quad \begin{cases} 1 & \text{if } \det(\boldsymbol{J}(p)) > 0 \\ -1 & \text{if } \det(\boldsymbol{J}(p)) < 0 \end{cases}$$

Then, we have

Theorem 3. *Suppose assumptions A1–A3 hold. Given any regular network, we have*

$$\sum_{p \in E} I(p) = (-1)^L$$

where L is the number of links.

We give two important consequences of this theorem.

Corollary 4. *Suppose assumptions A1–A3 hold. A regular network has an odd number of equilibria.*

Notice that Corollary 4 implies the existence of equilibrium. Although we have this via theorem 1 in a more general setting, this simple corollary shows the power of Theorem 3.

The next result provides a condition for global uniqueness. We say that an equilibrium $p^* \in E$ is *locally stable* if the corresponding Jacobian matrix $\boldsymbol{J}(p^*)$ defined in (8) is stable, that is, every eigenvalue of $\boldsymbol{J}(p^*) = \partial y(p^*)/\partial p$ has negative real part.

Corollary 5. *Suppose assumptions A1–A3 hold. The equilibrium of a regular network is globally unique if and only if every equilibrium point in E has an index $(-1)^L$. In particular, if all equilibria are locally stable, then E contains exactly one point.*

Finally we reveal that, under assumptions A1–A3, if the price mapping functions m_l^j are similar, then the equilibrium of a regular network is globally unique.

To state the result concisely, we need the notion of permutation. We call a vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_L)$ a *permutation* if each σ_l is distinct and takes value in $\{1, \ldots, L\}$. Treating $\boldsymbol{\sigma}$ as a mapping $\boldsymbol{\sigma} : \{1, \ldots, L\} \rightarrow \{1, \ldots, L\}$, we let $\boldsymbol{\sigma}^{-1}$ denote its unique inverse permutation. For any vector $a \in \Re^L$, $\boldsymbol{\sigma}(a)$ denotes the permutation of a under $\boldsymbol{\sigma}$, i.e., $[\boldsymbol{\sigma}(a)]_l = a_{\sigma_l}$. If $a \in \{1, \ldots, L\}^L$ is a permutation, then $\boldsymbol{\sigma}(a)$ is also a permutation and we often write $\boldsymbol{\sigma} a$ instead. Let $\boldsymbol{l} = (1, \ldots, L)$ denote the identity permutation. Then $\boldsymbol{\sigma l} = \boldsymbol{\sigma}$. See [8] for more details. Finally, denote dm_l^j/dp_l by $\dot{m}_l^j$.

Theorem 6. *Suppose assumptions A1–A3 hold. If, for any vector $\boldsymbol{j} \in \{1, \ldots, J\}^L$ and any permutations $\boldsymbol{\sigma}, \boldsymbol{k}, \boldsymbol{n}$ in $\{1, \ldots, L\}^L$,*

$$\prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{k}(\boldsymbol{j})]_l} + \prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{n}(\boldsymbol{j})]_l} \geq \prod_{l=1}^{L} \dot{m}_l^{[\boldsymbol{\sigma}(\boldsymbol{j})]_l} \qquad (15)$$

then the equilibrium of a regular network is globally unique.

IV. Conclusion

When sources sharing the same network react to different pricing signals, the current duality model no longer explains the equilibrium of bandwidth allocation. We have introduced a mathematical formulation of network equilibrium for multi-protocol networks and studied several fundamental properties, such as existence, local uniqueness, number of equilibria, and global uniqueness. We prove that equilibria exist, and are almost always locally unique. The number of equilibria is almost always finite and must be odd. Finally the equilibrium is globally unique if the price mapping functions are similar.

References

[1] L. Grieco and S. Mascolo. Performance Evaluation and Comparison of Westwood+, New Reno, and Vegas TCP Congestion Control ACM SIGCOMM Computer Communications Review,34(2):25-38, April 2004

[2] D. Wei, C. Jin, S. H. Low and S. Hegde FAST TCP: motivation, architecture, algorithms, performance. *to appear in IEEE/ACM Transactions on Networking*, 2007. `http://netlab.caltech.edu/FAST/`

[3] F. Kelly, A. Maoulloo, and D. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49:237–252,1998.

[4] S. Kunniyur and R. Srikant. End-to-end congestion control: utility functions, random losses and ECN marks. *IEEE/ACM Transactions on Networking*, 11(5):689-702, Oct 2003.

[5] K. Kurata, G. Hasegawa, and M. Murata. Fairness comparisons between TCP Reno and TCP Vegas for future deployment of TCP Vegas. *Proceedings of INET*, 2000.

[6] S. H. Low. A duality model of TCP and queue management algorithms. *IEEE/ACM Trans. on Networking*, 11(4):525-536, August 2003.

[7] S. H. Low and D. Lapsley. Optimization flow control, I: basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, 7(6):861-874, December 1999.

[8] C. Meyer. *Matrix Analysis and Applied Linear Algebra*. SIAM, 2000.

[9] J. Mo, R. La, V. Anantharam, and J. Walrand. Analysis and comparison of TCP Reno and Vegas. *Proceedings of IEEE Infocom*, March 1999.

[10] J. Mo and J. Walrand. Fair end-to-end window-based congestion control. *IEEE/ACM Transactions on Networking*, 8(5):556-567, October 2000.

[11] A. Simsek, A. Ozdaglar and D. Acemoglu. Generalized Poincare-Hopf Theorem for compact nonsmooth regions. submitted for publication, 2005.

[12] A. Tang, J. Wang, S. H. Low and M. Chiang. Network Equilibrium of Heterogeneous Congestion Control Protocols. *Proceedings of IEEE Infocom*, 2005.

[13] A. Tang, J. Wang, S. H. Low and M. Chiang. Equilibrium of Heterogeneous Congestion Control Protocols. *Caltech CS Technical Report CaltechCSTR:2005.005*, April 2005. `http://resolver.caltech.edu/CaltechCSTR:2005.005`

[14] A. Tang, J. Wang, S. Hedge and S. H. Low. Equilibrium and fairness of networks shared by TCP Reno and Vegas/FAST. Telecommunication Systems, 30(4):417-439, December 2005.

[15] H. Yaiche, R. Mazumdar, and C. Rosenberg. A game theoretic framework for bandwidth allocation and pricing in broadband networks. *IEEE/ACM Transactions on Networking*, 8(5), October 2000.

[2] It is recently shown in [11] that A3 is not necessary and one can generalize Theorem 3 to

$$\sum_{p \in E} (-1)^{\hat{L}(p)} I(p) = 1$$

where $\hat{L}(p)$ is the number of links of the active constraint set associated with equilibrium p. Clearly, if $\hat{L}(p) = L$, it reduces to Theorem 3. This generalized theorem also allows [11] to conclude the number of equilibria is odd (and therefore existence) without A3. In this paper, although A3 is imposed, all results can be viewed as with respect to a fixed active constraint set with appropriate modifications. In particular, the global uniqueness result theorem 6 directly apply without A3 since $\hat{\boldsymbol{J}}$ has a similar structure as $\boldsymbol{J}$ except with a smaller dimension.