Optimal Traffic Engineering Via Newton's Method

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Outline

1. Background
2. Network Entropy Maximization (NEM)
3. Solve NEM with Newton's Method
4. Performance Evaluation
5. Summary

Outline

Minimum-cost Multicommodity Flow

- Minimum-cost Multicommodity Flow Problem
  - Classical Convex Optimization problem
  - Aliases
    - ⋆ Optimal Routing: *Data Networks* [Bertsekas-Gallager]
    - ⋆ Optimal Traffic Engineering: IP congestion control
    - ⋆ ...
- Question: can we realize Optimal Routing with link-state routing?

City Traffic Control

- Big cities suffer from traffic congestion during rush hours
- The traffic to a same destination is a commodity

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- The traffic to a same destination is a commodity
- Traffic control to realize optimal commodity solution:
  - Explicit Routing
  - Road Price

Traffic Control with Explicit Routing

- At intersection A
  - Use Expressway I-95 if you go to Manhattan and your plate number is divisible with 7
  - Use Somewhere Lane if you go to Princeton and your plate number is divisible with 11
  - ...
- At intersection B ...

Traffic Control with Explicit Routing

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  - ...
- At intersection B ...
- Challenging even for drivers with Ph.D. degree

Traffic Control with Road Price

- Balance traffic by setting price for each road segment
- More feasible than Explicit Routing

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- Assumption I: all drivers choose the "cheapest" path (even splitting if multiple cheapest paths)
  $\Rightarrow$ Impossible to achieve optimal routing and NP-hard to find road prices [Fortz-Thorup, Infocom-00]

Traffic Control with Road Price

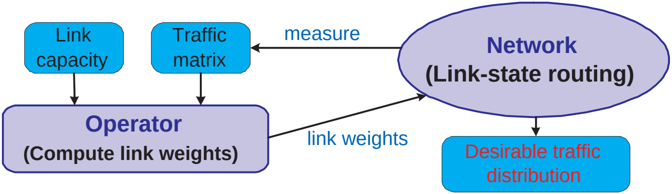
- Balance traffic by setting price for each road segment
- More feasible than Explicit Routing
- Assumption I: all drivers choose the "cheapest" path (even splitting if multiple cheapest paths)
  $\Rightarrow$ Impossible to achieve optimal routing and NP-hard to find road prices [Fortz-Thorup, Infocom-00]
- Assumption II:
  - More drivers choose the "cheapest" path
  - Fewer drivers choose more "expensive" path expecting less congestion (delay)

  $\Rightarrow$ Always achieve optimal routing and Convex Optimization to find road prices [Xu-Chiang-Rexford, Infocom-08]

Link-State Routing

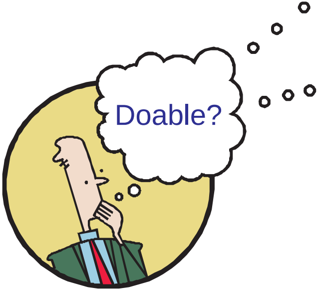
- Routers
  - Exchange link weights (states) with Interior Gateway Protocols (IGPs): e.g. OSPF (Open Shortest Path First)
  - Distributively determine "next hop" to forward a packet/split traffic
- Network operator configures link weights to guide routing
  $\Rightarrow$ Traffic Engineering

Tuning Link Weights



- Traffic Engineering (TE): based on the offered traffic matrix
  - Traffic matrix: rate of traffic between each node pair from measurement
  - Centralized and off-line
  - Network-wide convex optimization objective: minimizes key metrics like max link utilization, sum of M/M/1 delay at each link, etc.

Open Questions by 2008



Optimal TE with ONLY link weights?

Find link weights in a tractable way?

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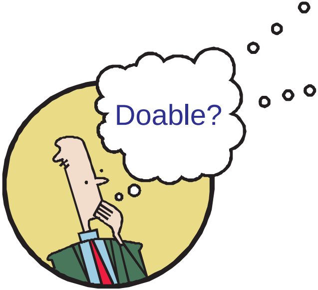

Optimal TE with ONLY link weights?

Find link weights in a tractable way?

YES !

NEM/PEFT [Xu-Chiang-Rexford, Infocom-08]

Outline

Notation

- Directed graph: N nodes and E links
- Inputs

$D(s,t)$	Traffic demand from s to t
$c_{u,v}$	Capacity of link (u,v)

- Variables

$w_{u,v}$	Weight for link (u,v)
$f^t_{u,v}$	Commodity flow on link (u,v) destined to t
$f_{u,v}$	$\triangleq \sum_t f^t_{u,v}$, Total flow on link (u,v)

Optimal TE Via Multicommodity-Flow

COMMODITY Problem:

minimize	$\Phi(\{f_{u,v}, c_{u,v}\})$	convex objective
subject to	$\sum_{v:(s,v)\in\mathbb{E}} f^t_{s,v} - \sum_{u:(u,s)\in\mathbb{E}} f^t_{u,s} = D(s,t)$	flow conservation
	$f_{u,v} \triangleq \sum_{t\in\mathbb{V}} f^t_{u,v} \leq c_{u,v}$	capacity constraint
variables	$f_{u,v} \geq f^t_{u,v} \geq 0.$	link flow, commodity flow
input	$D(s,t), c_{u,v}$	demand, capacity

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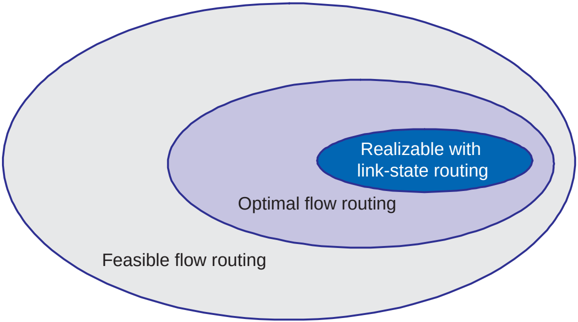
- Convex optimization (efficiently solvable).
- Can be realized with explicit routing: set up N^2E tunnels
- Link-state routing: E parameters

Necessary Capacity

- Necessary Capacity
  - $\widetilde{c}_{u,v} \triangleq f_{u,v}$: Total traffic on each link in optimal solution of COMMODITY
  - Minimal set of link capacities to realize optimal TE
- Set link weights with only necessary capacities

Intuition Behind the Theory



- Numerous ways of flow-level routing to realize optimal TE (different traffic distribution on the paths)
- Choose the flow-level routing which can be realized with link-state routing.
- How? Pick an additional objective function for these optimal flow-level routings

Network Entropy Maximization

- Assume we can enumerate all the paths from s to t, $P^i_{s,t}$. (only for analysis purpose)
- $x^i_{s,t}$: probability (fraction) of forwarding a packet of demand $D(s,t)$ to the i-th path ($P^i_{s,t}$)

subject to	$\sum_{s,t,i:(u,v)\in P^i_{s,t}} D(s,t)x^i_{s,t} \leq \widetilde{c}_{u,v}$	capacity constraint
	$\sum_i x^i_{s,t} = 1$	flow conservation
variables	$0 \leq x^i_{s,t} \leq 1.$	forwarding probability

Network Entropy Maximization

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- $x^i_{s,t}$: probability (fraction) of forwarding a packet of demand $D(s,t)$ to the i-th path ($P^i_{s,t}$)
- $z(x) = -x \log x$: Entropy function

Network Entropy Maximization (NEM)

maximize	$\sum_{s,t} D(s,t) \left(\sum_{P^i_{s,t}} z(x^i_{s,t}) \right)$	total entropy
subject to	$\sum_{s,t,i:(u,v)\in P^i_{s,t}} D(s,t) x^i_{s,t} \leq \widetilde{c}_{u,v}$	capacity constraint
	$\sum_i x^i_{s,t} = 1$	flow conservation
variables	$0 \leq x^i_{s,t} \leq 1.$	forwarding probability

NEM features

- NEM problem always has a global optimal solution.
  - Feasible solution: any optimal solution of COMMODITY problem
  - $z(x)$ is a concave function
  - Convex Optimization
- Solving directly is not efficient (Infinite path enumeration with cycles)

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- Solving directly is not efficient (Infinite path enumeration with cycles)
- Prim-dual method (with E dual variables)

Optimal Solution of NEM

- Necessary Condition

$$\frac{x^i_{s,t}}{x^j_{s,t}} = \frac{e^{-\sum_{(u,v)} K^{(u,v)}_{P^i_{s,t}} \lambda_{u,v}}}{e^{-\sum_{(u,v)} K^{(u,v)}_{P^j_{s,t}} \lambda_{u,v}}}.$$

- $\lambda_{u,v}$: dual variable for necessary capacity constraint
- $K^{(u,v)}_{P^i_{s,t}}$: number of times $P^i_{s,t}$ passes through link (u, v)

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Penalizing Exponential Flow-spliTting (PEFT)

$$\text{PEFT:} \qquad x^i_{u,t} = \frac{e^{-p^i_{u,t}}}{\sum_j e^{-p^j_{u,t}}}.$$

- $p^i_{u,t}$: sum of $\lambda_{u,v}$ along the ith path

Algorithm for Optimizing Link Weights

Optimize Over Link Weights

Compute necessary capacities $\widetilde{\mathbf{c}}$ by solving COMMODITY problem
$\mathbf{w} \leftarrow$ Any set of link weights
$\mathbf{f} \leftarrow$ Traffic_Distribution($\mathbf{w}$)
while $\mathbf{f} \neq \widetilde{\mathbf{c}}$ **do**
    $\mathbf{w} \leftarrow$ Link_Weight_Update($\mathbf{f}$)
    $\mathbf{f} \leftarrow$ Traffic_Distribution($\mathbf{w}$)
end while

Solve NEM Dual with Gradient Descent

- Solve NEM Dual problem using gradient descent

$$\boldsymbol{\lambda}(q+1) = [\boldsymbol{\lambda}(q) - \alpha(q)\nabla Q(\boldsymbol{\lambda}(q))]^+$$

- $$\begin{aligned}&\frac{\partial Q}{\partial \lambda_{u,v}}(q)\\ &= \widetilde{c}_{u,v} - \sum_{s,t,i} D(s,t) K^{(u,v)}_{P^i_{s,t}} x^i_{s,t}(q)\\ &= \widetilde{c}_{u,v} - f_{u,v}(q)\end{aligned}$$

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Solve NEM Dual with Newton's Method

- Gradient is scaled by the inverse of $\nabla^2 Q(\boldsymbol{\lambda}(q))$

$$\boldsymbol{\lambda}(q+1) = \left[\boldsymbol{\lambda}(q) - \nabla^2 Q(\boldsymbol{\lambda}(q))^{-1} \nabla Q(\boldsymbol{\lambda}(q))\right]^+ .$$

- $$\frac{\partial^2 Q}{\partial \lambda_{u,v} \partial \lambda_{u',v'}}(q) = \sum_{s,t,i} D(s,t) K_{P^i_{s,t}}^{(u,v)} K_{P^i_{s,t}}^{(u',v')} x^i_{s,t}(q).$$

Hessian of NEM Dual with Cycles

- $\psi^{t}_{u,v}$: Splitting fraction (destined to t) on link (u, v)
- $\eta^{s,t}_{u}$: Total flow at node u for unit traffic demand from s to t
- Compute $\psi^{t}_{u,v}$ and $\eta^{s,t}_{u}$ by solving N sets of $N \times N$ equations (see paper).

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- Compute $\psi^t_{u,v}$ and $\eta^{s,t}_u$ by solving N sets of $N \times N$ equations (see paper).

Theorem (Hessian of NEM Dual can be found in polynomial time)

$$\frac{\partial^2 Q}{\partial \lambda_{u,v} \partial \lambda_{u',v'}} = \\ = \begin{cases} \sum_{t \in \mathbb{V}} \left(f^t_{u,v} \eta^{v,t}_{u'} \psi^t_{u',v'} + f^t_{u',v'} \eta^{v',t}_{u} \psi^t_{u,v} \right) & \text{if } (u,v) \neq (u',v') \\ \sum_{t \in \mathbb{V}} \left(f^t_{u,v} (1 + 2\eta^{v,t}_u \psi^t_{u,v}) \right) & \text{if } (u,v) = (u',v'). \end{cases}$$

- Total time complexity $O(N^4 + NE^2)$

Hessian of NEM Dual without Cycles

- Optimal routing should contain no cycles
- Downward PEFT: approximation by forwarding traffic only on next hops closer to the destination
- Total time complexity $O(N^3 + N^2 E)$

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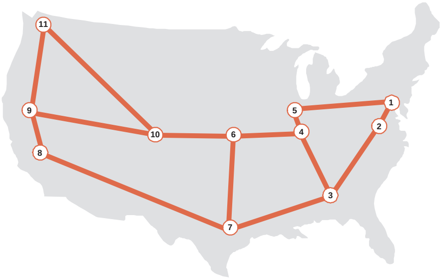
Various Methods of Solving NEM

- Exact PEFT with Gradient Descent
- Downward PEFT with Gradient Descent
- Exact PEFT with Newton's Method
- Downward PEFT with Newton's Method

Network Topologies



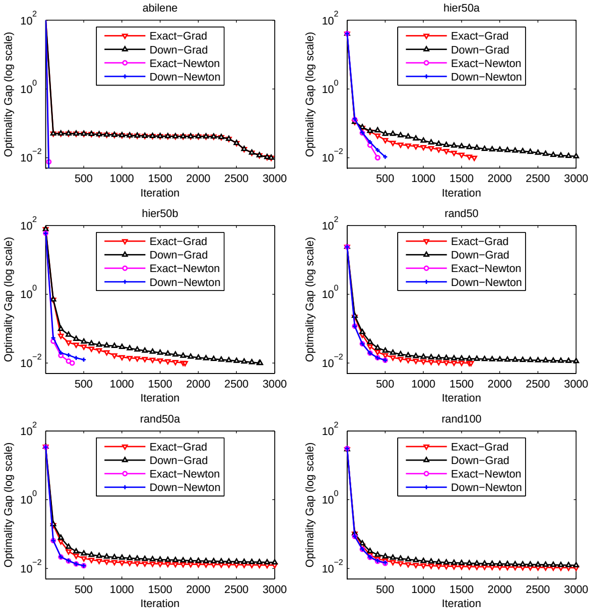
Abilene Network

Name	Topology	Node #	Link #	Link Capacity
abilene	Backbone	11	28	10Gbps
hier50a	2-level	50	148	local access(200), long-haul (1000)
hier50b	2-level	50	212	local access(200), long-haul (1000)
rand50	Random	50	228	1000
rand50a	Random	50	245	1000
rand100	Random	100	403	1000

Traffic Matrices

- Abilene Network: measured data on Nov. 15th, 2005
- Other networks: same as [Fortz-Thorup-2000]
- Uniformly scale the traffic matrix with maximum link utilization close to 100% in optimal TE.

Convergence Behavior



Average Running Time per Iteration

Net. ID	Node #	Link #	Time per Iteration (millisecond)			
			Gradient		Newton	
			Exact	Down	Exact	Down
abilene	11	28	3.4	2.1	7.9	3.6
hier50a	50	148	13.7	5.9	57.1	17.7
hier50b	50	212	13.8	6.3	82.4	36.4
rand50	50	228	19.4	7.1	94.8	43.9
rand50a	50	245	24.3	7.6	110.8	50.0
rand100	100	403	175.5	36.7	911.7	282.3

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- Time per iteration: Newton's method > Gradient descent: inverse of the Hessian
- Can be considerably simplified by adopting quasi-Newton methods

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Conclusion

- Minimum-cost multicommodity flow can be realized by a link-state routing protocol (PEFT) from solving NEM.
- Efficient Newton's methods to solve the NEM problem with an infinite number of variables.

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- Minimum-cost multicommodity flow can be realized by a link-state routing protocol (PEFT) from solving NEM.
- Efficient Newton's methods to solve the NEM problem with an infinite number of variables.
- Open Problems
  - Computational Complexity of NEM/PEFT: Polynomial?
  - Solve NEM/PEFT + COMMODITY problem altogether?
  - Whether DEFT [Xu-Chiang-Rexford, Infocom-07] can achieve optimal traffic engineering as well?
- More Information
  http://www.research.att.com/∼dahaixu