# Equilibria and Price of Anarchy in Parallel Relay Networks with Node Pricing

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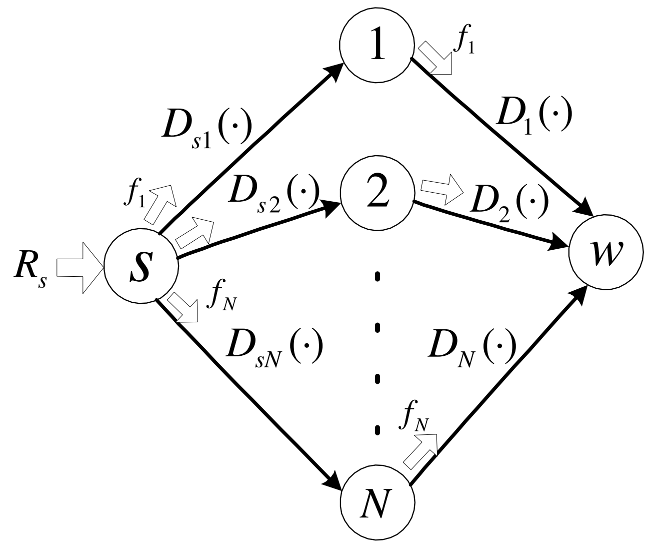
# Parallel Relay Networks




- Source $s$, destination $w$.
- No direct connection from $s$ to $w$.
- Relay nodes $1, \ldots, N$ needed to forward $R_s$.
- Relay nodes selfish and strategic.

# Selfish and Strategic Nodes

- Relays need incentive to forward traffic.
- Pricing: relays charge for their service.
- Competition: peer relays compete for traffic.
- Multipath routing: source allocates traffic to minimize payment.
- Wholesale market; long-term prices and traffic rates.

# Overview

- (Nonlinear) pricing game with atomic source.
- Nash equilibria of pricing game.
- Efficiency and inefficiency of equilibria (price of anarchy).
- Inefficiencies result from monopoly on traffic flow.
- Congestion control at source alleviates inefficiency.

# Pricing in Communication Networks

- Shadow prices: optimal dual variables for global optimization (Kelly, Maulloo and Tan 98).

- Selfish routing in general causes arbitrarily large efficiency loss (Roughgarden and Tardos 02, Roughgarden 05).

- Marginal cost pricing induces social optimum under selfish routing (Cole et al. 03).

- Pricing network services (Shu and Varaiya 03, He and Walrand 05, Basar and Srikant 05, Shakkottai and Srikant 05, Acemoglu and Ozdaglar 06, 07).

- Payment as incentive for cooperation in relay networks (Buttyan and Hubbaux 00, Crowcroft et al. 04, Ileri et al. 05, Blanc et al. 05, Marbach and Qiu 05, Zhong et al. 07)

# Price Competition

- Price competition in parallel-serial networks with non-atomic users (infinitesimal traffic) and constant unit price can cause arbitrarily large efficiency loss (Acemoglu and Ozdaglar 06, 07).

- Prices set by links independently.

- Exchange of prices potentially non-local.

- Non-atomic users: cannot split traffic; use one path.

- For parallel relay network, social surplus (utility - cost) at NE/optimal social surplus $\geq 5/6$.

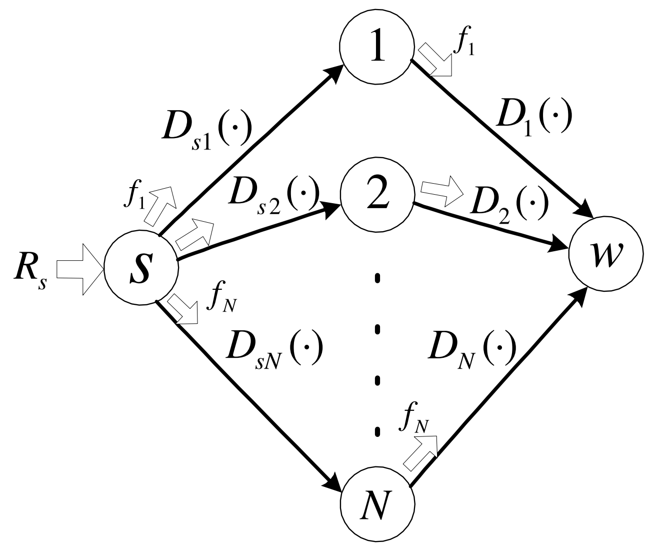
# Pricing Game in Parallel Relay Networks

- Relays compete for traffic from source.
- Source allocates traffic to multiple relays.
- Atomic source with non-negligible and infinitely divisible traffic.
- Node-based pricing.
- Relays bid to source:

  1) (possibly nonlinear) charging function,

  2) proposed share of traffic.

# Network Model




- Network topology: $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
- $\mathcal{N} = \{s, 1, \ldots, N, w\}$.
- $\mathcal{E} = \{(s,i), (i,w) | i = 1, \ldots, N\}$.

# Link Cost Functions

- Each $(i, w)$ has cost function $D_i(f_i)$.
- $D_i(f_i)$ strictly increasing and convex, $D_i(0) = 0$.
- $D_i(f_i)$ represents queuing delay or transmission power on $(i, w)$.

$$D_i(f_i) = \frac{f_i}{c_{iw} - f_i} \quad \text{or} \quad D_i(f_i) = 2^{f_i/W} - 1.$$

- $D_i(f_i)$ cost to $i \Rightarrow i$ needs incentive to forward traffic.
- Assume $d_i(f_i) = D_i'(f_i)$ positive and strictly increasing.
- Same for links $(s, i)$ and cost functions $D_{si}(f_i)$.
- Network cost $\sum_i D_{si}(f_i) + D_i(f_i)$.

# Socially Optimal Routing

- Routing $\boldsymbol{f}^* = (f_i^*)_{i=1,\ldots,N}$ socially optimal if $\boldsymbol{f}^*$ is solution to

$$\begin{aligned} \text{minimize} \quad & \sum_i D_{si}(f_i) + D_i(f_i) \\ \text{subject to} \quad & f_i \geq 0 \ \forall i, \sum_i f_i = R_s. \end{aligned}$$

- Cooperation needed to reach socially optimal routing.
- $\boldsymbol{f}^*$ and $\sum_i D_{si}(f_i^*) + D_i(f_i^*)$ are benchmarks.

# Bidding

- Relay $i$ announces bid $(P_i(\cdot), \gamma_i)$ to source $s$:
  - $\gamma_i \in [0, 1]$: proposed traffic share (in case $s$ is indifferent).
  - $P_i(f_i) = i$'s charge to $s$ for forwarding $f_i$ to $i$, $P_i(0) = 0$, $P_i(\cdot)$ can be nonlinear.
  - $s$'s total cost for forwarding to $i$: $B_i(f_i) = D_{si}(f_{si}) + P_i(f_i)$.
  - Equivalent view: $s$ pays $B_i(f_i)$ and $i$ pays $D_{si}(f_i)$.
  - Assume $B_i(\cdot)$ continuously differentiable: $\beta_i^h(f) = \frac{\mathrm{d}}{\mathrm{d}f} B_i(f)$ – pricing function.
- $w$ does not bid.

# Selfish Routing

- Incoming traffic $R_s$.
- Routing variables: $\phi_i = \dfrac{f_i}{R_s}$.
- $s$ adopts minimum-cost routing:

$$(\phi_i^*)_{i=1,\ldots,N} \in \underset{\phi_i \geq 0\ \forall i, \sum_i \phi_i = 1}{\arg\min} \sum_i B_i(R_s \phi_i).$$

# Tie-Breaking Rule

- $\mathcal{A} \triangleq \arg\min_{(\phi_i)} \sum_i B_i(R_s\phi_i)$ may contain multiple elements.
- $s$ applies tie-breaking rule (Maskin 86) when indifferent:
  - $\bar{\boldsymbol{\gamma}} =$ normalized version of $\boldsymbol{\gamma} = (\gamma_i)_{i=1,\ldots,N}$.
  - $\bar{\gamma}_i = 1/N$ if $\boldsymbol{\gamma} = \mathbf{0}$.
  - Choose optimal $\boldsymbol{\phi}^* = (\phi_i^*)$ closest to $\bar{\boldsymbol{\gamma}}$:

$$\boldsymbol{\phi}^* \in \mathcal{A}^* \triangleq \arg\min_{\boldsymbol{\phi}\in\mathcal{A}} \|\boldsymbol{\phi} - \bar{\boldsymbol{\gamma}}\|.$$

  - In case of tie, choose lexicographically largest $\boldsymbol{\phi}^* = (\phi_i^*)$.
- $s$'s traffic allocation $\boldsymbol{\phi}^*$ a function of $R_s$ and relay bids:

$$\boldsymbol{\phi}^* = \Phi(R_s, (B_i(\cdot), \gamma_i)_{i=1,\ldots,N}).$$

- Traffic from $s$ to $i$: $f_i = R_s\phi_i^*$.

# Payoff Function and Pricing Game

- $\boldsymbol{Q}_i \triangleq (B_i(\cdot), \gamma_i)$, $\boldsymbol{Q}_{-i} =$ bids by other relays.

- Relay $i$'s payoff $=$ $i$'s profit

$$\Gamma_i(\boldsymbol{Q}_i, \boldsymbol{Q}_{-i}) \triangleq B_i(R_s\phi_i^*) - D_{si}(R_s\phi_i^*) - D_i\left(R_s\phi_i^*\right).$$

- Formal definition of (static) pricing game:
  - Set of players: relays $\{1, \ldots, N\}$.
  - Strategy of $i$: continuously differentiable $B_i(\cdot)$ and $\gamma_i \in [0, 1]$.
  - Payoff to $i$: $\Gamma_i(\boldsymbol{Q}_i, \boldsymbol{Q}_{-i})$.

- Static game: one-shot competitions occur simultaneously.

- Consider pure strategies only.

# Best Response

- Given $\boldsymbol{Q}_{-i}$, $i$ can compute best response set

$$\mathcal{B}_i(\boldsymbol{Q}_{-i}) = \arg\max_{\boldsymbol{Q}_i \in \mathcal{B}\times[0,1]} \Gamma_i(\boldsymbol{Q}_i, \boldsymbol{Q}_{-i}).$$

- Consider charging function of virtual competitor $\hat{i}$:

$$B_{\hat{i}}(r) \triangleq \min_{\sum_{j\neq i} f_j = r} \sum_{j\neq i} B_j(f_j)$$

  - $B_{\hat{i}}(\cdot)$ is best bid that $i$'s competitors jointly offer.
  - Equivalent view: $i$ competes only with $\hat{i}$ for $s$'s traffic.

- $s$ pays no more than $B_{\hat{i}}(R_s)$.
- Relay $i$'s best charging function makes $s$ pay same amount $B_{\hat{i}}(R_s)$.

# Best Response

**Lemma 1** $\boldsymbol{Q}_i \in \mathcal{B}_i(\boldsymbol{Q}_{-i})$ if and only if

(i) $B_i(t) \geq B_{\hat{i}}(R_s) - B_{\hat{i}}(R_s - t)$ for all $t \in [0, R_s]$;

(ii) $B_i(\tilde{f}_i) = B_{\hat{i}}(R_s) - B_{\hat{i}}(R_s - \tilde{f}_i)$, where $\tilde{f}_i$ maximizes

$$\bar{\Gamma}_i(f_i; \boldsymbol{Q}_{-i}) \triangleq B_{\hat{i}}(R_s) - B_{\hat{i}}(R_s - f_i) - D_{si}(f_i) - D_i\,(f_i)\,;$$

(iii) $(\gamma_i, (\gamma_j)_{j \neq i})$ induces $\phi_i^* = \tilde{f}_i / R_s$.

- $\bar{\Gamma}_i(f_i; \boldsymbol{Q}_{-i})$ = maximum profit of $i$ if it manages to win $f_i$ in competition with $\hat{i}$.
- $\tilde{f}_i = \arg\max \bar{\Gamma}_i(f_i; \boldsymbol{Q}_{-i})$ = most profitable "market shares" for $i$.
- $B_i(\tilde{f}_i) = B_{\hat{i}}(R_s) - B_{\hat{i}}(R_s - \tilde{f}_i)$: $s$ indifferent to not using $i$ and giving $\tilde{f}_i$ to $i$.
- $B_i(t) \geq B_{\hat{i}}(R_s) - B_{\hat{i}}(R_s - t)$: gives $s$ no better alternatives.
- $\gamma_i$ induces $s$ to allocate $\tilde{f}_i$.

# Nash Equilibrium

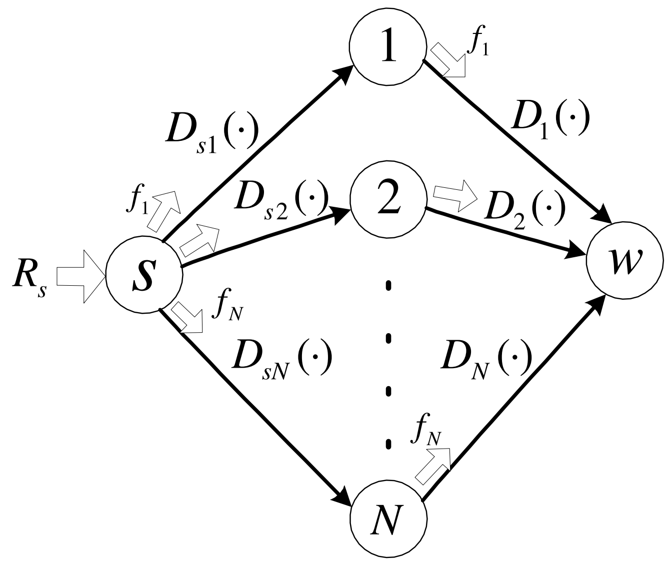
- Nash Equilibrium (NE) is fixed point of best response mapping.

**Definition 1** $\boldsymbol{Q}_i, i \in \mathcal{I}$ constitute an equilibrium if $\boldsymbol{Q}_i \in \mathcal{B}_i(\boldsymbol{Q}_{-i})$, $\forall i$.

- Efficient equilibrium:

**Definition 2** Equilibrium $(\boldsymbol{Q}_i)_{i \in \mathcal{I}}$ is *efficient* if it induces socially optimal routing.

# Oligopoly




- Relay $i$'s marginal cost function: $\lambda_i(t) \triangleq D'_{si}(t) + D'_i(t) = d_{si}(t) + d_i(t)$ strictly increasing.
- Relay $i$'s bid: $\gamma_i \in [0, 1]$ and $\beta_i(t) \triangleq B'_i(t) = d_{si}(t) + p_i(t)$.

# Best Response in Oligopoly

- Pricing function of $i$'s virtual competitor $\beta_{\hat{i}}(t) = B'_{\hat{i}}(t)$.
- Relay $i$ finds $f_i^*$ that maximizes

$$\int_0^{f_i} \beta_{\hat{i}}(R_s - r) - \lambda_i(r)\, dr.$$

- Designs $\beta_i(\cdot)$ s.t.

$$\begin{aligned} \int_0^{t} \beta_i(r)\, dr &\geq \int_0^{t} \beta_{\hat{i}}(R_s - r)\, dr, \quad 0 \leq t \leq R_s \\ \int_0^{f_i^*} \beta_i(r)\, dr &= \int_0^{f_i^*} \beta_{\hat{i}}(R_s - r)\, dr. \end{aligned}$$

- Sets $\gamma_i$ to induce $f_i^*$.

# Pricing Game in Oligopoly

- Socially optimal routing $(r_i^*)_{i=1}^N$ uniquely characterized by

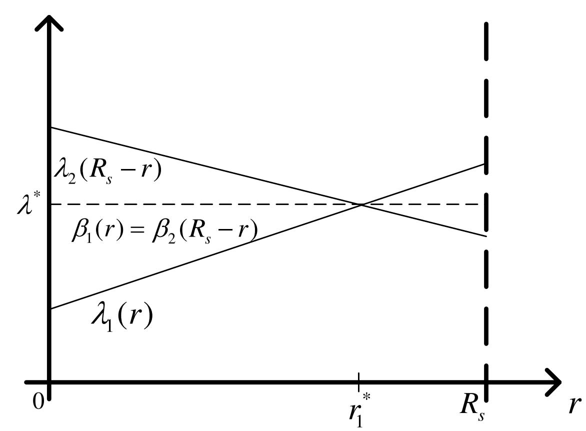
$$\lambda_i(r_i^*) = \min_{j=1,\cdots,N} \lambda_j(r_j^*), \quad \text{if } r_i^* > 0.$$

**Theorem 1** Socially optimal routing of oligopoly can always be induced by an equilibrium, i.e., (efficient) equilibrium always exists.

- Price of Stability $= (\min \text{Cost at an NE})/(\min \text{Cost}) = 1$.

# Efficient Equilibrium in Oligopoly

- Denote $\lambda^* \triangleq \min_{j=1,\cdots,N} \lambda_j(r_j^*)$.
- Let $\gamma_i = r_i^*/R_s$, $\beta_i(r) \equiv \lambda^*$ for all $i$ (linear $B_i(t)$ but not linear $P_i(t)$).
- Thus, $\beta_{\hat{i}}(t) \equiv \lambda^*$.
- $r_i^*$ maximizes $\int_0^{r_i} \beta_{\hat{i}}(R_s - r) - \lambda_i(r)\, dr$.
- $\int_0^t \beta_i(r)\, dr \geq \int_0^t \beta_{\hat{i}}(R_s - r)\, dr$, $0 \leq t \leq R_s$,
  $\int_0^{r_i^*} \beta_i(r)\, dr = \int_0^{r_i^*} \beta_{\hat{i}}(R_s - r)\, dr$.

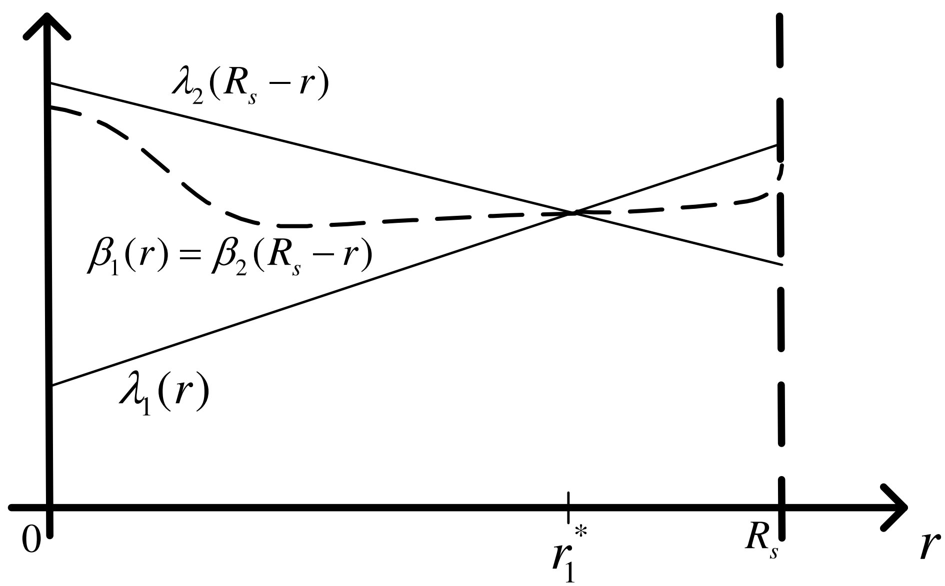



- $s$ induced to allocate $r_i^*$ to $i$.

# General Equilibrium in Oligopoly

- Efficient equilibrium can be induced by nonlinear $B_i(t)$.

- Example: duopoly pricing game with $\beta_{\hat{1}}(t) = \beta_2(t)$, $\beta_{\hat{2}}(t) = \beta_1(t)$, $\gamma_1 = r_1^*/R_s$, $\gamma_2 = (R_s - r_1^*)/R_s$.



# Competitive Equilibrium in Oligopoly

- Routing $(f_i)_{i=1}^N$ *monopolistic* if $f_m = R_s$ for some $m$ and $f_j = 0$ for all $j \neq m$, $m$: *dominant* relay.

- Equilibrium is *monopolistic* if it induces monopolistic routing.

- Routing $(f_i)_{i=1}^N$ is *competitive* if at least two relays $i, j$ have $f_i, f_j > 0$.

- Equilibrium is *competitive* if it induces competitive routing.

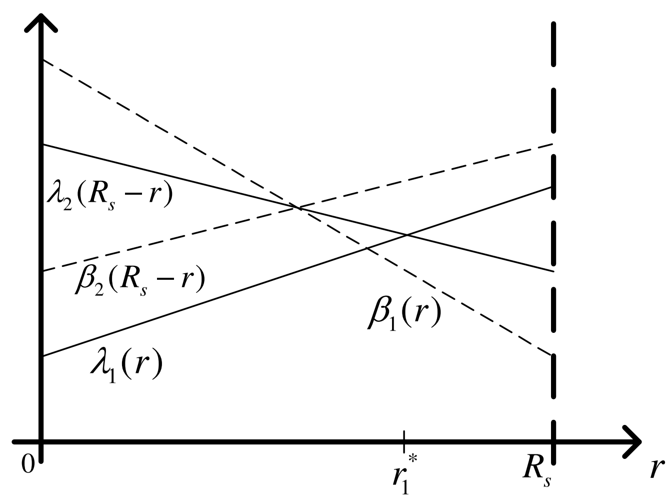
**Theorem 2** Any competitive equilibrium in oligopoly is efficient.

# Inefficient Equilibrium in Oligopoly

- Monopolistic equilibrium may be inefficient.
- Example: duopoly with competitive socially optimal routing $(r_1^*, R_s - r_1^*)$.
  - $\beta_2(R_s - r) > \lambda_1(r)$, $\forall r \in [0, R_s]$.
  - Relay 1 wants all the flow because

$$R_s = \arg\max_{0 \le r_1 \le R_s} \int_0^{r_1} \beta_2(R_s - t) - \lambda_1(t)\ dr.$$

  - Relay 1 uses $\gamma_1 = 1$ and $\beta_1(\cdot)$ s.t. $\int_0^t \beta_1(R_s - r)\ dr < \int_0^t \lambda_2(r)\ dr$ and $\int_0^t \beta_1(r)\ dr > \int_0^t \beta_2(R_s - r)\ dr$, $\forall t \in (0, R_s)$ and $\int_0^{R_s} \beta_1(R_s - r)\ dr = \int_0^{R_s} \beta_2(r)\ dr$.
  - $\beta_1(\cdot)$ leaves relay 2 no incentive to acquire any traffic, hence $\gamma_2 = 0$.



# Monopolistic Equilibrium in Oligopoly

**Theorem 3** If $m$ is dominant relay at monopolistic equilibrium, then

$$\int_0^{R_s} \lambda_m(r)\, dr \leq \int_0^{R_s} \lambda_j(r)\, dr$$

for any other relay $j$.

**Corollary 1** If socially optimal routing is monopolistic, then every equilibrium is monopolistic and efficient.

# Monopolistic Equilibrium in Oligopoly

**Corollary 2** If socially optimal routing is competitive, then there exists an inefficient (monopolistic) equilibrium.

Proof:

- Let all $\beta_j(t) = \beta(t)$ be strictly decreasing.
- Design $\beta(\cdot)$ such that
  - $\int_0^t \beta(R_s - r)dr \leq \int_0^t \lambda_j(r)dr$ for all $j$ and $t \in [0, R_s)$,
  - $\int_0^{R_s} \beta(R_s - r)dr = \int_0^{R_s} \lambda_m(r)dr$, $m \in \arg\min_j \int_0^{R_s} \lambda_j(r)dr$.
- $\gamma_m = 1$, $\gamma_j = 0$ for all $j \neq m$.

# Price of Anarchy

**Definition 3** Price of anarchy

$$\rho(\mathcal{G}, (\lambda_i(\cdot)), R_s) \triangleq \frac{\max_{(f_i)\in\mathcal{F}^E} \sum_i D_{si}(f_i) + D_i(f_i)}{\sum_i D_{si}(f_i^*) + D_i(f_i^*)},$$

where $\mathcal{F}^E$ = all possible equilibrium routings and $(f_i^*)$ = socially optimal routing.

**Theorem 4** For oligopoly $(N, (\lambda_i(\cdot)), R_s)$, if $(\lambda_i(\cdot))$ are non-negative, strictly increasing, and concave, $\rho(N, (\lambda_i(\cdot)), R_s) \leq N$. The upper bound is achieved when $(\lambda_i(\cdot))$ are identical and linear.

- More intense (larger $N$) competition implies more inefficiency when traffic is monopolized.

# Price of Anarchy (Concave Marginal Costs)

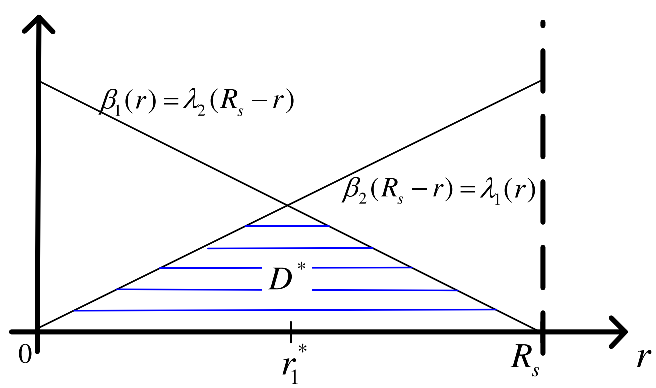
- Assume socially optimal routing is $(r_i^*)_{i=1}^N = (\alpha_i R_s)_{i=1}^N$, where $\sum_i \alpha_i = 1$.
- Optimal cost $D^* = \displaystyle\sum_{i=1}^N \int_0^{\alpha_i R_s} \lambda_i(r)\ dr$.
- By concavity of $\lambda_i(r)$, $\displaystyle\int_0^{\alpha_i R_s} \lambda_i(r) dr \geq \alpha_i^2 \int_0^{R_s} \lambda_i(r) dr$.
  - Equality holds when $\lambda_i(r)$ linear.
- $D^* \geq \displaystyle\sum_{i=1}^N \alpha_i^2 \int_0^{R_s} \lambda_i(r)\ dr$.
- $D^{ME}$: cost at inefficient (monopolistic) equilibrium (dominant relay $m$).

$$\frac{D^{ME}}{D^*} \leq \frac{\int_0^{R_s} \lambda_m(r)\ dr}{\sum_{i=1}^N \alpha_i^2 \int_0^{R_s} \lambda_i(r)\ dr} \leq \frac{\int_0^{R_s} \lambda_m(r)\ dr}{\sum_{i=1}^N \alpha_i^2 \int_0^{R_s} \lambda_m(r)\ dr} = \frac{1}{\sum_{i=1}^N \alpha_i^2} \leq N.$$

- Upper bound attained when all $\lambda_i(r)$ same and linear.

# Price of Anarchy (Concave Marginal Costs)

- Assume $\lambda_i(\cdot) = \lambda(\cdot)$ and linear for all $i$.
- Let $\beta_i(r) = \lambda(R_s - r)$ for every $i$.
- Thus, $\beta_{\hat{i}}(r) = \lambda(R_s - r)$ for every $i$.
- Since $\beta_{\hat{i}}(R_s - r) = \lambda(r)$, every relay is indifferent to having any amount of flow.
- Let $\gamma_1 = 1$ and $\gamma_i = 0$ for all $i \neq 1$.
- Monopolistic equilibrium is established.

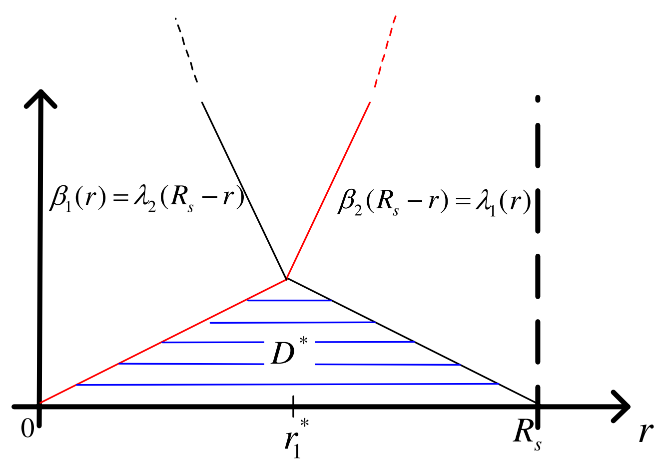


# Price of Anarchy (Convex Marginal Costs)

**Theorem 5** For $N$ relays and any $M > 0$, there exists oligopoly $(N, (\lambda_i(\cdot))_{i=1}^N, R_s)$ with convex $(\lambda_i(\cdot))$ s.t. $\rho(N, (\lambda_i(\cdot)), R_s) \geq M$.

Proof (sketch):

- Construct $N$-relay oligopoly with competitive socially optimal routing.
- Inefficient (monopolistic) equilibrium exists (Corollary 2).
- Dominant relay $m$ gets all the traffic at equilibrium.
- One can choose convex $\lambda_m(r)$ to make $\int_0^{R_s} \lambda_m(r)\, dr \geq M D^*$.

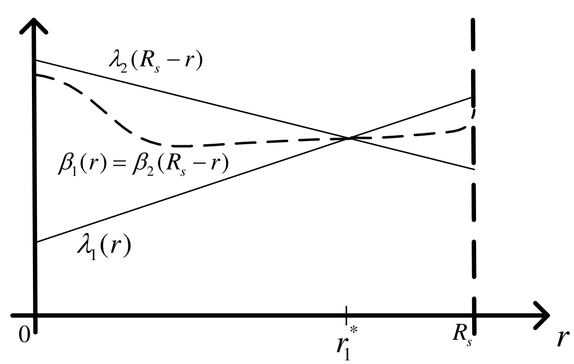


# Focal Equilibria

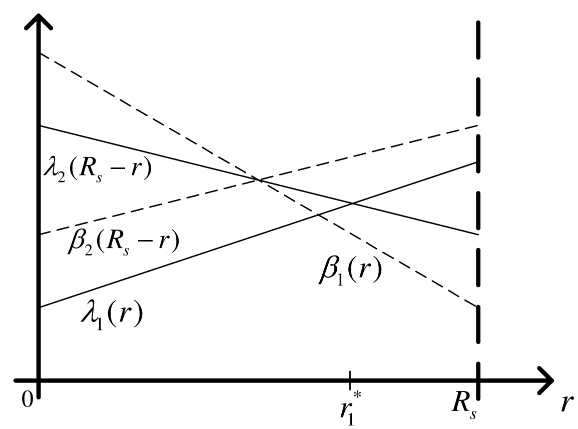
- Inefficient equilibria pathological.
- Consider following refinement of equilibria:

**Definition 4** An equilibrium is *focal* if every relay $i$ adopts replicating response, i.e.,

$$\beta_i(t) = \beta_{\hat{i}}(R_s - t), \quad t \in [0, f_i^*]$$
$$\int_{f_i^*}^{t} \beta_i(r)\, dr \geq \int_{f_i^*}^{t} \beta_{\hat{i}}(R_s - r)\, dr, \quad t \in (f_i^*, R_s].$$




Replicating Response

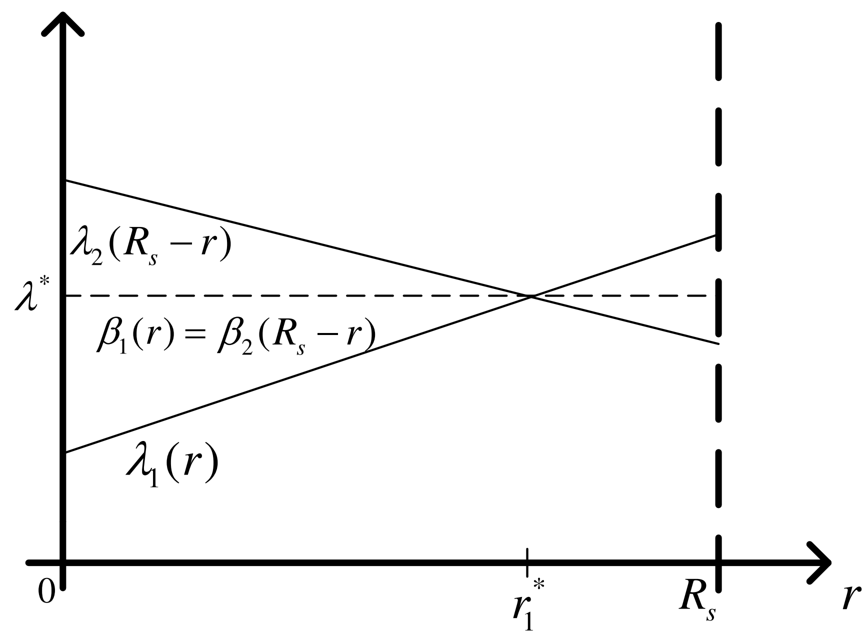



Non-replicating Response

# Focal Equilibria in Oligopoly

**Theorem 6** Socially optimal routing of oligopoly can always be induced by focal equilibrium.

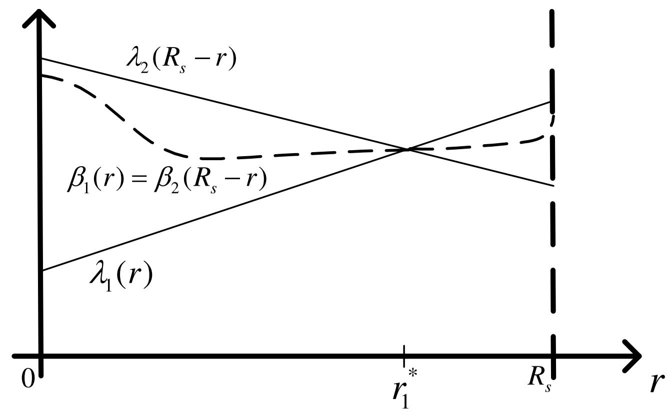
Proof: every $\beta_i(t) \equiv \lambda^*$, $\gamma_i = r_i^*/R_s$ constitutes focal equilibrium.



# Focal Equilibria in Oligopoly

**Theorem 7** Every focal equilibrium of oligopoly is efficient.

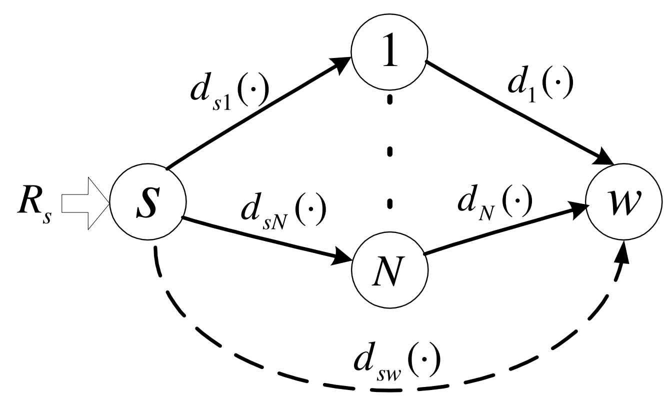
Proof: example of duopoly




- $r_1^*$ ideal to $1 \Rightarrow \beta_2(R_s - r_1^*) = \lambda_1(r_1^*)$.
- $R_s - r_1^*$ ideal to $2 \Rightarrow \beta_1(r_1^*) = \lambda_2(R_s - r_1^*)$.
- Replicating response $\Rightarrow \beta_1(r_1^*) = \beta_2(R_s - r_1^*)$.
- $\lambda_1(r_1^*) = \lambda_2(R_s - r_1^*) \Rightarrow (r_1^*, R_s - r_1^*)$ is socially optimal routing.
- In general, marginal cost $=$ price at focal equilibrium.

# Elastic Source

- Elastic source with utility function $U_s(r_s)$.
- $U_s(r_s) = U_s(R_s)$, $\forall r_s \geq R_s$: $R_s$ = maximum desired service rate.
- $U_s(\cdot)$ strictly increasing, concave on $[0, R_s]$ with derivative $u_s(\cdot)$.
- Define $f_{sw} \triangleq R_s - r_s$ = overflow rate (Bertsekas and Gallager 92).
- Overflow cost $= D_{sw}(f_{sw}) \triangleq U_s(R_s) - U_s(r_s)$ strictly increasing, convex with derivative $d_{sw}(\cdot)$.
- Additional competitor $w$ with *uniformly zero charging function* $P_w(f_{sw}) = 0$ or $B_w(f_{sw}) = D_{sw}(f_{sw})$.



# Elastic Source

**Theorem 8** If overflow link has $f^*_{sw} > 0$ at an equilibrium of oligopoly with elastic source, the equilibrium must be efficient.

- Whenever congestion control is exercised at equilibrium, equilibrium is efficient.
- Monopolistic equilibria can exist iff a relay has "vastly superior" cost function.
- In most cases, presence of overflow link removes inefficient equilibria.

# Summary of Contributions

- Game theoretic study of pricing, competition, and routing in parallel relay networks.
- Atomic users.
- Node-based pricing.
- Competition by using nonlinear charging functions.
- Efficient equilibrium always exists.
- Competitive equilibria are efficient.
- Focal equilibria are efficient.

# Summary of Contributions

- Inefficient equilibria are monopolistic.
- Price of anarchy $\leq N$ for concave marginal costs.
- Price of anarchy arbitrarily large for convex marginal costs.
- Elastic source: congestion control exercised at equilibrium $\Rightarrow$ efficiency.

# Multi-hop Relay Networks

- Single-source, single-destination, multi-hop layered relay network.
- Relay nodes are service providers to upstream nodes and customers of downstream nodes.
- Socially optimal routing can be induced by an equilibrium.
- Inefficiency due to multi-hop structure can be arbitrarily high.
- Everywhere competitive equilibrium is efficient.
- To appear at Infocom 2008.