Optimal Resource Allocation for OFDM Uplink Communication: A Primal-Dual Approach

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OFDM Systems

- Frequency band divided into several parallel orthogonal carriers/tones.
- High spectrum efficiency.
- Eliminate inter-symbol-interference (ICI) due to multi-path fading.
- Applications: WiMAX (802.16), Wi-Fi (802.11a/g), DSL, etc.
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- Applications: WiMAX (802.16), Wi-Fi (802.11a/g), DSL, etc.
- How to perform distributed and efficient resource allocation?
Resource Allocation

- Assign each user a utility, $U_i(\cdot)$, depending on delay, throughput, etc.
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- Scheduler maximizes first order change in total utility.

$$\max_{r \in \mathcal{R}(e)} \nabla U(X(t)) \cdot r = \max_{r \in \mathcal{R}(e)} \sum_i \dot{U}_i(X_i(t)) r_i,$$

- Motivated by channel-aware and gradient-based scheduling
- Myopic policy, requires no knowledge of channel or arrival statistics.
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- Resource allocation = **weighted rate maximization**
  - “Network”: assigning carriers to users
  - “User”: assign power over carriers
Achievable Rate Per Carrier

If a carrier $j$ is allocated to a single user $i$:

- $r_{ij} = \log(1 + \text{SNR}) = \log(1 + p_{ij}e_{ij})$
- $p_{ij} = \text{power user } i \text{ allocates to carrier } j.$
- $e_{ij} = \text{received SNR/unit transmit power.}$
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- By allowing time-sharing:
  - $r_{ij} = x_{ij} \log \left( 1 + \frac{p_{ij}e_{ij}}{x_{ij}} \right)$
  - $x_{ij} \in [0, 1] = \text{fraction of carrier } j \text{ allocated to user } i$. 

Why “time-sharing”?
- Convexify the problem.
- Can be achieved in practice.
- Not affecting the optimal objective value with large number of carriers.
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OFDMA rate region

- Rate region:

\[
\mathcal{R}(\mathbf{e}) = \left\{ \mathbf{r} : r_i = \sum_j x_{ij} \log \left( 1 + \frac{p_{ij} e_{ij}}{x_{ij}} \right), \forall i \right\}
\]

where \((\mathbf{x}, \mathbf{p}) \in \mathcal{X}\) such that

- \((\mathbf{x}, \mathbf{p}) \geq 0\)
- Each user \(i\): \(\sum_j p_{ij} \leq P_i\) (uplink transmission power constraint)
- Each carrier \(j\): \(\sum_i x_{ij} \leq 1\) (channel allocation constraint)
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Variation: sub-channelization (bundle carriers to reduce overhead)

- Interleaved (802.16 standard mode)
- Adjacent (Band AMC mode)
- Random (e.g. frequency hopped)
Weighted Rate Maximization Problem

\[
\max_{r \in \mathcal{R}(e)} \mathbf{w} \cdot \mathbf{r} = \max_{(x, p) \in \mathcal{X}} \sum_i w_i \sum_j x_{ij} \log \left( 1 + \frac{p_{ij} e_{ij}}{x_{ij}} \right) \quad \text{(OPT)}
\]
Weighted Rate Maximization Problem

\[
\begin{align*}
\max_{r \in \mathcal{R}(e)} \mathbf{w} \cdot \mathbf{r} &= \max_{(x,p) \in \mathcal{X}} \sum_i w_i \sum_j x_{ij} \log \left(1 + \frac{p_{ij}e_{ij}}{x_{ij}}\right) \\
&= \text{(OPT)}
\end{align*}
\]

- Concave maximization problem
- Non-strictly concave: typically many local/global optimal allocations
Weighted Rate Maximization Problem

\[
\max_{r \in R(e)} w \cdot r = \max_{(x,p) \in X} \sum_i w_i \sum_j x_{ij} \log \left( 1 + \frac{p_{ij} e_{ij}}{x_{ij}} \right)
\]  

(OPT)

- Concave maximization problem
- Non-strictly concave: typically many local/global optimal allocations
- What is known: centralized optimal and (very good) heuristic algorithms [HSBA’07]
- What we will show: distributed optimal algorithm [This talk]
Centralized Optimal Algorithm [HSBA’07]

- Problem (OPT) is convex and satisfies Slater’s condition
  \[ \Rightarrow \text{No duality gap.} \]
- Consider Lagrangian:

\[
L(x, p, \lambda, \mu) := \sum_i w_i \sum_j x_{ij} \log \left( 1 + \frac{p_{ij}e_{ij}}{x_{ij}} \right) \\
+ \sum_i \lambda_i \left( P_i - \sum_j p_{ij} \right) + \sum_j \mu_j \left( 1 - \sum_i x_{ij} \right).
\]

- Dual function:

\[
L(\lambda, \mu) = \max_{(x, p) \in \mathcal{X}} L(x, p, \lambda, \mu)
\]

- Optimal objective value to Problem (OPT):

\[
V^* = \min_{(\lambda, \mu) \geq 0} L(\lambda, \mu)
\]
A Dual-Based Centralized Algorithm [HSBA’07]

1. For fixed $\lambda$, analytically solve for $\mu(\lambda)$, $x(\lambda)$ and $p(\lambda)$.
2. Multi-dimensional subgradient search of optimal $\lambda^*$.
3. Find optimal $x^*$ and $p^*$ by solving a system of linear equations (multiple solutions).
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Properties:
- **Centralized** computation (step 3)
- **Slow** convergence (step 2)
- **High** signaling overhead (steps 1, 2 and 3): all channel information and power constraints
Distributed Primal-Dual Algorithm

- Try to reach a saddle point of Lagrangian $L(x, p, \lambda, \mu)$:

$$
\dot{x}_{ij} = k_{ij}^{x} \cdot \partial L / \partial x_{ij} = k_{ij}^{x} \left( f_{ij}(x_{ij}, p_{ij}) - \mu_{j} \right)_{x_{ij}}^{+}, \text{(mobile)}
$$

$$
\dot{p}_{ij} = k_{ij}^{p} \cdot \partial L / \partial p_{ij} = k_{ij}^{p} \left( g_{ij}(x_{ij}, p_{ij}) - \lambda_{i} \right)_{p_{ij}}^{+}, \text{(mobile)}
$$

$$
\dot{\lambda}_{i} = k_{i}^{\lambda} \cdot \partial L / \partial \lambda_{i} = k_{i}^{\lambda} \left( \sum_{j} p_{ij} - P_{i} \right)_{\lambda_{i}}^{+}, \text{(mobile)}
$$

$$
\dot{\mu}_{j} = k_{j}^{\mu} \cdot \partial L / \partial \mu_{j} = k_{j}^{\mu} \left( \sum_{i} x_{ij} - 1 \right)_{\mu_{j}}^{+}, \text{(base station)}
$$
Distributed Primal-Dual Algorithm

**Advantages:**

- **Distributed** and simple updates by mobiles and base station
- Low communication overhead: user feedback $x_{ij}$’s and base station announces $\mu_j$
Distributed Primal-Dual Algorithm

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Challenge:
- How to achieve global convergence with non-strictly concave objective functions
  - Difficult in general
  - Example: multi-path routing [Voice’06]
Convergence Result: Part I

**Theorem (Convergence to an Invariant Set)**

- All trajectories of the primal-dual system converge to an invariant set $V_0$ **globally and asymptotically**.
- All **optimal** solutions of Problem (OPT) are contained in set $V_0$. 
Convergence Result: Part I

Theorem (Convergence to an Invariant Set)

- All trajectories of the primal-dual system converge to an invariant set $V_0$ globally and asymptotically.
- All optimal solutions of Problem (OPT) are contained in set $V_0$.

**Proof**: constructing a proper Lyapunov function.

**Question**: Will set $V_0$ contain non-optimal solutions?
Convergence Result: Part II

Theorem (Convergence to a Global Optimal Solution)

All trajectories of the primal-dual system globally converge to optimal solutions of Problem (OPT) under properly chosen stepsizes.
Convergence Result: Part II

**Theorem (Convergence to a Global Optimal Solution)**

All trajectories of the primal-dual system globally converge to optimal solutions of Problem (OPT) under properly chosen stepsizes.

- Set $V_0$ only contains optimal solutions.
- Stepsize choice is easy: e.g., not all $k_j^{\mu}$’s are the same.
- **Proof:**
  - Over set $V_0$, the nonlinear system reduces to a linear one.
  - The linear system in marginal stable.
  - Set $V_0$ contains only the optimal solution if $(\lambda, \mu)$ is completely observable from $B[x^T, p^T]^T$. 
Relationship

Invariant set  Optimal solutions

Complete Observability

Invariant set  Optimal solutions
Simulation Set-up

- Single cell, \( M = 4 \) users.
- \( e_{ij} = (\text{fixed location-based term}) \times \text{(frequency selective fast fading)} \)
  - Fixed term = empirical distribution.
  - frequency selective term = block fading in time (\( 2\text{msec} \) coh. time \( \approx 250\text{Hz} \) Doppler); standard ref. mobile delay spread (6 taps, 1 \( \mu\text{sec} \)).
- 5 MHz BW, 512 carriers.
- Adjacent channelization, 8 carriers/subchannel.
- Resource allocation over 20 OFDM symbols.
- Randomly generated weights in \([0, 1]\).
Convergence of Primal and Dual Variables

Dual variables $\lambda$

Dual variables $\mu$

Total power allocation of all users

Total channel allocation of all channels

J. Huang (CUHK)
Convergence of Total Weighted Rate

Achieve 90% within 500 iterations
Conclusions

- **Topic**: optimal resource allocation for uplink OFDM systems

- **Algorithm**: Primal-Dual algorithm

- **Properties**
  - Distributed and simple updates
  - Low communication overhead per iteration

- **Future Work**:
  - Convergence with delay and asynchronous updates
  - Faster convergence
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