Bandwidth Allocation Games under Budget and Access Constraints

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***Abstract*—We study bandwidth allocation at base stations of a wireless network as a non-cooperative game between different wireless users. A user demands bandwidth from the base stations it has access to by submitting bids. Once the bidding process is complete, a base station distributes its bandwidth to the users in proportion to their bids. Users are assumed to be budget constrained and price anticipating, and split their wealth across base stations so as to maximize their individual bandwidths. In this paper, we study the properties of Nash Equilibrium (NE) for this bandwidth allocation game. For the special case where each user can access all base stations, we argue that there exists a unique NE, at which the bandwidth obtained by any user is proportional to its wealth. For a more general scenario where a user may be able to access only a subset of all base stations, we compare the NE of the game to the max-min fair bandwidth allocation. We show that although the NE may not be max-min fair in general, the bandwidth allocation at NE becomes arbitrarily close to max-min fair allocation as the number of users is increased, while keeping the number of base stations fixed.**

I. Introduction

We consider the bandwidth allocation problem for a multi-consumer multi-provider system, where a consumer may have access to only a subset of all providers. The scenario we consider is particularly relevant to wireless cellular or access point networks, where wireless users are interested in obtaining bandwidth from a set of base stations, but an individual wireless user may not, in general, have access to all base stations. We envision the problem of bandwidth allocation as a competition between different wireless users. Each user with a fixed amount of wealth competes for bandwidth on the base stations to satisfy its own requirements and is willing to pay a certain amount for these services. In order to 'buy' bandwidth, each user submits bids to the base stations. The base stations then distribute bandwidth to different users in proportion to the bids. The aim of a user is to bid in such a way that its own bandwidth its maximized. In this paper, we study the properties (existence, uniqueness and fairness) of the Nash Equilibrium of the general as well as one special case of this game.

The paper is organized as follows. We review related literature on bandwidth and resource allocation in section II. In section III we formulate our problem as a game between budget constrained users competing for bandwidths at different base stations. We prove the uniqueness of NE for the special case where every user can access every base station. We also find a closed form expression for the NE in this case and argue that the NE results in fair bandwidth allocation among users. Section IV addresses the more general case where a user can access only a subset of all base stations. We provide an example to show that NE may not result in max-min fair bandwidth allocation in this general scenario. We however prove that the NE results in a bandwidth allocation that becomes arbitrarily close for max-min fair allocation as the number of users in the system is increased. We conclude in section V by discussing some related open questions.

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II. Background

The problem we consider falls under the general class of resource allocation problems. Auctioning is a common way of allocating resources to users using a central agent. In an auction, each player *bids* a certain amount to *buy* the given resource(s). The role of the agent is to allocate the resources based on the bids obtained. In the most common type of auctions, the highest bidder wins the resource and pays as much as the bid. From the perspective of the player, it may not be optimal to bid as much as the resource is worth for this type of auction. That is the players may have an incentive to *lie* about their true value of the resource. In another auctioning scheme, termed the VCG auction, the highest bidder gets the (indivisible) resource but the amount that the winner pays is only as much as the second highest bidder. The VCG scheme is popular because for all players, the dominant strategy is to play truthfully. VCG scheme has been adapted to wireless fading channels in [1]. They show that the NE is Pareto optimal as well as no worse than 75% of the social optimum.

Another scenario of price anticipating players and a single resource is addressed by Maheswaran et.al. in [2], [3], [4], [5]. For a single divisible resource the authors have shown the existence and uniqueness of NE. Efficient mechanisms for maximizing the overall utility gained by players are developed. They also study the effect of collusion on the social optimum. In [6] the authors show that NE is no worse than 75% of the optimal for the case where players take into account the effect of their own bid on the price of the resource. This bound holds even for multiple resource case (i.e., a network with multiple paths between source and destination in general). However they do require the users to bid individually at each resource on the path to the destination. In [7] the efficiency loss is proved to be no worse than 33% if the players submit a single bid for the entire path. The utility model is also slightly different from that in [6] in the sense that the utility obtained by user is assumed independent of cost paid.

For a two player game and a single divisible resource, the bound in [6] has been improved to 87.5% by Hajek et. al. [8]. Their scheme however charges different price/unit resource for different players to achieve the bound. In [9], the authors show the stability of NE and compare the performance of different pricing schemes (discriminatory vs uniform). In [10], it is shown that for price anticipating users, a NE may not exist for a general network; however, a stable NE point is shown to exist for price taking users.

The case of budget constrained users bidding at multiple resources has been considered in [11], who show the existence of NE for this case. They also show that the NE can be non-unique for arbitrary utility functions. The efficiency of the NE is shown to decrease as $O(\frac{1}{\sqrt{m}})$. A greedy algorithm to achieve NE is proposed in [12]. A greedy algorithm does not always converge for this case; however, simulations show that after a certain number of iterations, the utility obtained is close to NE.

Our model for the bandwidth allocation game is similar to the one used by Zhang et.al., although we use the special utility function $U(x) = x$. In absence of access contraints, we prove the uniqueness and (weighted) max-min fairness of the NE. The case of access constrained users can also be viewed as a special case of the model developed in [11]. We prove that NE rates can be made arbitrarily close to max-min fair rates as the number of users in the system increase.

III. Bandwidth Allocation Games Without Access Constraints

Consider a wireless network consisting of N users and M base-stations. Each of the base-stations "own" some part of the usable spectrum (bandwidth); the spectrum owned by different base-stations are disjoint. The bandwidth owned by base-station $j \in M$ is denoted by B_j. Wireless users are interested in "buying" this bandwidth; A user $i \in N$ has a total amount (wealth) of $W_i > 0$ that it can spend for this purpose.

As an important special case, let us first assume that all users are in the range of all base-stations. Each user bids an amount w_{ij} to base-station j, with the constraint $\sum_{j \in M} w_{ij} = W_i$. Bandwidth allocation by base stations to are done in a proportionally fair manner, based on these bids. More specifically, each base-station divides its bandwidth amongst the users in proportion to the bids, i.e., base-station j allocates a bandwidth of $B_j w_{ij} / (\sum_{i \in N} w_{ij})$ to user i. Let $\mathbf{w}_i = [w_{ij}, j \in M]$ denote the vector of all bids by user i. Then, assuming that the users are interested in maximizing their allocated bandwidths, the spectrum allocation problem can be formulated as a game given by

$Game1$: (user i)

$$\max_{\mathbf{w}_i} \sum_{j \in M} B_j \frac{w_{ij}}{w_{ij} + \sum_{i' \in N \setminus \{i\}} w_{i'j}},$$

subject to

$$\sum_{j \in M} w_{ij} \leq W_i,$$

$$w_{ij} \geq 0, \ \forall j \in M.$$

The natural questions to ask are whether this bandwidth allocation game has Nash Equilibrium (NE), and what properties the NE satisfies. In the following we show that a unique Nash equilibrium exists for this game.

A. Nash Equilibrium

We shall only consider the case where there are at least two users since there is no competition in a one user game. It can be easily verified that $w_{ij} = \frac{B_j W_i}{\sum_{j' \in M} B_{j'}} \forall i, j$, is a NE for this game. Hence we only need to show that this is the only Nash equilibrium possible.

We first point out that the objective function of the user has a discontinuity at $w_{ij} = 0 \ \forall i$. To resolve this issue, we impose an additional restriction on the game that a user gets no bandwidth at a base station if its bid is zero at that base station *irrespective* of the bids of other users. Using this allocation scheme we first show that at least two bids at any base station j must be non-zero.

Our proof of uniqueness involves two basic steps. In lemma 1 we prove uniqueness of NE for $Game2$, which is similar to $Game1$ in all aspects except that the constraint $w_{ij} \geq 0, \forall j \in M$, of $Game1$ is replaced by $w_{ij} > 0, \forall j \in M$, in $Game2$. We then show that any NE for $Game1$ cannot have $w_{ij} = 0$ for any i, j. This together with uniqueness of NE for $Game2$ completes the proof.

Lemma 1. *There is a unique Nash equilibrium to the game if we assume all the bids by users to all base stations are non-zero, i.e.,* $w_{ij} > 0 \ \forall i, j$.

Proof: Consider any two users i_1 and i_2 and any two base stations j_1 and j_2. The objective for user i_1 can be written as

$$\max \sum_{j \in M} \frac{B_j w_{i_1 j}}{\sum_{i \in N} w_{ij}},$$

subject to the constraint

$$\sum_{j \in M} w_{i_1 j} = W_{i_1}.$$

We will write down the necessary conditions for a particular set of w_{ij} to be a NE. The Lagrange function $L(\mathbf{w}_{i_1}, \lambda)$ can be written as

$$\sum_{j \in M} \frac{B_j w_{i_1 j}}{\sum_i w_{ij}} - \lambda (\sum_{j \in M} w_{i_1 j} - W_{i_1}).$$

Taking the partial derivative with respect to $w_{i_1 j_1}$ and equating it to zero for $w_{i_1 j_1}$ to be a NE gives the following condition:

$$\frac{B_{j_1} (\sum_{i \in N \setminus \{i_1\}} w_{ij_1})}{(\sum_{i \in N} w_{ij_1})^2} = \lambda.$$

Repeating this argument for base station j_2 gives

$$\frac{B_{j_2} (\sum_{i \in N \setminus \{i_1\}} w_{ij_2})}{(\sum_{i \in N} w_{ij_2})^2} = \lambda.$$

Hence,

$$\frac{B_{j_1}(\sum_{i\in N\setminus\{i_1\}} w_{ij_1})}{(\sum_{i\in N} w_{ij_1})^2} = \frac{B_{j_2}(\sum_{i\in N\setminus\{i_1\}} w_{ij_2})}{(\sum_{i\in N} w_{ij_2})^2}.$$

This can be rewritten as

$$\frac{B_{j_2}(\sum_{i\in N\setminus\{i_1\}} w_{ij_2})}{B_{j_1}(\sum_{i\in N\setminus\{i_1\}} w_{ij_1})} = \frac{(\sum_{i\in N} w_{ij_2})^2}{(\sum_{i\in N} w_{ij_1})^2}. \quad (1)$$

Repeating this argument for user i_2 gives

$$\frac{B_{j_2}(\sum_{i\in N\setminus\{i_2\}} w_{ij_2})}{B_{j_1}(\sum_{i\in N\setminus\{i_2\}} w_{ij_1})} = \frac{(\sum_{i\in N} w_{ij_2})^2}{(\sum_{i\in N} w_{ij_1})^2}, \quad (2)$$

$$\frac{B_{j_2}(\sum_{i\in N\setminus\{i_1\}} w_{ij_2})}{B_{j_1}(\sum_{i\in N\setminus\{i_1\}} w_{ij_1})} = \frac{B_{j_2}(\sum_{i\in N\setminus\{i_2\}} w_{ij_2})}{B_{j_1}(\sum_{i\in N\setminus\{i_2\}} w_{ij_1})}. \quad (3)$$

Since i_1, i_2, j_1 and j_2 are completely arbitrary, we can write

$$\frac{(\sum_{i\in N\setminus\{i'\}} w_{ij})}{(\sum_{i\in N\setminus\{i'\}} w_{ij'})} = \alpha, \ \forall i' \in N. \quad (4)$$

Since $a/b = c/d \Rightarrow (a+b)/(c+d) = a/b = c/d$, summing all the numerators and denominators over the number of users gives

$$\frac{(N-1)(\sum_{i\in N} w_{ij})}{(N-1)(\sum_{i\in N} w_{ij'})} = \alpha. \quad (5)$$

Substituting the value of α in Eq. 4,

$$\frac{w_{ij}}{w_{ij'}} = \frac{\sum_{i\in N} w_{ij}}{\sum_{i\in N} w_{ij'}} \Rightarrow \frac{w_{ij}}{w_{ij'}} = \frac{w_{i'j}}{w_{i'j'}}.$$

Substituting this in the constraint equation for users i and i' gives $\frac{w_{ij}}{w_{i'j}} = \frac{W_i}{W_{i'}}$. We can rewrite this as

$$\frac{w_{ij}}{W_i} = \frac{w_{i'j}}{W_{i'}} = \cdots = \alpha_j \quad (6)$$

Substituting this in Eq. 2 yields $\frac{B_{j_1}}{\alpha_{j_1}} = \frac{B_{j_2}}{\alpha_{j_2}}$ or $\frac{B_{j_1}}{w_{ij_1}} = \frac{B_{j_2}}{w_{ij_2}}$. Using the constraint equation for user i gives $w_{ij} = \frac{W_i B_j}{\sum_{j'\in M} B_{j'}}$ as the only solution. ■

Note that the lemma only proves uniqueness of NE if $w_{ij} > 0$ for all i, j. We now prove that this NE is unique for a general bidding scenario. We first prove a useful lemma.

Lemma 2. *For any base station j, $\exists$ distinct users i, i' such that $w_{ij} > 0$ and $w_{i'j} > 0$.*

Proof: Suppose not, i.e., $\exists$ a base station j and a user i such that $w_{i'j} = 0 \, \forall \, i' \neq i$.(Note that we have merely assumed that all bids to base station j except for w_{ij} be zero; w_{ij} itself may be zero which covers the case of all bids to j being zero.)

If $w_{ij} > 0$ then i can decrease its bid at base station j by an infinitesimally small amount and increasing its bid to some other base station without changing its allocation at base station j. Hence user i can unilaterally increase its bandwidth which is not possible at a NE.

If $w_{ij} = 0$ then let m be some base station such that $w_{im} > 0$. It is trivial to show that such a m always exists. Since the allocation function at m is continuous due to w_{im} being non zero, there exists an $\epsilon > 0$ such that $B_m \frac{w_{im}-\epsilon}{\sum_{i'\in N} w_{i'm}-\epsilon} + B_{ij} > B_m \frac{w_{im}}{\sum_{i'\in N} w_{i'm}}$. Therefore the user i can unilaterally increase its bandwidth which is not possible at a NE. ■

Denote the original game by by $\{N, M, W, B\}$ where $W = [W_i]$ and and $B = \{B_j\}$. Denote the vector of all user bids as $\mathcal{W} = [w_{ij}]$. Define a Base Station Removal (*BSR*) operation on the original game to generate a new game in the following manner.

$BSR_T(\{N, M, W, B\}) \triangleq \{N, M-1, W_{T^-}, B_{T^-}\}$ where $W_{T^-} = [\tilde{W}_i] = [W_i - w_{iT}]$ and $B_{T^-} = B - B_T$.

Before presenting the details, we give a brief outline of the proof.

- Assume that there is at least one user which bids zero at some base station at NE.
- Use the BSR operation defined above to generate a new game with reduced numbers of base stations.
- In lemma 4 we prove a key property of the reduced game that will help us arrive at a contradiction.
- In lemma 3 we show that the BSR operation never results in a user with 0 wealth in the reduced game. Hence while the number of base stations is decreasing in a series of BSR operations, the number of users stays the same.

Lemma 3. *If T is a base station such that $w_{kT} = 0$ for some k at NE for the game $\{N, M, W, B\}$ then the game $\{N, M-1, W_{T^-}, B_{T^-}\}$ is such that $\tilde{W}_i > 0 \, \forall i$.*

Proof: Suppose not. Then $\exists$ at least one user r such that $\tilde{W}_r = 0$. Since $\tilde{W}_r = W_r - w_{rT}$, we get $W_r = w_{rT}$, i.e., $w_{rj} = 0 \, \forall j \neq T$, as the NE bids for user r in the original game $\{N, M, W, B\}$. Since user k cannot increase its bandwidth unilaterally in the original game, we get for some small $\epsilon > 0$, and for some j such that $w_{kj} > 0$:

$$B_j \frac{w_{kj}}{w_{kj} + \sum_{i'\in N\setminus\{k\}} w_{i'j}} \geq$$
$$B_j \frac{w_{kj} - \epsilon}{w_{kj} - \epsilon + \sum_{i'\in N\setminus\{k\}} w_{i'j}} + B_T \frac{\epsilon}{\epsilon + \sum_{i'\in N\setminus\{k\}} w_{i'T}}$$

Since ϵ can be arbitrarily small, the above condition translates to

$$\frac{B_j \sum_{i'\neq k} w_{i'j}}{(\sum_i' w_{i'j})^2} \geq \frac{B_T}{\sum_{i'} w_{i'T}}.$$

Since $w_{kj} > 0$ we get

$$\frac{B_j}{\sum_{i'} w_{i'j}} > \frac{B_T}{\sum_{i'} w_{i'T}}. \quad (7)$$

Note that we can write this condition for any user k and base stations j, T as long as $w_{kj} > 0$ and $w_{kT} = 0$. Therefore, since $w_{rT} > 0$ and $w_{rj} = 0$, we can write a similar condition for user r:

$$\frac{B_j}{\sum_{i'} w_{i'j}} < \frac{B_T}{\sum_{i'} w_{i'T}}. \quad (8)$$

Comparing Eq 7 and Eq. 8 gives us a contradiction. ■

In words, lemma 3 implies that if there is a base station j such that there exists a user who bids zero at this base station at NE, then there is no other user who bids its entire amount at this base station at NE. This is intuitive since a user that bids zero at a particular base station at NE means it perceives that base station ‘unfavorably’ as compared to other base stations. Hence there cannot be some other user who perceives this particular base station so 'favorable' so as to bid its entire wealth at it.

Lemma 4. *If $\mathcal{W} = [w_{ij}]$ are NE bids (possibly non-unique) for the game $\{N, M, W, B\}$, then the bids obtained by deleting the T^{th} column of $\mathcal{W}$, denoted by $\mathcal{W}_T$, is a NE for $\{N, M-1, W_{T^-}, B_{T^-}\}$.*

Proof: Suppose not, i.e., $\mathcal{W}_T$ are not NE bids for the new game $\{N, M-1, W_{T^-}, B_{T^-}\}$. Hence there must be at least one user who can unilaterally increase its bandwidth gain by changing the bids. Clearly this user can increase its bandwidth gain in the original game since the other user bids for base stations in B_{T^-} are the same in the original game $\{N, M, W, B\}$. This contradicts the fact that $\mathcal{W}$ are NE bids for the original game. ■

Lemma 5. *If $\mathcal{W}$ are NE bids for $\{N, M, W, B\}$ and T be a base station such that $w_{kT} = 0$ for some k, then at least one element of $\mathcal{W}_{T^-}$ is 0.*

Proof: Suppose not, i.e., $\mathcal{W}_{T^-}$ is such that all w_{ij} in $\mathcal{W}_{T^-}$ are strictly greater than 0. Since $\mathcal{W}$ are assumed to be NE bids, we have by lemmas 1 and 4,

$$w_{ij} = \frac{B_j \tilde{W}_i}{\sum_{j' \neq T} B'_j}, \ \forall j \neq T.$$

Let k be a user such that $w_{kT} = 0$ while $w_{kj} > 0 \ \forall j \in B_{T^-}$. Then we can write the condition given by Eq. 7 for this user. Also let r be a user such that $w_{rT} > 0$. Such a user r always exists by lemma 2. Also $w_{rj} > 0 \ \forall j \in B_{T^-}$ by assumption. Hence for user r we can write down one of the NE conditions as

$$\frac{B_j \sum_{i' \neq r} w_{i'j}}{(\sum_{i'} w_{i'j})^2} = \frac{B_T \sum_{i' \neq r} w_{i'T}}{(\sum_{i'} w_{i'T})^2}. \tag{9}$$

Comparing Eq. 7 and Eq. 9 gives

$$\frac{\sum_{i' \neq r} w_{i'j}}{\sum_{i'} w_{i'j}} < \frac{\sum_{i' \neq r} w_{i'T}}{\sum_{i'} w_{i'T}}. \tag{10}$$

Rearranging the terms we get

$$\frac{w_{rj}}{\sum_{i'} w_{i'j}} > \frac{w_{rT}}{\sum_{i'} w_{i'T}}. \tag{11}$$

Although we have written this equation for user r such that $w_{rT} > 0$, notice that the equation trivially holds for a user with $w_{rT} = 0$ as well. Hence Eq. 11 is holds for all users. This however is a contradiction since summing both LHS and RHS over all users results in $1 < 1$. ■

Theorem 1. *There is a unique NE to the game $\{N, M, W, B\}$, and the NE bid by a user i with total wealth W_i at base station j with bandwidth B_j is given by $w_{ij} = \frac{B_j W_i}{\sum_{j \in M} B_j}$.*

Proof: We have already shown that there is only one NE such that all the user bids to each base station are non-zero (lemma 1). Hence we need to show that there is no NE such that the a user has zero bid at some base station. Suppose not. Then there exists at least one user k and a base station T such that $w_{kT} = 0$ at NE. Use the operation BSR_T on this game to get a new game. By lemma 4 the new game is at NE and by lemma 5 there is at least one bid $w_{k'T'} = 0$. Hence we can keep applying the BSR operation on such base stations till we have just one base station left. Also by lemma 3, each base station removal operation leaves a strictly positive wealth with each user so that after a series of these operations, we get a game with N users with strictly positive wealth but just one base station. However this game has just one trivial NE with all bids non-zero which contradicts lemma 5. Since $w_{ij} = \frac{B_j W_i}{\sum_{j \in M} B_j}$ is a NE for $Game1$, it is the unique NE. ■

Since the NE bids for $Game1$ are the same as those for $Game2$, we have the closed form expression for NE bids as $w_{ij} = \frac{B_j W_i}{\sum_{j \in M} B_j}$. The sum total of bids at base station j is given by $\sum_i w_{ij} = B_j \frac{\sum_i W_i}{\sum_j B_j}$. Hence the revenue obtained by a base station is in proportion to the bandwidth it provides. The total bandwidth obtained by user i is given by

$$\sum_{j \in M} \frac{w_{ij} B_j}{\sum_{i' \in N} w_{i'j}} = W_i \frac{\sum_{j \in M} B_j}{\sum_{i' \in N} W_{i'}}.$$

Hence the bandwidths allocated to a user is proportional to the total wealth of the user. Hence the bandwidth allocations of users at NE is weighted max-min fair [13], with the weights being equal to the user wealths. An alternative interpretation of this is that at NE, the 'return' (i.e., the ratio of bandwidth obtained and price paid) is the same across all users.

IV. Bandwidth Allocation Games with Access Constrained Users

In this section we address the more general and realistic scenario where a user may not have access to every base station, but can instead access only a nonempty subset of all base stations. A user's inaccessibility of certain base stations may be due to various reasons such as physical barriers, distance from base station or authorization issues. The inaccessibility of a base station to a user leads to the bids of the user at the base station being zero. Note that this was not the case when there are no access constraints, as studied in the previous section.

Let the set of all users that can access a base station j be denoted by Γ_j. A strongly competitive game implies $|\Gamma_j| \geq 2$ for all j. In our model, we allow users to bid at any base station irrespective of accessibility constraints. However, no matter what the bid be, the bandwidth obtained from a base station which a user does not have access to, is zero.

Let $U_{ij}(x)$ be defined as follows:

$$U_{ij}(x) = \begin{cases} x & \text{if } i \in \Gamma_j, \\ 0 & \text{otherwise.} \end{cases}$$

We can now define the access constrained game as the following optimization problem for each user.

$Game3$: (user i)

$$\max_{\mathbf{w}_i} \sum_{j \in M} U_{ij} \left(B_j \frac{w_{ij}}{w_{ij} + \sum_{i' \in N \setminus \{i\}} w_{i'j}} \right),$$

subject to

$$\sum_{j \in M} w_{ij} \leq W_i,$$

$$w_{ij} \geq 0.$$

A. Existence of Nash Equilibrium

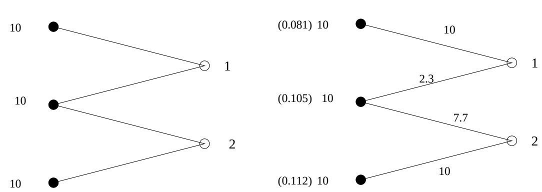
The existence of a NE is in general not guaranteed in this case. However, in [11] it is shown that if the game is strongly competitive, that is each base station can be accessed by at least two users, then the game has a NE.

B. Fairness Properties of Nash Equilibria

We have not been able to show uniqueness of NE for this case which remains an interesting open question for future research. Next we investigate the fairness properties of the NE by comparing the user bandwidth allocation at NE with the max-min fair [13] bandwidth allocation in the system.

First we define the notion of *max-min fairness* that we consider. Using the notation in previous section, let W_i be the wealth of user i; also let C_i be the total bandwidth obtained by user i. Let the 'return' for user i be $R_i = \frac{C_i}{W_i}$ and $R = [R_i]$. The 'return vector' R is (lexicographically) max-min fair if it is not possible to increase any particular component R_i without decreasing a component R_j that is less than or equal to R_i. A bandwidth allocation is said to be max-min fair if the corresponding return vector is max-min fair. Note that max-min fairness is a property of the given system, and in general depends on the given system parameters (number of users, number of base stations, user wealths, base station bandwidths, and the access constraints); the notion of max-min fairness is independent of the definition of our game. The max-min fair bandwidth allocation can be viewed as the "most fair" allocation among all possible bandwidth allocations to users in the system.

Fig. 1 shows that bandwidth allocation to users at NE need not be max-min fair in general .



(a) The accessibility graph (b) Nash Equilibrium

Fig. 1. Counterexample to show that bandwidth allocation at NE may not be max-min fair. In (a), the numbers across the users represent their wealths, and the numbers across the base stations represent their bandwidths. In (b), the numbers across the edges represent the NE bids, and the numbers in parentheses represent the returns of the users at NE. In the max-min fair bandwidth allocation, each user obtains 1 unit of bandwidth, and the return of each user is 0.1.

Now we consider $Game3$ as the number of users grows without bound, while the number of base stations remains fixed. In this case, we show that the bandwidth allocation to users at NE becomes arbitrarily close to max-min fair bandwidth allocation. To prove this result, we need some additional assumptions which are stated next.

Assumptions

A.1 $|W_{i_1} - W_{i_2}| < \Delta W$ for some fixed ΔW independent of N and any i_1, i_2.

A.2 A user can access at least one base station. Users that cannot access any base station can safely be removed from consideration.

A.3 No base station is starved of wealth. Formally the number of users that can access a base station j, denoted by n_j, is greater than αN for some fixed α. Notice that we are proving an asymptotic result. However an increase in the number of users may not necessarily imply an increase in the number of users that can access a particular base station. The assumption is needed to address this issue.

A.4 The amount of wealth that a user can have is lower bounded by some positive number, say W_{min}.

A.5 The amount of bandwidth owned by any base station j grows with the number of users as $\Theta(N)$ where N is the number of users. Since we are proving an asymptotic result, a fixed amount of bandwidth in the system will lead to progressively lower bandwidth to each user. The assumption allows each user to obtain a near constant bandwidth as number of users increase.

C. Main Result

Theorem 2. *For $Game3$ under assumptions A.1-5, given any $\xi > 0$, $\exists\, N_0$ such that for all $N > N_0$, $|R_i(N) - R_i^*(N)| \le \xi$, where $R_i(N)$ is the return for user i at NE, and $R_i^*(N)$ is the return for user i at max-min fairness.*

The result implies that the returns of the users at NE approach the max-min fair returns as the number of users in the system increases.

D. Proof of Theorem 2

Lemma 6. *The total wealth in the system grows linearly with the number of users N, i.e., there exist positive constants a, b such that the total wealth denoted by W obeys*

$$bN < W < aN. \tag{12}$$

Proof: Applying assumption A.1 for user 1, we get $W_i - \Delta W < W_1 < W_i + \Delta W$ for any i and a fixed ΔW. Note that the inequality holds for $i = 1$ trivially. Summing up all the inequalities over i gives $W - N\Delta W < NW_1$. Also by A.4 we have $W > NW_{\text{min}}$. We can rewrite this as $N(W_{\text{min}}) < W < N(\Delta W + W_1)$. ■

Lemma 7. *The sum of NE bids by users at any base station j grows linearly with the number of users $n_j \le N$ that can access the base station j.*

Proof: Let $f_j(n_j)$ be the asymptotic growth rate function of the sum of Nash equilibrium bids at base station j. Then by definition of $f_j(n_j)$, $c_j f_j(n_j) + d_j \le \sum_i w_{ij} \le a_j f_j(n_j) + b_j$. Summing up all these inequalities over j gives

$$\sum_j (c_j f_j(n_j) + d_j) \le \sum_i \sum_j w_{ij} \le \sum_j (a_j f_j(n_j) + b_j).$$

By assumption 2, we can write this as

$$\sum_j (c_j f_j(n_j) + d_j) < W < \sum_j (a_j f_j(n_j) + b_j).$$

By lemma 6, W grows linearly with N. Hence, we must have $\sum_j f_j(n_j)$ grow linearly with N. This implies any $f_j(n_j)$ grows linearly or sublinearly in N and $\exists$ at least one J such that $f_J(n_J)$ grows linearly with N.

We now show that sublinear growth is not possible. Suppose not, i.e., let $f_j(n_j)$ be sublinear. Using an argument similar to lemma 6 and assumption A.3, it is easy to see that $\sum_{i\in\Gamma_j} W_i$ increases linearly for any j. Hence if $f_j(n_j)$ is sublinear then $\exists\ i \in \Gamma_j$ such that $w_{ik} > 0$ for some $k \neq j$. There are two possibilities for $f_k(n_k)$:

1. $f_k(n_k)$ grows linearly with N. This however leads to a contradiction since $\frac{B_k \sum_{i'\neq i} w_{i'k}}{(\sum_{i'} w_{i'k})^2}$ grows as $\Theta(1)$ for k and $\Omega(1)$ (or strictly faster than $\Theta(1)$) for j. Hence $\exists$ N_0 such that $\forall N > N_0$, $\frac{B_k \sum_{i'\neq i} w_{i'k}}{(\sum_{i'} w_{i'k})^2} < \frac{B_j \sum_{i'\neq i} w_{i'j}}{(\sum_{i'} w_{i'j})^2}$. Hence $w_{ik} > 0$ cannot be a NE.
2. $f_k(n_k)$ grows sublinearly with N. In this case $\sum_{i\in\Gamma_j\cup\Gamma_k} W_i$ grows linearly with N. Hence $\exists\ i \in \Gamma_j \cup \Gamma_k$ such that $w_{il} > 0$ and $l \neq j, k$. Hence we can repeat this procedure till we get a contradiction. Hence in all cases $f_j(n_j)$ grows linearly with N.

■

Let τ_i denote the set of base stations such that for any $j \in \tau_i$, we have $w_{ij} > 0$ at NE. By A.2, the set τ_i is non-empty for arbitrary i. Based on NE bids lets divide the graph into some key disjoint sets. Start with an arbitrary user i and put it in a set say I_1. Put all base stations in τ_i in $\mathcal{B}_{I_1}$. Next put all users with strictly positive bids at any base stations in $\mathcal{B}_{I_1}$ in I_1. Now repeat the process for the new users in I_1 till we either exhaust the entire graph or no more users can be put in I_1 or no more base stations can be put in $\mathcal{B}_{I_1}$. If there are still more users left in the graph that are not in I_1 then pick an arbitrary user and put it in I_2 and continue the process. By construction it is easy to see that we have divided the bi partite graph into possibly many subgraphs. See Fig. 2 for an illustration. In the figure, an edge exists between user i and base station j iff the NE bid $w_{ij} > 0$.

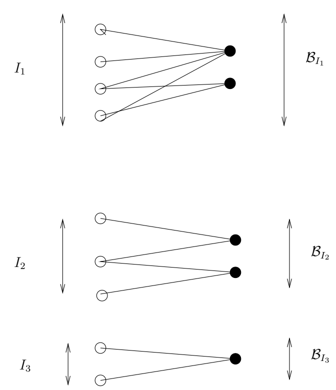


Fig. 2. The division of the user-base station NE bids graph into subgraphs.

Lemma 8. *There exists N_1 such that for all $N > N_1$ and for any two users i_1, i_2 in the same subgraph I, $|R_{i_1} - R_{i_2}| \leq (|\mathcal{B}_I| - 1)\xi'$ where ξ' is arbitrary.*

Proof: There are two cases to consider:

1. $|\mathcal{B}_I| = 1$. In this case it is easy to see that all users have the same return $\frac{B_j}{\sum_{i'\in I} w_{i'j}}$ and the result holds trivially.
2. $|\mathcal{B}_I| > 1$. In this case $\exists\ i \in I$ and base stations $j_1, j_2 \in \tau_i$ such that $w_{ij_1}, w_{ij_2} > 0$. Let $\mathcal{R}_j \equiv \frac{B_j}{\sum_{i'\in I} w_{i'j}}$. Writing the NE condition as in the previous section yields

$$\frac{\partial}{\partial w_{ij_1}}\left(\frac{w_{ij_1}B_{j_1}}{w_{ij_1} + \sum_{i'\in N\setminus i'\neq i} w_{i'j_1}}\right) = \frac{\partial}{\partial w_{ij_2}}\left(\frac{w_{ij_2}B_{j_2}}{w_{ij_2} + \sum_{i'\in N\setminus i'\neq i} w_{i'j_2}}\right) = \lambda_i$$

$$\frac{B_{j_1}\sum_{i'\in I, i'\neq i} w_{i'j_1}}{(\sum_{i'\in I} w_{i'j_1})^2} = \frac{B_{j_2}\sum_{i'\in I, i'\neq i} w_{i'j_2}}{(\sum_{i'\in I} w_{i'j_2})^2}$$

$$\frac{B_{j_1}\sum_{i'\in I} w_{i'j_1} - w_{ij_1}}{(\sum_{i'\in I} w_{i'j_1})^2} = \frac{B_{j_2}\sum_{i'\in I} w_{i'j_2} - w_{ij_2}}{(\sum_{i'\in I} w_{i'j_2})^2}$$

$$\frac{B_{j_1}}{\sum_{i'\in I} w_{i'j_1}} - \frac{B_{j_1}w_{ij_1}}{(\sum_{i'\in I} w_{i'j_1})^2} = \frac{B_{j_2}}{\sum_{i'\in I} w_{i'j_2}} - \frac{B_{j_2}w_{ij_2}}{(\sum_{i'\in I} w_{i'j_2})^2}$$

By lemma 7, $\frac{B_j w_{ij}}{(\sum_{i'\in I} w_{i'j})^2}$ is $O(\frac{1}{N})$, for $j = j_1, j_2$. Therefore

$$|\mathcal{R}_{j_1} - \mathcal{R}_{j_2}| = O(1/N) \leq \xi', \tag{13}$$

when N is sufficiently large.

Hence, every base station-user-base station chain in I causes a 'spread' of at most ξ' in the $\mathcal{R}_j$s of the base stations. Therefore the spread between any two base stations in I is at most $(|\mathcal{B}_I| - 1)\xi'$. The total bandwidth C_i gained by the user is

$$C_i = \sum_{j\in\tau_i} w_{ij}\mathcal{R}_j.$$

The return for the same user can be written as $R_i = \sum_{j\in\tau_i} \frac{w_{ij}}{W_i}\mathcal{R}_j$; in other words, the return of users in I is the weighted average of $\mathcal{R}_j$s in $\mathcal{B}_I$. Hence the 'spread' of returns for users is at most the same as the 'spread' in base station $\mathcal{R}_j$s. ■

Now we combine the subgraphs together in the following manner. Consider the subgraph I_1 (constructed according to the procedure described earlier, as in Fig. 2); the user with minimum return in I gets a return R_{I_1}. Then by the preceding lemma, all the users in I have returns in the range $R_{I_1} + (|\mathcal{B}_{I_1}| - 1)\xi'$. Hence in terms of user returns, the subgraphs can be represented by a range $[a, b]$. We combine any two subgraphs if the ranges overlap. For example in Fig. 3, I_1 and I_2 will combine while I_3 , I_4 will not. Let the new subgraphs be denoted by S_ks. Let S be any such S_k. The corresponding base station sets $\mathcal{B}_I$s combine to give $\mathcal{B}_S$. It is easy to see by construction of the Is and lemma 8 that for any users i, i' in S, $|R_i - R_{i'}| \leq (|\mathcal{B}_S| - 1)\xi'$. Substituting $\xi' = \frac{\xi}{M}$ yields $|R_i - R_{i'}| \leq (|\mathcal{B}_S| - 1)\frac{\xi}{M}$. Since $\mathcal{B}_I$ can have atmost M base stations, we have

$$|R_i - R_{i'}| \leq \xi \quad \forall i, i' \in S. \tag{14}$$

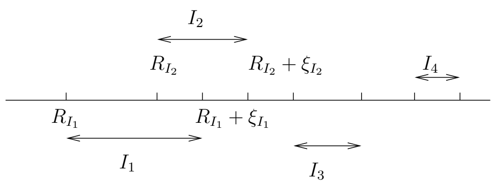


Fig. 3. The combining procedure; $\xi_{I_k} = R_{I_1} + (|\mathcal{B}_{I_1}| - 1)\xi'$, for any k.

Let R_i^* denote the max-min fair returns. We remind the reader that both the NE returns and max-min fair returns are functions of number of users. Assume W.L.O.G. that S_ks are arranged in increasing order of return ranges. (These ranges are non-overlapping by construction of S_ks). Let the range of returns for S_k be $[R_{S_k},\ R_{S_k} + \xi_{max}^k]$. In other words, we can write down the return of any user $i \in S_k$ as $R_{S_k} + \xi_i$ where $0 \le \xi_i \le \xi_{max}^k \le \xi$. By construction of S_ks, we must have $R_{S_k} > R_{S_{k-1}} + \xi_{max}^{k-1}\ \forall\ k$.

Consider an arbitrary user i_1 in S_1. We claim that i_1 cannot access any base station in $M - \mathcal{B}_{S_1}$. For if i_1 can access any base station $b \in M - \mathcal{B}_1$ then clearly this is not a Nash equilibrium since i_1 can increase its return by bidding at base station $b \in M - \mathcal{B}_{S_1}$. We remind the reader that a user may choose to bid zero at a base station or may be forced to bid zero at a base station due to access constraints. We have shown that in this case the user cannot access base stations in $M - \mathcal{B}_{S_1}$ and hence bids zero. The users in $N - S_1$ however *may have chosen* to bid zero at base stations in $\mathcal{B}_{S_1}$ due to lower return.

In general we can write that any user in S_k cannot access any base station in $M - (\mathcal{B}_{S_1} \cup \mathcal{B}_{S_2} \cdots \mathcal{B}_{S_k})$.

We will now compare the NE returns with max-min fair returns. We begin by proving some important properties that max-min fair returns must satisfy.

Lemma 9. *(i) If two users say i_1, i_2 are getting positive bandwidths from a single base station at max-min fair bandwidth allocation, then their returns must be equal. (ii) If the max-min fair returns for i_1 and i_2 are not equal and $R_{i_1}^* > R_{i_2}^*$, then i_2 cannot access any base station that is providing bandwidth to i_1 in the max-min fair bandwidth allocation.*

Proof: (i) Suppose not. Let i_1 be the user with higher return. Then some bandwidth that is being allocated to i_1 can be allocated to i_2 while keeping the returns of all the users other than i_1 constant. This increases the return of i_2 at the expense of i_1 which already had higher return. Hence the original allocation was not max-min fair.

(ii) Again assume that there is a base station accesible to i_2 that is serving i_1 in the max-min fair bandwidth allocation. Then some of the bandwidth allocated to i_1 can be allocated to i_2 there by increasing the return of i_2 at the expense of i_1 which means that $R_{i_1}^*$, $R_{i_2}^*$ are not max-min fair. ■

Since a user must get bandwidth from some base station, and every user getting bandwidth from a particular base station has same return, there can only be M possible values of max-min fair return for users. Let r_ks denote these values. W.L.O.G. assume that the r_ks are in ascending order of magnitude. It is clear by lemma 9 that the graph gets divided into many possible subgraphs at max-min fair allocation similar to the NE. Let U_k denote the set of all users with return r_k and $\mathcal{B}_{U_k}$ denote the set of base stations serving the users in U_k. By lemma 9, users in U_k cannot access any base station in $M - \mathcal{B}_{U_1} \cup \mathcal{B}_{U_2} \cdots \mathcal{B}_{U_k}$. Note that a similar property is satisfied by users in S_k.

Lemma 10. $S_1 \cap U_1 \neq \phi$.

Proof: Suppose not, i.e., $S_1 \cap U_1 = \phi$. By definition of U_1,

$$r_1 \sum_{i \in U_1} W_i = \sum_{j \in \mathcal{B}_{U_1}} B_j,$$

$$r_1 = \frac{\sum_{j \in \mathcal{B}_{U_1}} B_j}{\sum_{i \in U_1} W_i}. \tag{15}$$

Note that every user in $N - S_1$ gets a return of at least $R_{S_1} + \xi_{max}^1$ at NE. Hence if $S_1 \cap U_1 = \phi$ then at NE we can write for users in U_1 who cannot access any base station not in $\mathcal{B}_{U_1}$

$$(R_{S_1} + \xi_{max}^1) \sum_{i \in U_1} W_i \le \sum_{j \in \mathcal{B}_{U_1}} B_j,$$

$$R_{S_1} + \xi_{max}^1 \le \frac{\sum_{j \in \mathcal{B}_{U_1}} B_j}{\sum_{i \in U_1} W_i}. \tag{16}$$

Comparing Eq. 15 with Eq. 16 yields

$$r_1 \ge R_{S_1} + \xi_{max}^1. \tag{17}$$

Let $r > r_1$ be the minimum return that users in S_1 get at max-min fairness. Then we can write

$$r \sum_{i \in S_1} W_i \le \sum_{j \in \mathcal{B}_{S_1}} B_j,$$

$$r_1 < r \le \frac{\sum_{j \in \mathcal{B}_{S_1}} B_j}{\sum_{i \in S_1} W_i}. \tag{18}$$

Since for any user in S_1, the return at NE lies in $[R_{S_1},\ R_{S_1} + \xi_{max}^1]$, let the return R_i be denoted by $R_{S_1} + \xi_i$ where $0 \le \xi_i \le \xi_{max}^1$. At NE we can write,

$$\sum_{i \in S_1} (R_{S_1} + \xi_i) W_i = \sum_{j \in \mathcal{B}_{S_1}} B_j,$$

$$R_{S_1} \sum_{i \in S_1} W_i + \sum_{i \in S_1} \xi_i W_i = \sum_{j \in \mathcal{B}_{S_1}} B_j,$$

$$R_{S_1} + \sum_{i \in S_1} \frac{\xi_i W_i}{\sum_{i' \in S_1} W_{i'}} = \frac{\sum_{j \in \mathcal{B}_{S_1}} B_j}{\sum_{i \in S_1} W_i}. \tag{19}$$

Note that each $0 \le \xi_i \le \xi_{max}^1$. Let us first consider the case $\xi_{max}^1 > 0$. Since $\exists$ at least one user in S_1 with $\xi_i = 0$, we can write

$$R_{S_1} + \xi_{max}^1 > \frac{\sum_{j \in \mathcal{B}_{S_1}} B_j}{\sum_{i \in S_1} W_i}. \tag{20}$$

Note that Eq. 20 is derived without using the assumption $S_1 \cap U_1 = \phi$ and hence is true in general. Comparing Eq. 20 with Eq. 18 yields $r_1 < R_{S_1} + \xi_{max}^1$ which contradicts Eq. 17.

Next let us consider the case $\xi^1_{max} > 0$. From Eq. 19, we get $R_{S_1} = \frac{\sum_{j\in\mathcal{B}_{S_1}} B_j}{\sum_{i\in S_1} W_i}$, which then combined with Eq. 18 gives $r_1 < R_{S_1}$. However r_1 is the minimum return that any user gets at max-min fairness while R_{S_1} is the minimum return that any user gets at NE. Hence $r_1 \geq R_{S_1}$, which contradicts our earlier observation.

Since we arrive at a contradiction in either case, the assumption $S_1 \cap U_1 = \phi$ must be false. ∎

By lemma 10, $S_1 \cap U_1 \neq \phi$. Hence, the minimum return a user in S_1 gets at max-min fairness is r_1. Therefore at max-min fairness, for users in S_1 we can write

$$r_1 \sum_{i\in S_1} W_i \leq \sum_{j\in\mathcal{B}_{S_1}} B_j,$$

$$r_1 \leq \frac{\sum_{j\in\mathcal{B}_{S_1}} B_j}{\sum_{i\in S_1} W_i}. \tag{21}$$

Subtracting from Eq. 20 yields

$$r_1 - R_{S_1} < \xi^1_{max}. \tag{22}$$

We have argued earlier that $r_1 \geq R_{S_1}$. Combining with Eq. 22 gives

$$0 \leq r_1 - R_{S_1} < \xi^1_{max}. \tag{23}$$

Lemma 11. *The minimum return r_{S_2} that a user in $N - S_1$ gets in max-min fair allocation is at least $R_{S_2} > R_{S_1} + \xi^1_{max}$ if $N - S_1 \neq \phi$.*

Proof: At NE every user in $N - S_1$ gets a return of at least $R_{S_2} > R_{S_1} + \xi^1_{max}$ while getting bandwidth only from $M - \mathcal{B}_{S_1}$. Since users in S_1 can only access base stations in B_{S_1}, even at max-min fair allocation the users in $N - S_1$ get the entire bandwidth in $M - \mathcal{B}_{S_1}$. However at max-min fairness, users in $N - S_1$ can possibly use some bandwidth from $\mathcal{B}_{S_1}$ but that cannot decrease the minimum return obtained by a user in $N - S_1$. Hence $r_{S_2} \geq R_{S_2} > R_{S_1} + \xi^1_{max}$. ∎

Lemma 12. *The maximum return that any user in S_1 can get in max-min fair allocation, say r_{max}, is upper bounded by $R_{S_1} + \xi^1_{max}$.*

Proof: There are two cases to be considered for users in S_1 under max-min fair allocation.

1. All users get the same return at max-min fair aloocation. Then by lemma 10, this rate must be r_1 and by Eq. 23 the result holds.
2. There are users in S_1 with return strictly greater than r_1. Let U_{max} be the set of users with return r_{max}. By lemma 9, the users in S_1 not in U_{max} cannot access base stations in $\mathcal{B}_{U_{max}}$. We will compare returns of users in $U_{m1} = S_1 \cap U_{max}$ at max-min fair allocation and NE. Let $\mathcal{B}_{m1} = \mathcal{B}_{S_1} \cap \mathcal{B}_{U_{max}}$. Since $U_{m1} \subseteq S_1$, the users in U_{m1} can only access base station in $\mathcal{B}_{S_1}$. Hence, while $\mathcal{B}_{U_{max}}$ may contain base stations not in $\mathcal{B}_{S_1}$, the users in U_{m1} cannot access them. Hence the only base stations from which users in U_{m1} are getting bandwidth from are $\mathcal{B}_{m1} = \mathcal{B}_{S_1} \cap \mathcal{B}_{U_{max}}$. Therefore

$$r_{max} \sum_{i\in U_{m1}} W_i \leq \sum_{j\in\mathcal{B}_{m1}} B_j \tag{24}$$

At NE we know that no user in $N - S_1$ is getting any bandwidth from $\mathcal{B}_{S_1} \supseteq \mathcal{B}_{m1}$ while users in $S_1 - U_{m1}$ cannot access $\mathcal{B}_{U_{max}} \supseteq \mathcal{B}_{m1}$. Hence the entire bandwidth in B_{m1} is utilized by users in U_{m1}.

$$\sum_{i\in U_{m1}} (R_{S_1} + \xi_i) W_i \geq \sum_{j\in\mathcal{B}_{m1}} B_j \tag{25}$$

From Eq. 25 and Eq. 24, we have

$$\sum_{i\in U_{m1}} (r_{max} - R_{S_1} - \xi_i) W_i \leq 0$$

Hence $\exists\ i$ such that $r_{max} \leq R_{S_1} + \xi_i$. Since $0 \leq \xi_i \leq \xi^1_{max}\ \forall\ i \in S_1$, we have $r_{max} \leq R_{S_1} + \xi^1_{max}$. ∎

Combining Eq. 23 with lemma 12 gives the range of returns for users in S_1 at max-min fair allocation as $[R_{S_1},\ R_{S_1} + \xi^1_{max}]$. Hence the main result holds for users in S_1.

Lemma 13. *The bandwidth allocated to users in $N - S_1$ by base stations in B_{S_1} is zero at max-min fair allocation.*

Proof: By lemma 12, every user in S_1 in max-min fair allocation has a return less than $R_{S_1} + \xi^1_{max}$. Hence, $S_1 \subseteq (U_1 \cup U_2 \cdots U_{max})$. By lemma 11 every user in $N - S_1$ gets a return strictly greater than $R_{S_1} + \xi^1_{max}$ at max-min fairness. Hence $(N - S_1) \cap (U_1 \cup U_2 \cdots U_{max}) = \phi$, or

$$S_1 = (U_1 \cup U_2 \cdots U_{max}). \tag{26}$$

We can represent $N - S_1$ by

$$N - S_1 = U_{max+1} \cup U_{max+2} \cdots \tag{27}$$

We will now show that $\mathcal{B}_{S_1} = \mathcal{B}_{U_1} \cup \mathcal{B}_{U_2} \cdots \mathcal{B}_{U_{max}}$. Comparing Eq. 26 and Eq. 27, we can say that every user in $N - S_1$ is getting a return higher than any user in S_1 at max-min fairness. Hence any base station that is allocating bandwidth to a user in $N - S_1$ must be inaccesible to users in S_1 by lemma 9. Since $N - S_1 = U_{max+1} \cup U_{max+2} \cdots$ by Eq. 27, the base stations allocating bandwidth to $N - S_1$ are $\mathcal{B}_{U_{max+1}} \cup \mathcal{B}_{U_{max+2}} \cdots$. Since every base station in $\mathcal{B}_{S_1}$ is accessible to some user in $S_1 = U_1 \cup U_2 \cdots U_{max}$, it cannot have any common element with $\mathcal{B}_{U_{max+1}} \cup \mathcal{B}_{U_{max+2}} \cdots$. ∎

Remove users from S_1 and base stations in $\mathcal{B}_{S_1}$ and it is easy to see that the main result holds for users in S_2. We can remove the $S_1 \cup \mathcal{B}_{S_1}$ component of the graph because it is a disjoint component both at NE and max-min fairness. Hence in general we can write

$$|R^*_i - R_i| \leq \xi \quad \forall i \in N. \tag{28}$$

It is to be noted that we have shown that the bandwidth allocation to users in *any* NE is 'close' to the max-min fair allocations. Its worth mentioning, however, that if we assume that base stations split their bandwidths among users in proportion to the user bids (as we have assumed in our game formulations), then the bids that realize max-min fair bandwidth allocation may be non-unique, even without access constraints. This is illustrated in Fig. 4.

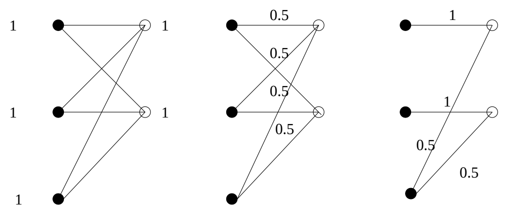


Fig. 4. A simple example where max-min fair bids are non-unique. In the leftmost graph, the numbers across the users represent their wealths, and the numbers across the base stations represent their bandwidths. The middle and rightmost graphs show two distinct sets of bids that results in max-min fair bandwidth allocation, if the base stations split bandwidths in proportion to the bids. The max-min fair bandwidth and the max-min fair return for each user are both $\frac{2}{3}$ in this example.

V. Discussion and Conclusions

We have presented several important results for bandwidth allocation games under budget constraints, *viz.* existence and uniqueness of NE for the case where every user can access every base station, and max-min fairness of the bandwidth allocation at NE for access constrained users, as the number of users increases without bound while the number of base stations remains fixed. The uniqueness of NE for the case of access constrained users remains an open question.

It is also possible to extend the definition of the game to include a general network topology, where each user is associated with a set of routes, each of which is possibly composed of multiple links (nodes). The user has to bid for bandwidth on each link on a route to secure the route. The overall bandwidth gained on a route is the same as the bandwidth on the bottlenecked link. The authors in [6] address a closely related problem, but with non budget-constrained users. Whether or not a similar analysis can be done for budget constrained users remains an open question.

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