Bandwidth Allocation Games under Budget and Access Constraints

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Outline

- Introduction
  - Motivation and System Model
- Bandwidth Allocation under Budget Constraints
  - Existence and Uniqueness of Nash Equilibrium
- Bandwidth Allocation under Additional Access Constraints
  - Asymptotic Fairness of the Nash Equilibrium
- Open Issues

Motivation

- System Model:
  - Users (*b/w consumers)* associated with individual budgets
  - Base-stations (*b/w providers)* own (disjoint) parts of spectrum
  - A user may not have access to all base-stations, and vice versa
  - Users split their wealth across base-stations (bids)
  - Base-stations split their bandwidth based on user bids
- Question:
  - What if users are *greedy* and base-stations are *fair*?
  - Existence, uniqueness and *fairness* properties of the Nash Equilibrium (NE)

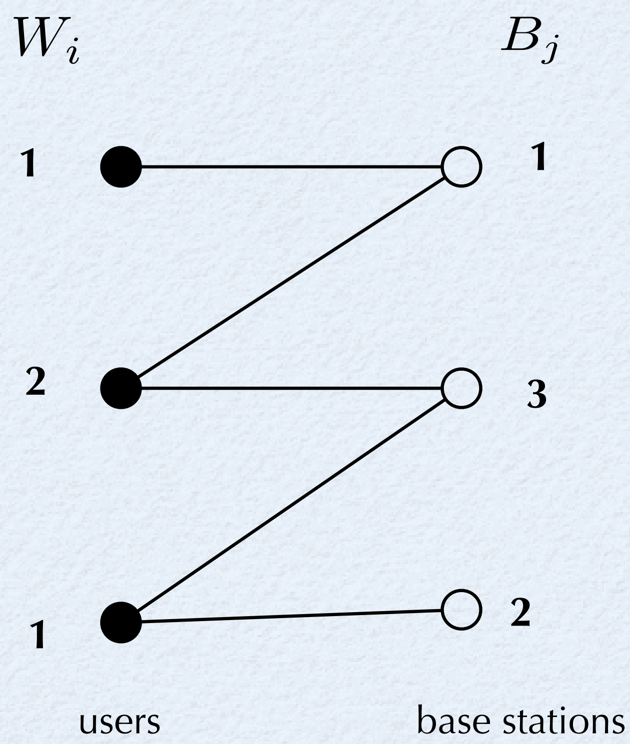
Related Work

- Utility (profit) maximization game
  - Lot of interest in recent years
  - Maheswaran and Basar '03 '04
  - Johari and Tsitisklis
  - Sanghvi and Hajek '04, Yang and Hajek '07 *etc.*

- Budget constrained version
  - Zhang '05, Feldman, Lai and Zhang '05

System Model

- N: set of users; M: set of base-stations
- User i has a budget (wealth) of W_i *(budget constraints)*
- Base-station (BS) j owns a total b/w of B_j
- A BS j is accessible to a subset Γ_j of all users *(access constraints)*



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A Greedy User - Fair BS Game

- Game Assumptions:
  - A user *i* bids w_{ij} on BS *j*
  - Each BS splits is bandwidth in proportion to the user bids (*fair BSs*)
  - Users choose the bids so as to maximize their overall bandwidth (*price-anticipating, greedy users*)
- Game definition: *(No access constraints)*

$Game1$: (user i)

$$\max_{\mathbf{w}_i} \sum_{j \in M} B_j \frac{w_{ij}}{w_{ij} + \sum_{i' \in N \setminus \{i\}} w_{i'j}}$$

subject to

$$\sum_{j \in M} w_{ij} \leq W_i,$$

$$w_{ij} \geq 0, \ \forall j \in M.$$

(Also: A user with zero bid gets zero b/w at a BS, irrespective of the bids of other users)

NE Properties (1/3)

- Questions of Interest:
  - Existence of the NE *(Yes!)*
  - Uniqueness of the NE *(Yes!)*
  - Can the NE be expressed in closed form *(Yes!)*
  - Is the NE "fair"? *(Yes!)*
- Existence and Uniqueness:
  - Assume that none of the w_{ij} are zero
  - Then the NE can be explicitly calculated as

$$w_{ij}^* = \frac{B_j W_i}{\sum_{j' \in M} B_{j'}}$$

  - Therefore NE is unique in this case

NE Properties (2/3)

- Question: How do we know $w_{ij} > 0 \;\; \forall i, j$
- Proof Outline:
  - Use contradiction: assume $\exists \; w_{kT} = 0$
  - Form a reduced game $\{N, M-1, W_{T-}, B_{T-}\}$ from the original game $\{N, M, W, B\}$ by removing BS T and bids to that BS
  - Show that there is a NE to the reduced game (easily derived from the NE of the original game) with some zero bid
  - Keep on iterating until we are left with one BS: it cannot have any zero bid at NE (contradiction)

NE Properties (3/3)

- Note: $w_{ij}^* = \frac{B_j W_i}{\sum_{j' \in M} B_{j'}}$ implies:
  - b/w obtained by user *i* is

$$b_i^* = W_i \left(\frac{\sum_{j' \in M} B_j}{\sum_{i' \in N} W_i'} \right)$$

  - Therefore, each user obtains b/w in proportion to its wealth
  - The NE leads to fair sharing of b/w among users (weighted max-min / proportional fairness)

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Game Generalization

- Game Assumptions:
  - A user *i* bids w_{ij} on BS *j*
  - Fair BSs
  - Price anticipating greedy users
  - A user can only obtain b/w from BSs it has access to
- Game definition:

$Game3$: (user i)

$$\max_{\mathbf{w}_i} \sum_{j \in M} U_{ij} \left(B_j \frac{w_{ij}}{w_{ij} + \sum_{i' \in N \setminus \{i\}} w_{i'j}} \right),$$

subject to

$$\sum_{j \in M} w_{ij} \leq W_i,$$

$$w_{ij} \geq 0.$$

(Also: A user with zero bid gets zero b/w at a BS, irrespective of the bids of other users)

$$U_{ij}(x) = \begin{cases} x & \text{if } i \in \Gamma_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$|\Gamma_j| \geq 2$$

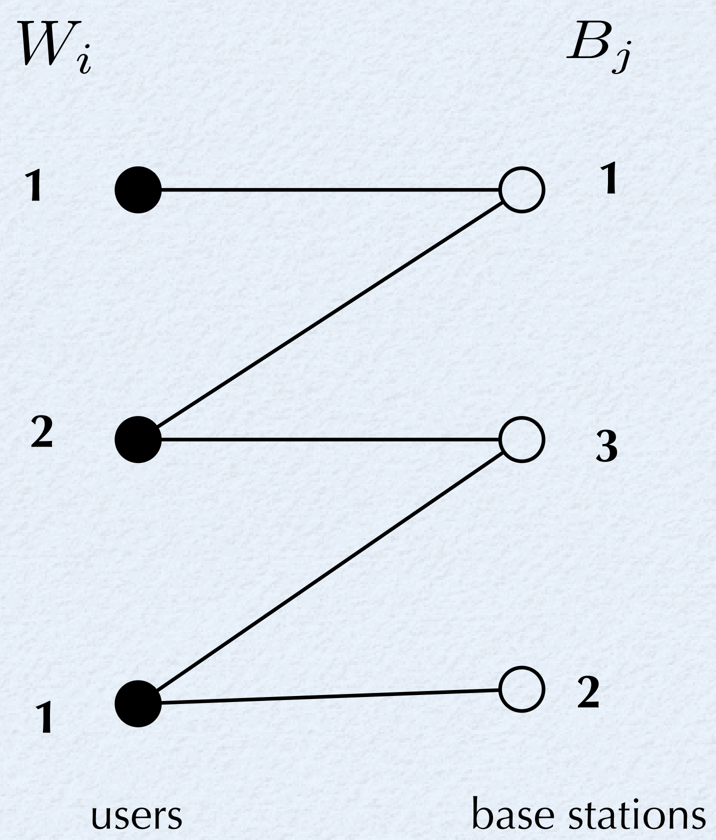
NE Properties

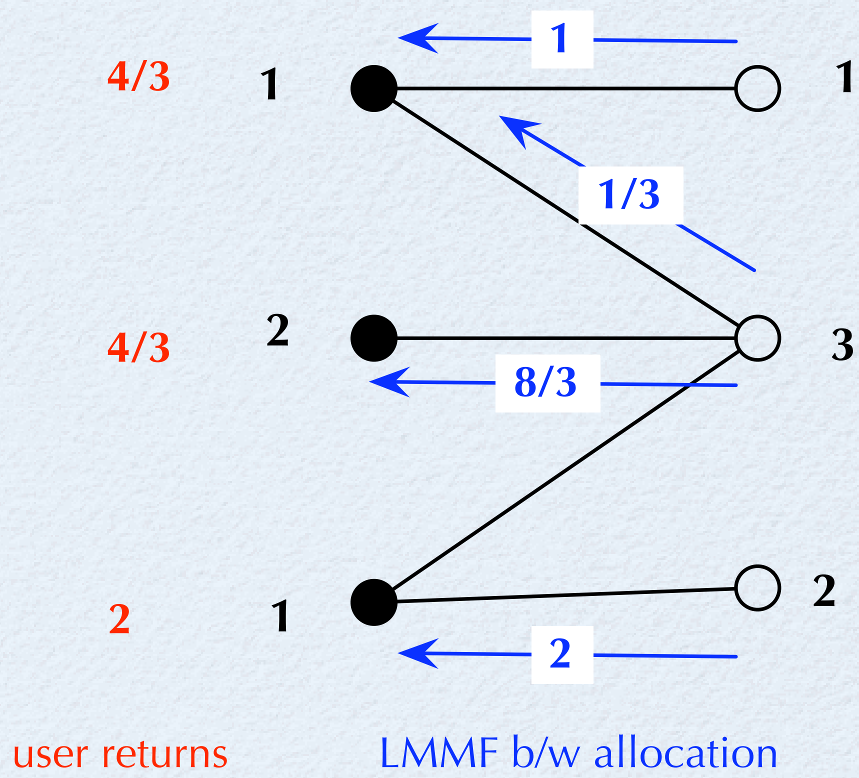
- Questions of Interest:
  - Existence of the NE *(Yes!* - shown by Zhang '05)
  - Uniqueness of the NE *(Remains open)*
  - Can the NE be expressed in closed form *(Appears unlikely)*
  - Is the NE "fair"? *(Yes, but only asymptotically!)*
- Fairness:
  - How do you measure fairness in this case?
  - It may not be possible to attain the same b_i^* / W_i across all users in this case (due to access constraints)
  - How about *(lexicographic) max-min fairness (LMMF)*?

LMMF

- Maximize the "user returns" b_i^* / W_i *lexicographically*
  - Maximize the minimum user return, then maximize the second minimum user return (subject to the minimum user return being the maximum), and so on
  - A user return cannot be increased further without decreasing a user return of equal or lesser value
  - "Ideal" fairness measure
- Can be considered as the *system optimum*
  - Property of the graph (users, BSs and accessibility constraints), the W_i and the B_j
  - Not related to the game (no notion of bids at all)
  - Can be computed by solving a sequence of flow problems

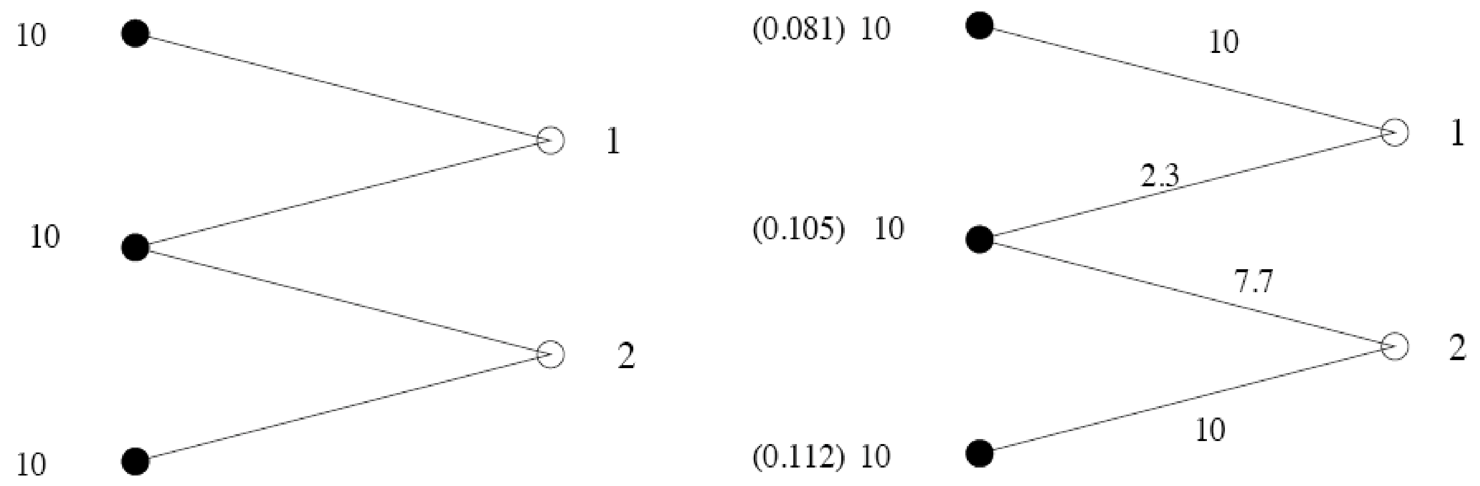
LMMF: Example





LMMF vs NE

- In general, NE may not be LMMF
    - NE can be very unfair in general
- Example:



(a) The accessibility graph (b) Nash Equilibrium

LMMF returns for this network is (0.1,0.1,0.1)

Asymptotic Case

- What happens if number of users grows w/o bound?
- Assumptions:
  - Number of users for *each* BS grows as $O(|N|)$, i.e., $aN <= \Gamma_j$ for some positive constant a
  - Bandwidths of BSs (B_j) grow as $O(N)$
  - User wealths (W_i) are positive and bounded
- LMMF vs NE:
  - User returns at NE are can be made *arbitrarily close* to their values at LMMF by making N sufficiently large

Asymptotic Fairness of NE

- Given any $\xi > 0,\ \exists n_0$ such that $\forall |N| > n_0$,

$$|R_i^* - R_i^{LMMF}| \leq \xi \quad \forall i$$

  - for all networks (graphs) $G(|N|)$ that satisfy the earlier (asymptotic) assumptions
  - R_i^* : return of user *i* at NE (b_i^*/W_i)
  - R_i^{LMMF} : return of user *i* optimality (LMMF)

Basic Intuition (1/2)

- If a user i can access two BSs, j_1 and j_2

$$\frac{\partial}{\partial w_{ij_1}}\left(\frac{w_{ij_1}B_{j_1}}{w_{ij_1}+\sum_{i'\in N\setminus i'\neq i} w_{i'j_1}}\right)=\frac{\partial}{\partial w_{ij_2}}\left(\frac{w_{ij_2}B_{j_2}}{w_{ij_2}+\sum_{i'\in N\setminus i'\neq i} w_{i'j_2}}\right)$$

$$\frac{B_{j_1}\sum_{i'\in I, i'\neq i} w_{i'j_1}}{(\sum_{i'\in I} w_{i'j_1})^2}=\frac{B_{j_2}\sum_{i'\in I, i'\neq i} w_{i'j_2}}{(\sum_{i'\in I} w_{i'j_2})^2}$$

$$\frac{B_{j_1}\sum_{i'\in I} w_{i'j_1}-w_{ij_1}}{(\sum_{i'\in I} w_{i'j_1})^2}=\frac{B_{j_2}\sum_{i'\in I} w_{i'j_2}-w_{ij_2}}{(\sum_{i'\in I} w_{i'j_2})^2}$$

For large $|N|$: $$\frac{B_{j_1}}{\sum_{i'\in I} w_{i'j_1}}\approx\frac{B_{j_2}}{\sum_{i'\in I} w_{i'j_2}}\approx\alpha \text{ (say)}$$

Basic Intuition (2/2)

$$
\begin{aligned}
b_i^* &= \sum_{j:i\in\Gamma_j} w_{ij} \frac{B_j}{\sum_{i'\in I} w_{i'j}} \\
&\approx \alpha \sum_{j:i\in\Gamma_j} w_{ij} \\
&= \alpha W_i
\end{aligned}
$$

Moral of the story: *User greed is good, as long as the providers are fair*

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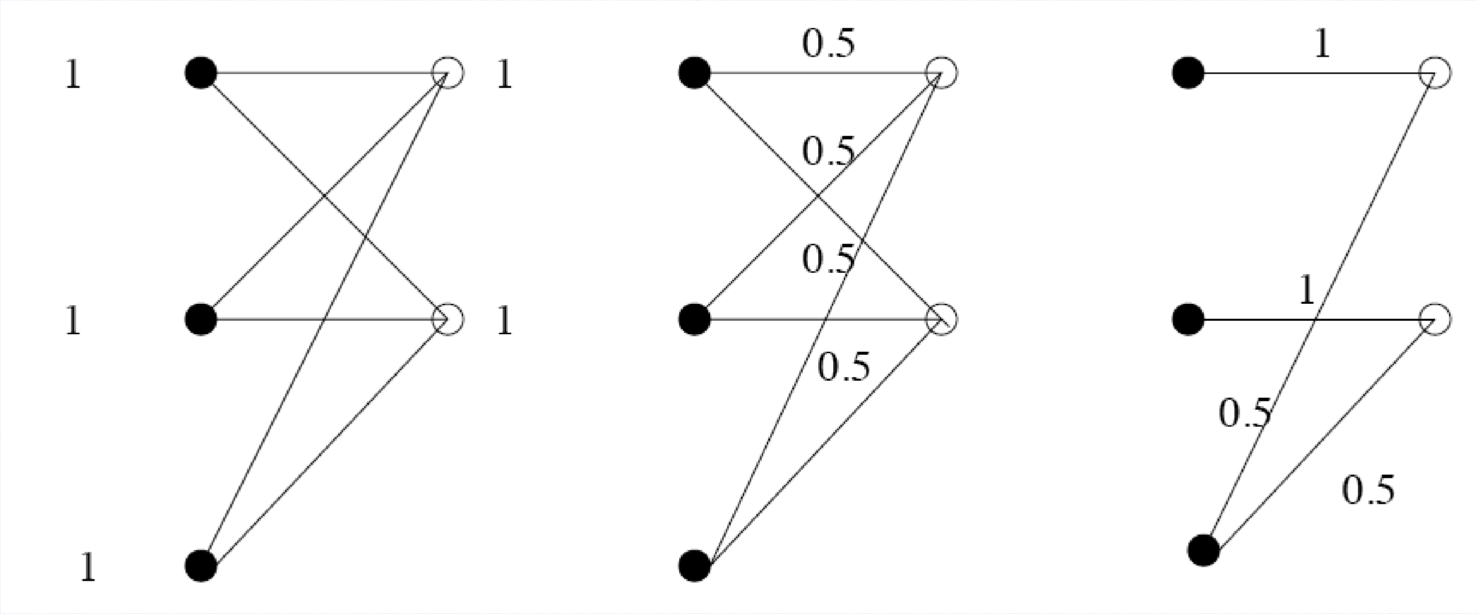
Open Issues

- With access constraints, is the NE unique?
- Note that LMMF b/w allocation may not be unique
  - The bids that result in LMMF (assuming fair BSs) may also be non-unique



Thank You!