Neutrino Mixing

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I pledge that this paper represents my own work in accordance with University regulations.
1 Introduction

The search for the elusive neutrino began, in an intellectual sense, with Pauli’s prediction of a neutral particle to spirit away the missing energy in beta decay. Since the first direct observation in 1956, neutrino physics has evolved to focus largely on neutrino mass, now one of the only well-established observations to clearly go beyond Standard Model physics. Direct measurements of the mass have met with little success, serving only to set weak (or sometimes imaginary!) upper limits. A more stringent constraint comes from cosmology. The recent WMAP results set an upper limit on the contribution of neutrinos to the energy density of the universe, corresponding to a maximum total mass of all three neutrino flavors \[ m_{\nu_e} + m_{\nu_{\mu}} + m_{\nu_{\tau}} \lesssim 0.71 \text{ eV}. \] (1)

The three neutrino flavors are the partners of the charged leptons \( e, \mu, \) and \( \tau \) in the Standard Model. Neutrino oscillations between flavors, which require a non-zero mass, gave the first indication that the upper limit on total mass would not be pushed down indefinitely. Solar and atmospheric neutrino detectors have consistently observed fewer neutrinos than expected, hinting that they oscillate from one flavor to another during their journey to the detector.

Oscillation not only requires neutrino mass, it implies a rich set of parameters and phenomenological behavior. Numerous experiments have gradually constrained these parameters, zooming in on the correct oscillation behavior, only to find that eventually every corner of parameter space had been eliminated. Explaining all of the experiments requires new physics more drastic than introducing neutrino mass. The most contested experimental result comes from the Liquid Scintillator Neutrino Detector (LSND), so far unconfirmed by other experiments. In this paper, we examine the implications of several widely proposed remedies to the LSND problem. If LSND is rejected, then next on the list of many neutrino physicists is the search for CP violation; if not, then even more exciting physics is on the horizon, perhaps including more than three neutrinos or violation of the CPT Theorem.

2 Neutrino Oscillations

Neutrinos exist in (at least) three “flavors,” \( \nu_e, \nu_\mu \) and \( \nu_\tau \), associated with the three known charged leptons, \( e, \mu, \) and \( \tau \). In weak interactions, each charged lepton is coupled to the neutrino of the corresponding flavor. If neutrinos are massive, then there is an alternative basis \( |\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle \) of “mass eigenstates” of the vacuum Hamiltonian with masses \( m_1, m_2, \) and \( m_3 \). The flavor states describe neutrinos seen in experiments (through weak interactions), while the mass eigenstates provide the fundamental time evolution of neutrinos in vacuum. There is no reason to assume that the mass eigenstates are identical to the flavors, but both sets of states span the same space so we can relate them by a unitary matrix \( U \).

Then the neutrinos of flavor \( \alpha = e, \mu, \tau \) are related to the mass basis by

\[
|\nu_\alpha\rangle = \sum_i U_{\alpha i}^* |\nu_i\rangle.
\] (2)

For the two-neutrino case, the mixing matrix \( U \) is parametrized by

\[
U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\] (3)

and in the three-neutrino case, a standard form is

\[
U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} \sin \delta & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.
\] (4)
where \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \).

With this formalism in place, neutrino oscillation is a straightforward result of quantum mechanics. In the laboratory frame, eigenstates of the Hamiltonian evolve according to

\[
|\nu_\alpha(t)\rangle = e^{-i(E_{\alpha} t - p_i L)} |\nu_\alpha(0)\rangle
\]

where \( E_i \) is the neutrino energy and \( L \) is the distance travelled. Assuming the neutrino moves near the speed of light, it travels a distance \( L \approx t \) and has energy \( E = \sqrt{p^2 + m_i^2} \approx p + m_i^2/2p \). Therefore the evolution of a definite flavor is

\[
|\nu_\alpha(L)\rangle \approx \sum_i e^{-i m_i^2 L/2E} |\nu_i\rangle .
\]

This solves the time evolution of a neutrino flavor, but what we really want for practical computations is the flavor oscillation probability; if a neutrino of flavor \( \alpha \) has energy \( E \), and we measure it after travelling a distance \( L \), what is the chance we will see the flavor \( \beta \)?

In the two-neutrino case, a newly created flavor state is

\[
|\nu_\alpha(0)\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle,
\]

so after travelling a distance \( L \), it becomes

\[
|\nu_\alpha(L)\rangle = e^{-i m_1^2 L/2E} \cos \theta |\nu_1\rangle + e^{-i m_2^2 L/2E} \sin \theta |\nu_2\rangle.
\]

After inverting the mixing matrix \( U \) to rewrite the right-hand side in terms of the flavor states, we find the transition probability \( |\langle \nu_\beta | \nu_\alpha(L) \rangle|^2 \) for \( \alpha \neq \beta \),

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \sin^2 2\theta \sin^2 (1.27 \Delta m^2_{12} L/E),
\]

where \( \Delta m^2_{12} \equiv m_1^2 - m_2^2 \). The numerical factor 1.27 comes from converting to practical units, with \( L \) in km, \( E \) in GeV, and \( \Delta m^2 \) in eV^2. With three neutrino flavors the derivation is identical but more tedious, yielding [2]

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha \beta} - 4 \sum_{i>j} \Re \left( U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \right) \sin^2 (1.27 \Delta m^2_{ij} L/E)
\]

\[
+ 2 \sum_{i>j} \Im \left( U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \right) \sin (2.54 \Delta m^2_{ij} L/E).
\]

In the limit where one mass-squared difference is much greater than another, the three neutrino oscillation probability reduces to Eq. (6) with an effective \( \sin^2 2\theta \) (and is therefore dubbed quasi-two neutrino oscillations). This lets us refer to “the” \( \Delta m^2 \) or mixing angle of particular experiments which fall in this limit, even though the exact solution must involve two independent mass-squared differences and three mixing angles.

We will also want to calculate oscillation probabilities for antineutrinos, following ref. [2]. According to the CPT Theorem, which relies only on very basic assumptions (but will be questioned in Section 8), the laws of physics do not change if we reverse the flow of time and substitute antineutrinos for neutrinos, so

\[
P_{\nu_\alpha \rightarrow \bar{\nu}_\beta} = P_{\nu_\beta \rightarrow \nu_\alpha} \quad \text{(With CPT)}.
\]

From the explicit probability (7), we also know

\[
P_{\bar{\nu}_\beta \rightarrow \nu_\alpha} (U) = P_{\nu_\alpha \rightarrow \nu_\beta} (U^*) \quad \text{(Always)}.
\]

Therefore

\[
P_{\bar{\nu}_\beta \rightarrow \nu_\alpha}(U) = P_{\nu_\alpha \rightarrow \nu_\beta}(U^*) \quad \text{(With CPT)}.
\]

This means we can calculate antineutrino oscillation probabilities by replacing \( U \rightarrow U^* \). If \( U \) is complex, then neutrinos and antineutrinos behave differently. The symmetry CP relates left-handed particles to right-handed antiparticles without reversing the flow of time. Since all neutrinos are left-handed and antineutrinos are right-handed, a complex mixing matrix \( U \) therefore breaks CP symmetry. This possibility is considered further in Section 7.
3 Experimental Results

Solar Neutrinos Neutrinos ($\nu_e$, specifically) are produced in the nuclear reactions that generate power inside the Sun. Neutrino oscillations were first suggested by underground solar neutrino detectors which did not see the predicted number of solar $\nu_e$. The Homestake chlorine experiment found roughly 1/3 the expected flux of solar $\nu_e$, and three gallium detectors (Gallex, GNO and SAGE), sensitive to lower-energy solar neutrinos, found about 1/2 the expected flux. The Kamiokande and Super-Kamiokande (Super-K) experiments corroborated the lack of $\nu_e$. More recently, the Sudbury Neutrino Observatory (SNO) found $\nu_e$ coming from the Sun, providing the first “appearance” observation of solar neutrino oscillations.

A number of solutions for $\Delta m^2_{\text{sun}}$ and $\theta_{\text{sun}}$ can reproduce these results in a quasi-two neutrino oscillation scenario [2]. For reasons discussed later, one solution in particular, the matter-enhanced large mixing angle (LMA) solution, is highly favored by the recent KamLAND observation of reactor $\bar{\nu}_e$ disappearance (with CPT conservation, this behaves identically to $\nu_e$ disappearance, $P_{\nu_e \rightarrow \nu_e} = P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}$). The best-fit LMA solution corresponds to $\Delta m^2_{\text{sun}} \approx 7.0 \times 10^{-5} \text{ eV}^2$, $\tan^2 \theta_{\text{sun}} = 0.79$ [3, 4].

Atmospheric Neutrinos Super-K also observed the flux of $\nu_\mu$, $\bar{\nu}_\mu$, $\nu_e$, and $\bar{\nu}_e$ produced in the decay of particles produced by cosmic ray collisions with the atmosphere. Without oscillations, conserving the total number of neutrinos of each flavor requires the flux of upward-going $\nu_\mu$ to match that of downward-going $\nu_\mu$. However, the upward flux is much lower (almost 50%) than the downward flux, so it seems that atmospheric $\nu_\mu$ oscillate into some other flavor while travelling through the Earth. Taking into account the energy distribution and limits from reactor experiments such as CHOOZ (which did not see any oscillations in $\bar{\nu}_e$, thereby cutting large regions out of the allowed parameter space), atmospheric oscillations are best explained by quasi-two neutrino $\nu_\mu \rightarrow \nu_\tau$ oscillations with $\Delta m^2_{\text{atm}} = (2.3 \pm 0.7) \times 10^{-3} \text{ eV}^2$, $\sin^2 2\theta_{\text{atm}} > 0.92$ [2]. These parameters have recently been corroborated by the accelerator-based K2K experiment, which observes oscillations of a man-made $\nu_\mu$ beam over 250 km (but so far has very low statistics) [5].

“Man-made” Neutrinos Many accelerator- and reactor-based experiments have not found any evidence for neutrino oscillation (notably CHOOZ, CDHS, and Bugey). Combining these with solar and atmospheric experiments, the mixing matrix $U$ is fairly well-constrained, and the remaining task is to narrow down the confidence regions. If there are just three neutrinos, then the two mass-squared differences from solar and atmospheric oscillations fix the relative locations of the mass eigenstates. Given $\Delta m^2_{\text{atm}} \gg \Delta m^2_{\text{sun}}$, we would infer the third mass-squared difference $\Delta m^2_3 \approx \Delta m^2_{\text{atm}}$. Our job is not over, however, because this is not the splitting observed by the “short-baseline” LSND accelerator experiment, where a $\bar{\nu}_\mu$ beam hits a detector just 30 m away. LSND reports a small fraction ($\sim 0.264\%$) of $\bar{\nu}_e$ coming from the $\bar{\nu}_\mu$ source [6]. We can determine the mixing parameters by solving the vacuum oscillation formula (6),

$$P_{\nu_\mu \rightarrow \nu_e} = \sin^2(2\theta_{LSND}) \sin^2(1.27 \Delta m^2_{LSND} \frac{L}{E}) = .00264$$

with distance $L = 30$ m and a known energy distribution from 20 – 60 MeV. This gives the characteristic best-fit curve shown in Fig. 1. Combining this with limits set by reactor experiments seeing no oscillations gives $1 \gtrsim \Delta m^2_3 \lesssim 10 \text{ eV}^2$, where the lower end of the mass range is preferred by the cosmological limit (1).
$\sin^2(2\theta)$

4 Neutrinos in Matter

We will look at several possible solutions to the LSND problem, including adding more neutrinos, abandoning LSND completely, and abandoning the CPT Theorem. Rather than simulating every experiment in full detail for all three cases, we content ourselves to understand (and accept) the solution to the solar neutrino problem, using it to constrain the mixing parameters under each scenario.

In the previous section, we glossed over the details of how the actual experimental data (in terms of oscillation probabilities at various energies) is translated into mass differences and mixing angles. For reactor and accelerator experiments, the analysis is, to simplify it greatly, a matter of finding the best-fit curve (as in Fig. 1) and then choosing the points that match the observed energy distribution. For solar and atmospheric neutrinos, the task is made much more daunting by matter effects inside the Sun and Earth.

Supposing for simplicity that there are just two neutrinos, $\nu_e$ and $\nu_\mu$, the Hamiltonian of a neutrino travelling through vacuum is, directly from the mixing matrix,

$$H_{\text{vac}} = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

in the flavor basis. In matter, however, interactions with neighboring particles change the effective mass of the neutrinos via the MSW effect, named after Wolfenstein, who first considered it in 1978, and Mikheyev and Smirnov, who discovered some of its most interesting effects in 1985 [7]. The correction depends on how the neutrino interacts with matter: In ordinary matter, where there are many electrons but no other charged leptons, $\nu_\mu$ and $\nu_\tau$ have small corrections from neutral current interactions, and $\nu_e$ has a larger correction from both neutral current and charged current interactions. The neutral current correction, common to all three flavors, induces an insignificant phase shift, while the charged current correction changes the Hamiltonian to

$$H = H_{\text{vac}} + \begin{pmatrix} \pm \sqrt{2}G_F N_e(r) & 0 \\ 0 & 0 \end{pmatrix}$$

(8)
where $G_F$ is the Fermi constant and $N_e(r)$ is the number density of electrons at location $r$. The positive sign is taken for matter, and the negative sign for antimatter. The eigenstates of the full Hamiltonian are labelled the “light” ($\nu_L$) and “heavy” ($\nu_H$) states. The time evolution of this Hamiltonian is solved analytically for certain electron densities in the Appendix, but the most important features can be understood with an appeal to the adiabatic theorem. When neutrinos are produced in the Sun, $N_e$ is much larger than the elements of $H_{vac}$, so $H$ is effectively diagonal. In other words, $\nu_e$ and $\nu_\mu$ are the “local mass eigenstates,” and $m_{\nu_e} \gg m_{\nu_\mu}$. According to the adiabatic theorem, if external conditions (that is, $N_e(r)$) change gradually enough, then a particle in an eigenstate of $H(t=0)$ will evolve into an eigenstate of the new Hamiltonian $H(t=t_0)$. A neutrino produced as a $\nu_e$, the heavier mass eigenstate inside the Sun, must therefore exit the Sun as the heavier vacuum eigenstate, $\nu_e \rightarrow \nu_H$. If the heavy vacuum eigenstate happens to be $\nu_H \sim \nu_\mu$, then we have complete $\nu_e \rightarrow \nu_\mu$ conversion.

We do not actually observe complete conversion, because the flavor states are not eigenstates, and the evolution of neutrinos travelling through the Sun is not quite adiabatic. There is a chance that a neutrino will “hop” non-adiabatically from one eigenstate to another, explaining the mix of $\nu_e$ and other flavors seen in real experiments. Combining the requirements of near-adiabatic conversion from the Appendix, limits from reactor experiments, and a fast computer, several solutions to the solar oscillation parameters could be found until recently.

The KamLAND experiment, observing reactor $\bar{\nu}_e$ disappearance over $\sim 180$ km, has now eliminated all but the large mixing angle (LMA) solution, with $\Delta m_{\text{sun}}^2 = 7.0 \times 10^{-5}$ eV$^2$, $\tan^2 \theta_{\text{sun}} = 0.79$. To see how KamLAND eliminated the other candidates, consider as an example the small mixing angle solution allowed before KamLAND: $\Delta m_{\text{sun}}^2 \approx 6 \times 10^{-6}$ eV$^2$, $\sin^2 2\theta_{\text{sun}} \approx 6 \times 10^{-3}$. From the vacuum oscillation formula (6), the KamLAND fraction of disappearing $\bar{\nu}_e$ would be at most $6 \cdot 10^{-3}$. However, KamLAND sees $P_{\bar{\nu}_e \rightarrow \bar{\nu}_e} \approx 0.4$, definitively eliminating the small angle solution. A small mixing angle with large oscillations only makes sense when matter effects play a large role, causing a resonant flavor change much larger than the expected vacuum oscillation. KamLAND, where the MSW effect is negligible, ruled out small mixing and the handful of other solutions, leaving only LMA.

Beyond adding to the complexity of the solar neutrino problem, the MSW effect adds a valuable tool to the arsenal of neutrino physicists: It is the first effect we have seen that is sensitive to the sign of $\Delta m^2$, and can distinguish between the mixing angles $\theta$ vs. $\pi/2 - \theta$, illuminating the aptly named “dark side” of parameter space (this is why solar mixing angles are often quoted in terms of $\tan^2 \theta$ instead of $\sin^2 2\theta$). Matter effects are therefore crucial in determining the mass hierarchy of neutrinos.

## 5 A Closer Look at Vacuum Oscillations

With the LMA solution in hand, we need only to elaborate on how the three-neutrino oscillation formula (7) gives concrete experimental predictions. We will then be able to evaluate the consistency of various theories incorporating the LSND anomaly, using the techniques developed here to test their predictions.

“Vacuum experiments” (which do not occur in vacuum at all, but matter effects are small enough to ignore) can be simulated reasonably well with the oscillation formula (6) or (7) and integrations over the energy spectra and distances. The average oscillation probability is

$$P_{\nu_\alpha \rightarrow \nu_\beta}^{\text{ave}} = \sum_{i,j} P_{\nu_\alpha \rightarrow \nu_\beta}^{\text{simple}}(L_i, E_j) \frac{\Delta N_{ij}}{N}$$

where $P_{\nu_\alpha \rightarrow \nu_\beta}^{\text{simple}}(E, L)$ is the oscillation probability derived in Section 2 and $\Delta N_{ij}/N$ weights the average over the proper energy spectrum. The sum over $i$ is over neutrino path lengths,
which can be interpreted as a sum over different parts of the detector (as in LSND, where the 8 m detector is just 30 m from the source) or as a sum over different sources (as in KamLAND, which observes neutrinos from 16 different reactors from 81 to 824 km away). The sum over $j$ is over the initial energy spectrum of the neutrinos. For example, the K2K experiment observes $\nu_\mu \to \nu_\mu$ with $L = 250$ km. Because there is only one neutrino source and the detector size is negligible compared to 250 km, the oscillation probability simplifies to

$$P^{\text{ave}}_{K2K} = \sum_j P_{\bar{\nu}_\mu \to \bar{\nu}_\mu}(L = 250, E_j) \frac{\Delta N_j}{N} \quad (9)$$

We can recreate the K2K oscillation probability with the atmospheric splitting $\Delta m^2_{\text{atm}} = 2.8 \cdot 10^{-3}$, $\sin^2(2\theta_{\text{atm}}) = 1$ and an energy range of 0.5 – 5 GeV. With these mixing parameters, Eq. (9) gives $P^{\text{ave}}_{K2K} \approx 0.67$, compared to the observed oscillation $P_{K2K} \approx 0.62$. The distortion of the energy spectrum in this calculation (Fig. 2) gives reasonable agreement with the observed spectrum (although statistics for K2K are still very low, and the error bars on this plot would be quite large).

![Figure 2: The spectral distortion of the K2K neutrinos. The curve shows the simulated distortion assuming atmospheric mixing; the data points give the observed distortion, as taken from Fig. 2 of ref. [5] by comparing K2K data points to the predictions of the K2K collaboration. Survival probability data points are cut off at 1 to make the graph readable, justified by the low statistics.](image)

The oscillation probabilities for LSND and KamLAND are calculated in much the same way, but for KamLAND the distances to 16 different reactors must be included in the double sum. Although the size of the LSND detector (8 m) is comparable to the neutrino path length (30 m), effects on the far side of the detector cancel those on the close side, so a single $L$ is a suitable approximation.

6 Constructing a Mixing Matrix: Sterile Neutrinos

In a world with just three mass eigenstates and no other extravagant neutrino behavior (CPT violation, etc.), we could now try to patch together the experimental observations to put limits on the mixing matrix. However, three independent mass-squared differences, all of different orders of magnitude ($\Delta m^2_{\text{LSND}} \gg \Delta m^2_{\text{atm}} \gg \Delta m^2_{\text{sun}}$), cannot fit together no matter how we tweak the mixing matrix. We may as well try to draw a triangle with sides of length 1, 10, and 1000; it simply cannot be done.

Since it is impossible to squeeze all the observations into our 3$\nu$ model, suppose there are in fact four neutrinos. From the results of Z production at LEP, the fourth neutrino does not participate in weak interactions, so it is labelled “sterile,” as opposed to the regular
three “active” flavors.¹ Sterile neutrinos participate in oscillations but do not show up in a detector, as they are bystanders to every known interaction. It might seem that a fourth neutrino would allow us to match the three mass-squared differences with ease, but given a few extra details about the experiments, it remains nearly impossible to accommodate all of the observations.

There are two basic 4ν schemes allowing \( \Delta m^2_{\text{SND}} \gg \Delta m^2_{\text{atm}} \gg \Delta m^2_{\odot} \). We can put both small gaps on the same side of the large gap, in what is called a 3 + 1 scheme, or on different sides of the large gap, in a 2 + 2 scheme (Fig. 3). Theorists tend to prefer 2 + 2 schemes, but neither can satisfy the experimentalists, for several reasons described below [9, 10, 11].

![Diagram of 4ν schemes](source: [2])

**Experimental Limits** Some experiments can indirectly measure the flux of sterile neutrinos. Charged leptons, seen in a detector, are produced through charged current interactions, 

\[
\nu_l + N \to l + X ,
\]

where \( N \) is a nucleon. At solar neutrino energies, only the \( l = e \) interaction is important, and both \( l = e \) and \( l = \mu \) are important for atmospheric neutrinos. The high mass of the \( \tau \) renders the \( \nu_\tau \) interaction negligible below several GeV. With the charged current interaction, solar neutrino experiments measure the survival probability \( P_{\nu_e \leftrightarrow \nu_e} \). This probability says nothing about the final product of the oscillations: Is it \( \nu_e \to \nu_\mu \), \( \nu_e \to \nu_\tau \), or oscillation into the sterile neutrino, \( \nu_e \to \nu_s \)? We cannot discern between the first two possibilities, but the third possibility can be checked with the help of neutral current interactions,

\[
\nu + N \to \nu + N .
\]

The neutral current interaction occurs among all three active flavors with equal likelihood, so SNO can determine the total active neutrino flux by measuring this reaction. The same holds for atmospheric oscillations at Super-K, \( \nu_\mu \to \nu_\tau \). Sterile neutrinos, by definition, do not participate in either type of interaction. We can write the final product of each oscillation as a linear combination of active (\( \nu_a \)) and sterile components,

\[
\begin{align*}
|\nu_e \rangle & \to \cos \alpha^\text{sun} |\nu_a \rangle + \sin \alpha^\text{sun} |\nu_s \rangle \\
|\nu_\mu \rangle & \to \cos \alpha^\text{atm} |\nu_a \rangle + \sin \alpha^\text{atm} |\nu_s \rangle,
\end{align*}
\]

for solar and atmospheric oscillations respectively. The SNO neutral current measurement of the active flux \( \phi_{NC} \) implies the sterile flux²

\[
\phi_{\text{sterile}} \approx \phi_{\text{theory}} - \phi_{NC} ,
\]

¹The cross section of \( Z \) production from \( e^+e^- \) collisions depends on the number of possible final states, which in turn depends on the number of non-sterile neutrinos. The observed cross section sets the number of active neutrinos at 3.00 ± 0.08 [8].

²The relation here is approximate because a rigorous treatment requires a definition more suitable to statistical analysis, where the sterile fraction is defined relative to the total flux, which is then defined relative to the theoretical flux, as in [3].
or, relating the parameter $\alpha^{\text{sun}}$ to observed quantities,

$$\phi_{\text{sterile}} = \phi_{\text{total}}(1 - P_{\nu_e \rightarrow \nu_e}(\alpha^{\text{sun}})) \sin^2 \alpha^{\text{sun}}$$

where $P_{\nu_e \rightarrow \nu_e}(\alpha^{\text{sun}})$ is averaged over the initial energy spectrum. Since SNO has found $\phi_{NC} \approx \phi_{\text{theory}}$, and $P_{\nu_e \rightarrow \nu_e} \neq 1$, then presuming $\phi_{\text{total}} \approx \phi_{\text{theory}}$, the sterile fraction $\sin^2 \alpha^{\text{sun}}$ of the oscillation result must be low. More precisely, the $1\sigma$ bound from [3] on the fraction of sterile neutrinos involved in solar oscillations is

$$\sin^2 \alpha^{\text{sun}} \lesssim 0.13.$$  \hfill (11)

Exactly the same argument carries over to atmospheric experiments, but the initial neutrinos are both $\nu_\mu$ and $\nu_\tau$. The angular distribution also plays a role. Super-K measures $\nu_e$ and $\nu_\mu$ flux separately, and finds that the $\nu_e$ flux matches predictions (without oscillation), while only the $\nu_\mu$ flux falls short. This leaves oscillations of $\nu_\mu \rightarrow \nu_\tau$ or $\nu_\mu \rightarrow \nu_\tau$. In the case of $\nu_\mu \rightarrow \nu_\tau$, the number of upward-going neutral current events should be less than the number of downward-going events. However, the data supports the $\nu_\mu \rightarrow \nu_\tau$ hypothesis in which the neutral current flux is isotropic [12]. All told, the maximum sterile component of atmospheric oscillations is [9]

$$\sin^2 \alpha^{\text{atm}} \lesssim 0.34.$$  \hfill (12)

**Restricting 4ν Schemes** Consider a $2 + 2$ scheme, with the solar and atmospheric oscillations essentially decoupled by the large $\Delta m^2_{\text{LSND}}$ splitting. Then the sterile component of the solar oscillation pair is separate from that of the atmospheric pair, and, since all four mass eigenstates are included in one pair or the other,

$$\sin^2 \alpha^{\text{sun}} + \sin^2 \alpha^{\text{atm}} = 1 \quad (2 + 2 \text{ Schemes}).$$  \hfill (13)

From SNO and Super-K, however, we know both pairs contain mostly active neutrinos. The limits (11) and (12) imply $\sin^2 \alpha^{\text{sun}} + \sin^2 \alpha^{\text{atm}} \lesssim 0.47$, leaving no room for a sterile neutrino in a $2 + 2$ scheme. This argument is completely independent of the LSND results.

A $3 + 1$ scheme leaves a bit more maneuvering room because regular 3ν oscillations are a limiting case (where the fourth neutrino is sterile and completely decoupled). So this scheme can certainly accommodate the solar and atmospheric data. However, it is still very difficult to fit the LSND accelerator data. We can no longer use the unitarity relation (13), because now the solar and atmospheric oscillations pairs share a common mass eigenstate. Sterile neutrinos involved in solar oscillations may also take part in atmospheric oscillations, and (13) becomes a useless inequality.

The strong, but not indisputable, argument against $3 + 1$ schemes comes instead from trying to reconcile LSND with other short-baseline (SBL) experiments, including CHOOZ, Bugey, and CDHS, which have not seen any oscillations [11]. The 2ν oscillation probability (ignoring possible CP violation) is

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{i>j} U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2(1.27 \Delta m^2_{ij} L_E)$$

The amplitude of LSND oscillations (in the sense of a sinusoidal amplitude, not the quantum amplitude), if the isolated state $\nu_4$ has approximately the same large mass difference $\Delta m^2_{4\nu} \approx \Delta m^2_{\text{LSND}}$ with respect to all three other states, is then

$$A_{\nu_\mu \rightarrow \nu_e} = -4 U_{e4} U_{\mu 4} \sum_{j \neq 4} U_{e j} U_{\mu j} = 4 T_{e4} T_{\mu 4},$$  \hfill (14)
where \( T_{\alpha j} = |U_{\alpha j}|^2 \) is the fraction of flavor \( \alpha \) contained in mass eigenstate \( j \), and we used the unitarity relation
\[
\sum_j U_{\alpha j} U_{\mu j} = 0.
\]
The disappearance signal measured by other reactor and accelerator experiments (which did not see oscillation) has amplitude
\[
A_{\nu_\alpha \rightarrow \nu_\alpha} = 4 T_{\alpha 4} (1 - T_{\alpha 4})
\]
where \( \alpha = e \) for Bugey and CHOOZ, and \( \alpha = \mu \) for CDHS. To fit the no-oscillation experiments, this amplitude is small, so \( T_{\mu 4} \) and \( T_{e 4} \) are very small or very close to one. Solar oscillations within the \( \nu_1 \leftrightarrow \nu_2 \) pair “use up” most of the e flavor, so \( T_{e 4} \) cannot be large, and Super-K puts a similar upper limit on \( T_{\mu 4} \). Therefore both of these quantities are very small, pushing the LSND oscillation magnitude (14) below the observed signal.

This is the basic argument against 3+1 schemes, but we can examine it more quantitatively. In the range \( 1 \lesssim \Delta m^2_{\text{LSND}} \lesssim 10 \text{ eV}^2 \), both CDHS and Bugey give oscillation bounds of order \( A_{\nu_\alpha \rightarrow \nu_\alpha} \lesssim 0.1 \), or \( T_{\alpha 4} \lesssim 0.025 \). The least stringent bound on the LSND amplitude \( A_{\nu_\alpha \rightarrow \nu_\alpha} = 4 T_{e 4} T_{\mu 4} \) occurs at \( \Delta m^2_{\text{LSND}} = 0.86 \text{ eV}^2 \) [1], where the wiggles in the allowed regions of both experiments happen to match up (where ‘wiggles’ refers to the oscillating vertical part of the allowed region; compare Fig. 1). At a 99% confidence level (CL), the Bugey and CDHS bounds allow, respectively,
\[
T_{e 4} \approx 0.02, \quad T_{\mu 4} \approx 0.03.
\]
Plugging these mixing parameters into the LSND \( \bar{\nu}_e \) appearance probability formula gives the marginally allowed solution shown in Fig. 4. The lone point is the LSND oscillation probability (with error bars), the dashed line is the oscillation probability using the parameters above, and the solid line shows the result of taking 90% CL bounds. LSND can only be accommodated at the 99% CL. Although we cannot yet reject 3+1 sterile neutrino schemes, this last corner of parameter space is certainly not promising. We rejected 2+2 schemes using only the limits on sterile neutrinos in SNO and Super-K, but severely limiting 3+1 schemes has required all of the major experimental players: SNO (to limit \( T_{e 4} \) above), Super-K (to limit \( T_{\mu 4} \) above), LSND (to limit \( T_{e 4} T_{\mu 4} \) below), and the no-oscillation experiments (to make the upper limits much tighter).

**More Sterile Neutrinos?** Not surprisingly, it becomes easier to fit the data as more sterile neutrinos are added to the mix. Some of the same arguments against fitting LSND still apply, but with more neutrino species it becomes easier to push the theoretical limits into the observed region. In a 3 + 2 scheme, where the isolated pair is largely sterile, the 90% CL bounds push well into the LSND allowed region [13]. Only time (and perhaps MiniBooNE) will tell just how much further the sterile neutrino argument should be pursued.

## 7 CP Violation Without LSND

Before moving on to the more radical suggestion that CPT may be violated, we consider a particularly interesting aspect of one of the most common approaches to resolving the LSND anomaly: abandoning it altogether. If LSND is not verified by MiniBooNE running in antineutrino mode, then it will be rejected and sterile neutrinos or CPT violation will no longer be necessary to build a consistent theory. Suppose for now that LSND is incorrect, CPT is upheld, and neutrino mixing is set by the solar and atmospheric oscillations, with \( \Delta m^2_{\text{sun}} \approx 7.0 \times 10^{-5} \text{ eV}^2 \), \( \Delta m^2_{\text{atm}} \approx 2.3 \times 10^{-3} \text{ eV}^2 \) and large mixing angles.
Figure 4: The best fit to LSND in a 3+1 scheme. The data point is the LSND oscillation probability, the dashed line shows possible oscillation with 99% CL bounds from Bugey and CDHS, and the solid line shows oscillation with 90% CL bounds. In 3+1 schemes, LSND can only be accommodated with 99% CL bounds in a small region around $\Delta m^2_{\text{LSND}} \approx 0.86 \text{ eV}^2$. This figure shows the marginal solution schematically, but should not be taken too seriously by itself because it ignores the important spectral distortion of the LSND neutrinos.

The mass splitting question may be resolved, but this is by no means the end of the story. A natural question is whether CP is violated by neutrinos. That is, we would like to know if

$$P_{\nu_\alpha \rightarrow \nu_\beta} = P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta} \quad (?)$$

From Section 2, this is equivalent to the question of whether the mixing matrix $U$ is necessarily complex, or whether the CP violating phase $\delta$ in (4) is non-zero. There is no known reason to assume $\delta = 0$, but there is one other prerequisite for CP violation to appear in oscillations: The two-neutrino oscillation probability (6) does not depend on $\delta$, so apparently CP violation can only be observed as a three-family effect. If one of the mass eigenstates contains only two of the three flavors, then all oscillations can be written as $2\nu$ oscillation and no CP violation will be observed. (There are, however, other CP violating phases in neutrino mixing that do not affect oscillations but may influence other processes [2].)

On the other hand, if all neutrino flavors are distributed among all three mass eigenstates, then future experiments may confirm CP violation (unless nature happens to have chosen $\delta = 0$). Solar oscillations (connecting mass states 1 and 2) and atmospheric oscillations (between states 2 and 3) both have large mixing, so the existence of three-family effects hinges on the fraction $T_{e3}$ of the flavor $\nu_e$ in eigenstate $\nu_3$, where in the standard parametrization (7)

$$T_{e3} = |U_{e3}|^2 = \sin^2(\theta_{13}).$$

Reactor experiments seeing no oscillation find

$$P_{\nu_e \rightarrow \nu_e} = 1 - 2T_{e3}(1 - T_{e3}) \approx 1,$$

which, combined with solar oscillations, sets

$$T_{e3} \lesssim 0.03.$$

Super-K, observing both atmospheric $\nu_\mu$ and $\bar{\nu}_\mu$ (although unable to distinguish between the two), sets a similar upper limit. The upcoming MINOS accelerator experiment will
observe $\nu_\mu$ and $\bar{\nu}_\mu$ separately, thereby probing deep enough to test parameter space down to roughly $T_{e3} \approx 0.01$ (Fig. 5).

A likely candidate to probe CP violation beyond the MINOS limit is an as yet unplanned very long baseline accelerator experiment [15]. With a neutrino path length of $\sim 2540$ km and energies from 1 – 6 GeV, multiple peaks would appear in the $\nu_\mu \rightarrow \nu_e$ oscillation signal. The magnitude and shape of the oscillations could be measured accurately enough to explore $T_{e3}$ to another order of magnitude.

The oscillation probabilities for such an experiment under various scenarios are plotted in Fig. 6 using the techniques of Section 5. Matter effects are handled numerically as described in the Appendix, assuming a uniform matter density in the Earth, $\rho = 3$ g/cm$^3$. We take the best-fit solar and atmospheric solutions and set $T_{e3} = 0.05$ (except where noted), a factor of two below the expected limit from MINOS. We consider the same two mass hierarchies as in ref. [16]: The atmospheric pair may lie above or below the solar pair. The first case corresponds to $m_1 < m_2 < m_3$, and is termed “natural ordering” (NO), and the second case is $m_2 > m_1 > m_3$, or “reverse ordering” (RO). The antineutrino NO scheme behaves like the neutrino RO scheme.$^3$

These plots highlight some key issues for a very long baseline experiment looking for CP violation. First, Fig. 6a shows that matter effects are certainly not negligible. At $E \approx 4 – 6$ GeV, the MSW effect moves the peaks, doubles the oscillation signal for NO neutrinos and suppresses it by $\sim 70\%$ for RO neutrinos. In Figs. 6b-c, several example values of $T_{e3}$ are shown with $\delta = 0$ (including matter effects). At high energies, the oscillation probability is responsive to both $T_{e3}$ and the mass hierarchy. However, some of the response to $T_{e3}$ is lost if the mass ordering turns out to be RO.

The main benefit of an experiment with a wide energy band comes from Figs. 6d-e showing CP violation for sample values of $\delta$.$^4$ The low energy region (1-3 GeV) is highly responsive to $\delta$, allowing this experiment to measure $\delta$, $T_{e3}$, and the mass ordering all at

$^3$Actually, the NO antineutrino scheme behaves exactly like the neutrino scheme $m_1 > m_2 > m_3$ which we do not consider, but this is qualitatively similar to the RO scheme. In general, neutrinos travelling through matter behave like antineutrinos in matter with the inverse mass hierarchy.

$^4$Our sign convention for $\delta$ from (4) is opposite that of ref. [16].
Figure 6: Oscillation response of a very long baseline (L = 2540 km) experiment with various parameters. NO denotes $m_1 < m_2 < m_3$ and RO denotes $m_2 > m_1 > m_3$. Matter effects are included numerically. See text for a description of each figure.
once by comparing the effects in different areas of the energy spectrum. A major advantage of a long-baseline experiment is that it can observe CP violation without an antineutrino beam. The obvious way to test CP violation would be to compare antineutrino oscillations to neutrino oscillations, but it is notoriously more difficult to collect adequate data on antineutrinos due to lower production rates and interaction cross sections.

8 CPT Violation

After a glimpse of neutrino physics without LSND, we now return to the less pessimistic view that what is needed is not to reject LSND, but to find new physics. We have seen that regular 3ν schemes cannot even begin to match all of the data, and even 4ν schemes give only a marginal fit. A third, more drastic proposal to remedy the problem of three independent mass-squared differences is to abandon CPT invariance. That is, we will allow $P_{\nu_\alpha \to \nu_\beta} \neq P_{\bar{\nu}_\beta \to \bar{\nu}_\alpha}$ and we will no longer require the mass-splits to be identical among neutrinos and antineutrinos. This is a big leap, because the CPT Theorem of quantum field theory relies only on some very basic assumptions: Lorentz invariance, locality, and Hermicity of the Hamiltonian. However, there is no experimental basis to assume CPT invariance at such low energy scales, and confirming its violation would certainly lead to interesting new physics.

With the freedom to choose masses for neutrinos and antineutrinos separately, there are numerous ways to set the masses in agreement with observed splits $\Delta m^2_{\text{sun}}$ (neutrinos), $\Delta m^2_{\text{LSND}}$ (antineutrinos), $\Delta m^2_{\text{KamLAND}} \approx \Delta m^2_{\text{sun}}$ (antineutrinos), and $\Delta m^2_{\text{atm}}$ (both types, but dominantly neutrinos). Before KamLAND, it was easy to find mixing matrices $U$ and $\bar{U}$ (different now for neutrinos and antineutrinos) to satisfy all of the experiments. Rather than trying to grasp four independent mass differences and six mixing angles, we can describe a CPT violating scheme using the bar graph diagrams of Fig. 7. The shading shows the relative fractions $T_{\alpha i} = |U_{\alpha i}|^2$ of each flavor $\nu_\alpha$ contained in each mass eigenstate $\nu_i$. Flavors co-occupying large portions of the same mass eigenstate have large mixing.

![Figure 7](image-url)

Figure 7: (a) A possible CPT violating scheme before KamLAND. This scheme is rejected because it predicts no oscillations at KamLAND. (b) A CPT violating scheme which fits KamLAND and all previous experiments. Here the KamLAND result comes from a splitting in the antineutrino sector rather than LMA solution to solar oscillations. (Source: [17])

The predominant CPT violating scheme before KamLAND is shown in Fig. 7a [18]. The neutrino spectrum accounts for solar and atmospheric oscillations, while the antineutrino spectrum accounts for LSND and the remaining atmospheric oscillations. Based on this scheme, it was predicted that KamLAND would not see solar-like oscillations in antineutrinos [19].

Recently, the KamLAND collaboration reported antineutrino oscillations consistent with CPT invariance and the LMA solution of the solar neutrino problem [20]. The mixing scheme of Fig. 7a must therefore be rejected. There is, however, at least one other candidate CPT
violating scheme that is consistent with most experimental bounds. From the discussion of CP violation, we know \( T_{	ext{e3}} \) is close to one or close to zero. Setting it close to one gives the pre-KamLAND scheme of Fig. 7a. Setting it close to zero gives mixing matrices depicted in Fig. 7b. This scheme leaves the neutrino spectrum alone (in the LMA region), but ascribes the KamLAND result to a new mass-squared splitting in the antineutrino sector, misinterpreted as solar-like LMA oscillations [17].

The post-KamLAND spectrum in Fig. 7b is defined by

\[
\bar{\theta}_{13} = 0.08, \quad \bar{\theta}_{23} = 0.5, \quad \bar{\theta}_{12} = 0.6, \quad \Delta \bar{m}^2_{12} = 5 \times 10^{-4} \text{ eV}^2, \quad \Delta \bar{m}^2_{13} = 1 \text{ eV}^2. \tag{15}
\]

The calculation explained in Section 5 gives the oscillation probabilities \( P_{\text{KamLAND}} \approx 0.62 \) and \( P_{\text{LSND}} \approx 0.003 \), in good agreement with the observed probabilities of \( \sim 0.611 \) and \( 0.00264 \) respectively. At first glance, this spectrum does not appear friendly to the Super-K results. A water Cherenkov detector like Super-K cannot distinguish between neutrinos and antineutrinos, so one might suspect the antineutrino spectrum to require a mass splitting of order \( \Delta \bar{m}^2_{\text{atm}} \approx 2.3 \times 10^{-3} \text{ eV}^2 \), far below \( \Delta m^2_{\text{atm}} \). However, atmospheric antineutrinos are slightly less common than regular neutrinos and have a smaller interaction cross section, so it may be possible to fit the Super-K data even with a very large \( \Delta \bar{m}^2_{\text{atm}} \). Some authors have found that these parameters reproduce both the up-down asymmetry and the angular distribution observed at Super-K when the LMA solution is assumed in the neutrino sector [17]. However, another author has contested this claim, insisting that CPT violation is no longer a viable option to explain all of the experimental data and rejecting the same parameters at 5\( \sigma \) [21].

CPT Violation and Super-Kamiokande

Our own numerical simulations of Super-K also lean toward rejecting the CPT violating scheme (Fig. 7b), but not strongly. Including some effects ignored at least in the qualitative arguments of ref. [21], we find that the disagreement in the up-down asymmetry is only \( \sim 1.8\sigma \). We also find that the asymmetry is quite sensitive to the method of calculation, explaining the large discrepancy between previous results.

The simple vacuum oscillation formula does not apply to Super-K, where neutrinos and antineutrinos, coming in from all directions, travel various distances. The path length through the Earth of neutrinos produced over Manhattan hitting Super-K in Japan is long enough for matter effects to play a non-negligible role. We also have to include the experimental differences between neutrinos and antineutrinos (including different fluxes from the atmosphere, different cross sections in the detector, and an extra minus sign in the MSW effect). Furthermore, Super-K does not see neutrinos directly, but infers their flux and angular dependence based on charged leptons (e and \( \mu \)) produced in interactions in the detector. The Super-K data is reported as the zenith angle distributions of the charged leptons. Based on their characteristic Cherenkov rings, events are classified as ‘\( \mu \)-like’ or ‘e-like’, depending on which lepton is seen, and further separated into sub-GeV (\( E_l < 1.33 \text{ GeV} \)) and multi-GeV (\( E_l > 1.33 \text{ GeV} \)) energy bins. Although data is reported in just two energy bins and ten angular bins, all of our analysis is done with 40 angular bins and 50 energy bins. We do not consider the other Super-K data (upward-going muons).

In our calculation, the final quantity we seek is the number of charged leptons per zenith angle bin, normalized to no-oscillations,

\[
F_{\alpha} = \frac{N^l_{\alpha}}{N^l_{\alpha\text{-no-osc}}}, \tag{16}
\]

where \( N^l_{\alpha} \) is the number of leptons in the zenith angle bin (a function of \( U \) and the \( \Delta m^2 \)'s) and \( N^l_{\alpha\text{-no-osc}} \) is the same quantity but with each \( \Delta m^2 \) set to zero (as in the Standard Model).
Isolating the contribution from each original neutrino flavor,

\[
N^l_e = N^l_{\mu} + N^l_{\nu e}, \quad N^l_{\nu} = N^l_{\mu e} + N^l_{\mu\mu}, \\
N^l_{\bar{\nu}} = N^l_{\bar{\mu}e} + N^l_{\bar{\mu}\bar{\nu}}, \quad N^l_{\bar{\nu}} = N^l_{\bar{\mu}e} + N^l_{\bar{\mu}\bar{\nu}},
\]

where the number \(N^l_{\alpha\beta}\) of charged \(\alpha\) leptons coming originally from atmospheric \(\beta\) neutrinos is calculated from

\[
\frac{dN^l_{\alpha\beta}}{d\cos \theta_{\alpha}} = A \int \frac{dN^\nu_{\alpha\beta}}{d\cos \theta_{\alpha}} dE_{\nu} \epsilon(E_{\alpha}) \frac{d\sigma(E_{\nu}, \cos \theta_{\alpha})}{d\cos \theta_{\alpha}} dE_{\nu} dE_{\nu} d\cos \theta_{\nu}.
\]

Here \(A\) is an overall normalization depending on detector size and running time, \(E_{\nu}, \theta_{\alpha}, \theta_{\nu}\), and \(\epsilon_{\alpha}\) are the lepton and neutrino energies and zenith angles, and \(\epsilon(E_{\alpha})\) is the detector efficiency for a lepton \(\alpha\) of energy \(E_{\alpha}\). The quantity \(dN^\nu_{\alpha\beta}\) is the differential flux of \(\nu_{\alpha}\) actually arriving at the experiment, but that began as \(\nu_{\beta}\) cosmic rays.

We start with \(dN^\nu_{\alpha\beta}\), the only quantity that depends on neutrino oscillations. The differential neutrino flux at the location of the detector is

\[
\frac{dN^\nu_{\alpha\beta}}{d\cos \theta_{\alpha}} dE_{\nu} = 2\pi \left(\frac{d\Phi_{\beta}}{d\Omega_{\nu} dE_{\nu}}\right) P_{\nu_{\beta} \rightarrow \nu_{\alpha}}(E_{\nu}, \cos \theta_{\nu}, U).
\]

The oscillation probability \(P_{\nu_{\beta} \rightarrow \nu_{\alpha}}\) is computed with a Monte Carlo simulation. Various reactions contribute to the Super-K charged lepton flux, but we consider only quasi-elastic scattering, which can be experimentally distinguished (to some extent) from the other contributions to the cross section. Other interactions tend to produce multiple rings of Cherenkov light, so the quasi-elastic scattering events are “single-ring” events. (Charged current pion production, though ignored here, also contributes significantly to single-ring events.) In quasi-elastic scattering, a neutrino scatters off of a heavy nucleon to produce a nucleon and a charged lepton,

\[
\nu_e + N \rightarrow e + N', \\
\nu_\mu + N \rightarrow \mu + N'.
\]

The total cross sections for this process were taken from ref. [25]. For multi-GeV events, the total neutrino cross section is roughly double the antineutrino cross section. The angular dependence of the cross section controls the smoothing of the final data. Because we are mostly concerned with multi-GeV muons, we can roughly approximate the angular dependence of \(\nu_e\) and sub-GeV \(\nu_\mu\) events as neutrino-quark scattering. In the center of mass frame, the angular distribution is uniform for neutrino-quark scattering and proportional to \((1 + \cos \theta_{cm})^2\) for antineutrino-quark scattering.
For multi-GeV muon events, the degree of smoothing (i.e., the degree of forward-peaking in the differential cross section) significantly affects the up-down asymmetry, as demonstrated below. Neutrino-quark scattering is not a sufficient approximation, so we used the average quasi-elastic angular distribution for multi-GeV atmospheric neutrinos and antineutrinos in Fig. 8. These distributions, going well beyond the means of a simple program, were computed with a commercial Monte Carlo package including nuclear effects (these calculations were provided by another source; ref. [26]). The resulting angular dependence of the outgoing leptons is strongly forward-peaked compared to the neutrino-quark approximation.

Figure 8: Quasi-elastic CM frame angular distribution for atmospheric neutrino-nucleon scattering (left), and antineutrino-nucleon scattering (right). These distributions are averaged over the atmospheric neutrino flux with the cutoff $E_\nu > 1.4$ GeV. (Calculation source: [26])

After choosing the CM scattering angle, the lepton energy $E_{\alpha}^{CM}$ is determined by the incident neutrino energy $E_\nu$. A Lorentz boost into the lab frame where the nucleon is at rest gives the measured lepton energy $E_\alpha$ and scattering angle $\theta_\alpha^{rel}$.

Figure 9: Geometry of the incoming neutrino and outgoing lepton in quasi-elastic scattering.

The azimuthal angle $\phi$ is uniformly distributed, and after a bit of geometry (Fig. 9) the zenith angle of the charged lepton in the lab frame is

$$\cos \theta_\alpha = \cos \theta_\nu \cos \theta_{\alpha}^{rel} - (\cos \phi) \sin \theta_\nu \sin \theta_{\alpha}^{rel}.$$ 

The differential cross section is calculated by simulating a large number of incident neutrinos with energy $E_\nu$. The final lepton state should also be smeared by the detector resolution,
but Super-K reports energy resolution $\lesssim 3\%$ and angular resolution $\sim 4\%$, both of which are ignored in our calculation of the zenith angle distribution with just ten angular bins.

This fills in the last piece of the puzzle, giving the final charged lepton distribution. The up-down asymmetry, defined for multi-GeV muons, is

$$A_\mu = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow},$$

where $N_\uparrow$ includes $\mu$-like multi-GeV events with $-1 < \cos \theta < -.2$, and $N_\downarrow$ events have $-.2 < \cos \theta < 1$, and ‘horizontal’ events with $|\cos \theta| < 0.2$ are ignored. In the CPT conserving solution with maximal mixing, down-going $\nu_\mu$ do not travel far enough to oscillate, while up-going $\nu_\mu$ oscillate rapidly, averaging to $50\%$ survival probability. This predicts $A_\mu \approx 1/3$, although it is lessened by the smearing of the final leptons. The up-down asymmetry observed by Super-K is [$27$]

$$A_\mu = 0.288 \pm 0.030 \quad (1\sigma).$$

For the CPT conserving case (with the LMA solution for both neutrinos and antineutrinos), the calculation described above gives the zenith angle distributions in Fig. 10, compared to Super-K data. The corresponding up-down asymmetry for multi-GeV muons is

$$A_\mu = 0.30.$$

This is higher than the result $A_\mu = 0.28$ from ref. [21] (for which the calculation details are not available), but below the result $A_\mu = 0.32$ from ref. [17]. In ref. [17], the differential cross section is not included in the calculation as it is here. Instead, they assume $E_\nu \approx E_\alpha$, and smooth the angular distribution of $\nu_\mu$ by a Gaussian of width $15^\circ$ to find the distribution of muons. This method bypasses the detailed conversion of neutrino flux to lepton flux, simply smoothing the incoming neutrinos to mimic the outgoing leptons. Using Gaussian smoothing instead of the differential cross sections, we find for our calculation $A_\mu = 0.31$, in closer agreement.

Now turning to the CPT violating parameters from (15), we find, using the calculated cross sections,

$$A_\mu = 0.235,$$

deviating from the experimental up-down asymmetry by only $\sim 1.8\sigma$. The zenith angle distributions (Fig. 11) also show reasonable agreement, although we have not performed a detailed statistical test. The analyses from refs. [17] and [21] give $A_\mu = 0.27$ and $A_\mu = 0.21$, respectively.

Computed values of the up-down asymmetry are compared in Table 1. Note that the smoothing method, or the chosen angular distribution for neutrino-nucleon scattering, plays a key role in the asymmetry. Compared to the detailed angular distributions from Fig. 8, simple Gaussian smoothing with width $\sigma = 15^\circ$ is a good approximation. Neutrino-quark scattering gives poor agreement with the results of the more detailed analysis.

Although the fit of the CPT violating scheme is marginal, it might not be rejected entirely (at $5\sigma$) as found in ref. [21]. The qualitative argument leading to the $5\sigma$ result comes from identifying a CPT violating scenario with $\Delta m^2_{\text{atm}} \gg \Delta m^2_{\text{atm}}$, with a CPT conserving scenario, by introducing an effective mixing angle defined by

$$\sin^2 2\theta_{\text{eff}} = \frac{4 \sin^2 2\theta_{\text{atm}}}{6 - \sin^2 2\theta_{\text{atm}}}. \quad (19)$$

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Considering the limiting cases for $\theta_{atm}$ and $\bar{\theta}_{atm}$, it is clear this argument assumes that neutrinos contribute twice as much data as antineutrinos, and that rapidly oscillating antineutrinos have no up-down asymmetry. Following the argument in ref. [21], this results in $\sin^2 2\theta_{eff} < 4/5$, 5σ below the observed maximal mixing ($\sin^2 2\theta_{eff} \approx 1$).

However, this argument ignores the second mass splitting, $\Delta \overline{m}^2_{KamLAND}$. Although $\Delta \overline{m}^2_{KamLAND} \ll \Delta \overline{m}^2_{atm}$ is negligible in a highly asymmetric scenario, it plays a critical role in the asymmetry when other effects disappear. In the CPT violating scheme, the rapid oscillations due to $\Delta \overline{m}^2_{atm}$ give no asymmetry, so $\Delta \overline{m}^2_{KamLAND}$ effects take over. Since we are using $\sin^2 2\theta_{atm} = 0.7$, downward-going neutrinos have average survival probability $P_\downarrow \approx 1 - 0.7/2 = 0.65$. With $\Delta \overline{m}^2_{KamLAND} = 0$, the up-going survival probability would be identical, but with our parameters, $P_\uparrow \approx 0.55$ (with matter effects). Therefore the antineutrino up-down asymmetry is $A_\mu \approx -0.08$, contradicting (19). Combining this with the LMA asymmetry $A_\mu \approx 1/3$ in the neutrino sector, the total “ideal” up-down asymmetry is $A_{\mu^{\text{ideal}}} \approx 0.27$, compared to the maximum of 1/4 when neglecting the second splitting. The ideal asymmetry is defined by (18), but with $N_\uparrow$ only including directly upward-pointing events ($\cos \theta = -1$), and similarly for $N_\downarrow$ ($\cos \theta = 1$).

It is not clear whether the second mass splitting, ignored in the qualitative arguments of ref. [21], was included in the numerical simulations that produced the disallowed $\chi^2 \approx 5^2$.

Based on the sensitivity we found to the smoothing method, a reliable statistical result will require a detailed simulation of the detector and precise knowledge of the Super-K cuts and ring classification. Also, although we considered only quasi-elastic scattering, other processes also result in $\mu$-like single ring events and should be included. If this scheme survives the more rigorous analysis, CPT violation may join the list of ‘marginally allowed’ solutions to all of the neutrino observations, along with 3 + 1 sterile neutrino schemes. In any case, it seems that the list of ‘perfect fit’ solutions stands conspicuously empty.

9 Future Experiments

MiniBooNE A direct test of CPT violation will come with MiniBooNE, which can run in both neutrino and antineutrino modes. It is on its way to settling the LSND question once and for all, currently taking data in neutrino mode. If the LSND oscillation does not appear, then antineutrino mode will serve to test CPT conservation. If the LSND oscillation is confirmed in neutrinos, then sterile neutrinos will likely reappear in full force in phenomenological studies. Although 4ν schemes are tightly constrained, one marginally allowed 3 + 1 scheme remains, and there is still the possibility of a 5ν scheme (or 6ν, or 7ν, or...). If the LSND signal is rejected outright, then MiniBooNE will still succeed in making a deeper cut into the allowed parameters.
Figure 10: Zenith angle distributions of the charged leptons observed by Super-K, normalized to no-oscillations. The Super-K data points are shown with error bars, and histogram shows the predicted flux for a CPT conserving scheme with $\sin^2 2\theta_{\text{sun}} = 0.75$, $\sin^2 2\theta_{\text{atm}} = 1$, $\sin^2 2\theta_{13} = .025$, $\Delta m^2_{\text{sun}} = 10^{-4}$ eV$^2$, $\Delta m^2_{\text{atm}} = 2.8 \times 10^{-3}$ eV$^2$.

Figure 11: Zenith angle distributions of the Super-K charged leptons. Histograms show the predicted flux for the LMA solution (see previous caption) in the neutrino sector, and the antineutrino parameters from (15), compared to actual Super-K data points.
MINOS Capable of untangling the behavior of $\nu_\mu$ and $\bar{\nu}_\mu$, MINOS will test CPT violation in the atmospheric sector. The MINOS detector is already observing cosmic ray neutrinos, and the accelerator beam is scheduled to turn on in 2006. In the scheme from Section 8 (Fig. 7b), the large difference between the neutrino and antineutrino mass splits will be readily apparent. MINOS will also explore the magnitude of $\sin^2 \theta_{13}$ down to $\sim 0.01$, further restricting CP violation in neutrinos.

Borexino A third test for CPT violation comes from solar neutrinos. Borexino will be able to distinguish between several of the pre-KamLAND solutions to the solar neutrino problem, but, unlike KamLAND, it detects neutrinos rather than antineutrinos. Sensitive to the lower energy $^7$Be solar neutrinos (SNO detects $^8$B neutrinos), Borexino will choose between the LMA solution, the LOW solution (which would cause a day/night asymmetry), and the VAC solution (resulting in seasonal variations). These solutions were somewhat disfavored even before KamLAND, but should Borexino point to any solution other than LMA, it may be witnessing CPT violation. Otherwise, it will continue the march towards well-constrained mixing parameters.

With or without sterile neutrinos, CP violation, or CPT violation, the current round of neutrino experiments is sure to provide important new insights into the nature of neutrino mixing. Long-standing questions as fundamental as the number of neutrinos, the consistency of physical conservation laws, and the origins of particle mixing will be moved one step further, and, as always, many more new questions will arise.

Appendix: Solving the MSW Hamiltonian

Two-Neutrino Analytical Solutions

The qualitative description of resonant neutrino oscillations in the Sun was given in Section 4. Here we show the analytical solution for two neutrino oscillations with mixing angle $\theta_\nu$. Following [28], write the flavor decomposition of an arbitrary neutrino travelling through matter as

$$|\nu(t)\rangle = a_e(t) |\nu_e\rangle + a_\mu(t) |\nu_\mu\rangle .$$

The matter Hamiltonian (8) in the flavor basis corresponds to the evolution

$$i \frac{d}{dx} \begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} 2E\sqrt{2}G_F N_e(x) - \Delta m^2 \cos 2\theta_\nu & \Delta m^2 \sin 2\theta_\nu \\ \Delta m^2 \sin 2\theta_\nu & 2E\sqrt{2}G_F N_e(x) + \Delta m^2 \cos 2\theta_\nu \end{pmatrix} \begin{pmatrix} a_e \\ a_\mu \end{pmatrix} ,$$

where $N_e(x)$ is the electron number density, $\Delta m^2 \equiv m_1^2 - m_2^2$, and we have added a multiple of the identity to the Hamiltonian to simplify the algebra. The light and heavy local mass eigenstates are defined by

$$|\nu_L\rangle = \cos \theta(x) |\nu_e\rangle - \sin \theta(x) |\nu_\mu\rangle$$

$$|\nu_H\rangle = \sin \theta(x) |\nu_e\rangle + \cos \theta(x) |\nu_\mu\rangle$$

where the local mixing angle $\theta(x)$ is set to diagonalize the Hamiltonian, with

$$\cos 2\theta(x) = \frac{\sin 2\theta_\nu}{\sqrt{T^2(x) + \sin^2 2\theta_\nu}}$$

$$\sin 2\theta(x) = \frac{-T(x)}{\sqrt{T^2(x) + \sin^2 2\theta_\nu}}$$

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where
\[ T(x) \equiv 2\sqrt{2}G_F N_e(x)E/\Delta m^2 - \cos 2\theta_\nu. \]

In terms of the local mass eigenstates, the neutrino state is
\[ |\nu(x)\rangle = a_L(x)|\nu_L(x)\rangle + a_H(x)|\nu_H(x)\rangle. \]

Rewriting the neutrino evolution in terms of this basis, we have, finally
\[ i d\frac{d}{dx} \begin{pmatrix} a_H \\ a_L \end{pmatrix} = \begin{pmatrix} \lambda(x) & i\alpha(x) \\ -i\alpha(x) & -\lambda(x) \end{pmatrix} \begin{pmatrix} a_H \\ a_L \end{pmatrix} \]

where
\[ \lambda(x) = \frac{\Delta m^2}{4E} \sqrt{T^2(x) + \sin^2 2\theta_\nu}, \]
\[ \alpha(x) = \frac{\sqrt{2}E G_F \sin 2\theta_\nu}{\Delta m^2(T^2(x) + \sin^2 2\theta_\nu)} \left( \frac{dN_e(x)}{dx} \right) \]

In constant matter density, \( \alpha(x) \equiv 0 \), so the evolution reduces to two-neutrino oscillations of the local mass eigenstates. Otherwise, we look to the adiabatic theorem. The adiabatic condition is that the local mass eigenstates change much more slowly than the neutrino state, or
\[ \gamma(x) = \left| \frac{\lambda(x)}{\alpha(x)} \right| \gg 1. \]

The evolution will be least adiabatic at the crossing point \( x_c \) of the two flavor states, where \( \lambda(x) \) achieves its minimum. In the Sun, the adiabatic condition is not fulfilled at the critical crossing point, but it is a good approximation elsewhere. Since most of the action occurs at \( x_c \), we can approximate the electron density as a linear or exponential function with the correct derivative \( dN_e(x_c)/dx \) at this point. Then the non-adiabatic part is solvable analytically, with the resulting survival probability [28]
\[ P_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}(1 - 2P_{hop}) \cos 2\theta_\nu \cos 2\theta_\nu(x = 0). \]

With a linear density approximation near the critical point,
\[ P_{hop} = e^{-\pi\gamma(x_c)/2}. \]

Suppose the light vacuum state is \( \nu_1 \sim \nu_e \), and the transition is adiabatic \( (\gamma(x_c) \gg 1, \text{ so } P_{hop} \approx 0) \). The neutrino is created in a very dense region, inducing the local heavy eigenstate to be \( \nu_H \sim \nu_e \), and implying \( \theta(x = 0) \approx \pi/2 \), so the survival probability is \( P_{\nu_e \to \nu_e} \approx 0 \) regardless of \( \theta_\nu \). Therefore, the MSW effect can lead to high flavor conversion even with a very small mixing angle.

**Three-Neutrino Numerical Solutions**

For general three-family effects, it is straightforward to solve the MSW Hamiltonian numerically by integrating the Schroedinger equation. Working in the mass eigenstate basis, the vacuum Hamiltonian is

\[ \mathcal{H}_{vac} = \frac{1}{2E} \text{diag}(m_1^2, m_2^2, m_3^2) \]
\[ = \frac{1}{2E} \text{diag}(0, -\Delta m_{sun}^2, -\Delta m_{atm}^2) \]
where the equivalence holds because we can add a multiple of the identity to $H$. Converting to the flavor basis with the transformation $H \rightarrow U H U^\dagger$ and adding the matter effect term,

$$H = \frac{1}{2E} U \begin{pmatrix} 0 & -\Delta m^2_{\text{sun}} & -\Delta m^2_{\text{atm}} \\ -\Delta m^2_{\text{sun}} & 0 & 0 \\ -\Delta m^2_{\text{atm}} & 0 & 0 \end{pmatrix} U^\dagger + \begin{pmatrix} \pm \sqrt{2} G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The positive sign is taken for neutrinos, and the negative sign for antineutrinos. The effective $\nu_e$ mass induced by matter effects is

$$\sqrt{2} G_F N_e = 3.81 \times 10^{-5} \left( \frac{\rho}{\text{g cm}^{-3}} \right) \left( \frac{\text{eV}^2}{\text{GeV}} \right)$$

where $\rho$ is the matter density, and we assume $Z/A = 1/2$. Writing the neutrino state in terms of this basis,

$$|\nu\rangle = a_e |\nu_e\rangle + a_\mu |\nu_\mu\rangle + a_\tau |\nu_\tau\rangle,$$

the time evolution is given by the Schroedinger equation,

$$i \frac{d}{dx} \begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix} = H \begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix} \quad \text{(A-1)}$$

The CP violation and some of the Super-K calculations in this paper were done by solving this equation with a standard numerical ODE package.

**Constant Density Solution**

Although the numerical solution is easy to implement and always applicable, it is very computationally demanding when the state oscillates rapidly, or $\Delta m^2 L/E \gg 1$. For the CPT-violating antineutrino oscillation probabilities in Section 8 with $\Delta m^2 L/E \approx 10^3$, we resort to an analytical solution from ref. [29], applicable only for piecewise-constant electron density.

The Schroedinger equation (A-1) with matter effects has three independent solutions, corresponding to the states starting in each pure flavor. Composing these three solutions into the rows of a 3x3 matrix $X(t)$, the time evolution in the mass basis satisfies

$$i \frac{dX}{dt} = HX$$

with the initial condition $X(t = 0) = 1_{3x3}$ because each row starts as a definite flavor. For constant electron density, the eigenstates are fixed, so there is a closed form solution,

$$X(t) = \exp(-iHt).$$

The quantity $X_{\alpha\beta}$ is the amplitude for a state starting in flavor $\alpha$ to oscillate into flavor $\beta$. The solution for $X(t)$ can be computed by Lagrange’s formula, whereby any function of a Hermitian matrix $S$ can be rewritten as a function of its eigenvalues $\lambda_i$,

$$f(S) = \sum_i \left( \prod_{j \neq i} \frac{S - \lambda_j}{\lambda_i - \lambda_j} \right) f(\lambda_i).$$

Therefore, in our case

$$X(t) = \sum_i \left( \prod_{j \neq i} \frac{H - \lambda_j}{\lambda_i - \lambda_j} \right) \exp(-i\lambda_i t).$$
where the $\lambda_i$ are eigenvalues of $H$, computed analytically or numerically. (There is an exact formula for the eigenvalues, but it is not very enlightening; see ref. [29]).

Since each row of $X(t)$ denotes the evolution of a particular flavor, the oscillation probabilities are given by $P_{\nu_\alpha \rightarrow \nu_\beta} = |X_{\alpha\beta}|^2$. If the initial wave function of a neutrino is the column vector $\vec{\nu}_0$ (in the flavor basis) then after time $t$ it becomes

$$\vec{\nu}(t) = X(t)^T \vec{\nu}_0.$$ 

This allows us to patch together several regions of constant density, using the result of one computation as the input to the next.

References