The Aharanov-Bohm Effect in a Superconducting Lattice

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Abstract

In the Aharanov-Bohm effect, the vector potential of a magnetic flux affects the wave function of electrons. In a superconducting lattice, quantization of fluxoid forces the electrons to form currents that maintain integer number of fluxoids through holes in the lattice. The presence of non-integer values of flux quanta, therefore, may lead to highly frustrated systems exhibiting massively degenerate ground states for certain flux levels. I examined this degeneracy in the kagomé superconducting lattice at the filling ratio \( f = \frac{1}{2} \)--half a flux quantum through each triangular cell. I attempted to find ground states of \( f = \frac{1}{2} \) with energies linear in \( (f - \frac{1}{2}) \) under small perturbations of \( f \) near \( \frac{1}{2} \). My results suggest that no such states that are localized to within two wires actually exist.

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This paper represents my own work in accordance with University regulations.
1 Introduction

A “superconductor” is a perfect conductor. Current patterns inside of a superconductor will not die out over time—at least not because of resistance. Superconductors also attempt to expel internal magnetic fields. Fields only extend small distances into the superconductor and drop off exponentially from the surface. Superconductors are, however, influenced by the vector potential of a magnetic field through the Aharanov-Bohm effect. Therefore, superconducting wires arranged in lattices allow magnetic flux through the plane of the wires and thus may exhibit interesting properties when placed in applied magnetic fields.

Frustration in superconducting systems is caused by the interplay between the magnetic field and the geometry of the lattice and has been studied extensively in recent years [7], [9], [10], [12]. Specifically, the phase transition between superconducting and normal (nonsuperconducting) is an accessible regime in which “competition between the periodic potential imposed by the lattice structure and the length scale provided by the magnetic field” [12, Ch. 3] creates a delicate system with unusual properties. The ground states of these systems—the patterns of supercurrents when the system first goes superconducting—sometimes have macroscopic entropy that scales with the size of the system [7]. The superconducting state behaves as if it were a single electron in the so-called tight binding model [1]. In fact, there is a direct mapping between the state of a superconductor and \psi, the wave function for a single electron in a potential of the same geometry. “The application of Ginzberg Landau theory to the superconducting wire networks leads to a mathematical description identical to that of the tight-binding model...[The] study of the networks therefore provides...insight into the electronic system” [12, Ch. 2]. Quantum mechanical systems in periodic potentials sometimes exhibit fractal properties and sensitivity to the rationality or irrationality of the magnetic flux [1], [5], [8].

The lattice examined in this paper is the kagome lattice, a tiling of the plane with hexagons and triangles. The spectrum of this lattice—roughly the temperature of the phase transition into superconductivity which is effectively equivalent to the spectrum of the single electron problem—has been analyzed both analytically [9] and experimentally [7], [12] and is rather well mapped out. However, the patterns of supercurrents at that phase transition are much less well understood. In fact, although we know that the spectrum, in certain regions, reacts linearly to a changing magnetic field, there is no specific pattern of supercurrents known to have such a linear response in its energy.

This paper attempts to find states at or near the edge of this phase boundary with energy linear in perturbations of the magnetic field. We will look at states localized in one dimension and determine the reaction of their critical temperatures to small changes in the magnetic field.
2 Aharanov-Bohm effect

The time independent Schrödinger equation for a single particle in a scalar potential field is

\[-\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi = E\psi.\]

(1)

If the particle has charge $e$, it will be affected by the presence of a magnetic field via the field’s vector potential. Assume the particle is subject to $B$ with potential $A$—i.e. $B = \nabla \times A$. The Schrödinger equation becomes [6, Ch. 13]

\[-\frac{1}{2m} \left(\frac{\hbar}{i} \nabla + \frac{e}{c} A\right)^2 \psi + V(r)\psi = E\psi.\]

(2)

If $\psi$ solves the Schrödinger equation with $A$, and $A' = A = \nabla f(r)$, then $\psi' = e^{i\frac{e}{\hbar c}f(r)}\psi$ will solve the Schrödinger equation under $A'$.

Because the interaction between the wave function $\psi$ and the magnetic field occurs via the vector potential and not the magnetic field directly, the Aharanov-Bohm effect, a purely quantum mechanical effect, may occur.

Consider an electron confined to a two dimensional circle of radius $r$. Let $\Phi$ be a flux through the center of the circle but such that there is no magnetic field on or outside of the circle. Let $A = \frac{\Phi}{2\pi r} \theta$. Defining a “flux quantum” as $\phi = \frac{\hbar c}{e}$, the Schrödinger equation reads

\[-\frac{\hbar^2}{2mr^2} \left(\frac{d}{d\theta} + i\frac{\Phi}{\phi}\right)^2 \psi = E\psi.\]

(3)

Using $\psi_n = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ as an ansatz results in eigenenergies of $\frac{\hbar^2}{2mr^2}(n + \frac{\Phi}{\phi})^2$. And the energies are dependent upon the flux through the center of the circle—despite the fact that the flux and the particle are never present at the same location. “The inner flux, in a region inaccessible to the electron, has an observable effect on the spectrum” [4, Ch. 4].

Before we can discuss the Aharanov-Bohm effect applied to superconductivity, we must first gain a basic understanding of superconductivity.

3 Superconductivity

The resistance of certain metals, mercury, lead, and tin, for example, vanishes at temperatures below a critical, material dependent, temperature. This zero resistance behavior is the characteristic property of superconductivity [11, Ch. 1]. H. Kamerlingh Onnes first observed this zero-resistance behavior in 1911. In 1933 Meissner and Ochsenfeld observed the property of perfect diamagnetism in superconductors—cooling a potential superconductor below its characteristic temperature, $T_c$, pushes magnetic fields almost completely out of the material [11]. Fields at the surface penetrate only short distances into a superconductor and decay exponentially with a characteristic length called the penetration
depth [11]. Therefore, strong enough magnetic fields—i.e. too strong to be expelled—will prevent a system from exhibiting superconductivity [11].

The superconductor then, at $T_c$ undergoes a phase transition between the normal state (no superconduction) and the superconducting state. The calculation of the spectrum of superconducting systems has been a primary focus of study in recent years [9], [12].

In 1957, Bardeen, Cooper, and Schrieffer [3] introduced a theory of superconductivity based upon superconducting electron pairing. In effect, the superconducting charge carriers in a superconductor are BCS electron pairs. Hence, the charge per charge carrier is $e^* = 2e$ where $e$ is the charge on the electron. Similarly, the mass per charge carrier, $m^*$, is equal to twice the electron mass: $m^* = 2m$ [11].

4 Ginzberg Landau theory

In 1950, before the BCS theory of superconductivity, Ginzberg and Landau introduced a phenomenological theory based upon a complex valued order parameter $\psi$ whose amplitude squared would be the density of superconducting electrons [11, Ch.4]. In Ginzberg Landau (GL) theory, $n_s(x) = |\psi(x)|^2$, where $n_s$ is the superconducting electron density.

The theory derives from expanding the free energy of the superconductor in orders of $|\psi|^2$. “The basic postulate of GL is that if $\psi$ is small and varies slowly in space, the free-energy density $f$ can be expanded in a series of the form” [11]

$$f = f_{no} + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla - \frac{e}{c} A \right) \psi^2 + \frac{\hbar^2}{8\pi}.$$  (4)

Examining the case in which there is no magnetic field, $A = 0$, and $\psi$ is spatially uniform, $\nabla \psi = 0$, the free energy is roughly quadratic in $|\psi|^2$. The system attempts to minimize free energy, so $\beta$ must be positive for a minimum to exist. For $\alpha > 0$ the minimum of $f$ occurs at $|\psi|^2 = 0$ and the superconductor must be in the normal, non-superconducting regime. For $\alpha < 0$, the material is superconducting. $\alpha$ should, thus, be a function of both the critical temperature and the actual temperature. In order to recover $T_c$ from $\alpha$ it is possible to use $\alpha \propto \frac{1-T^2}{1-T^2} \approx 1 - t$, with $t = \frac{T}{T_c}$ [11]. The constant of proportionality is left to experiment.

Now, the appropriate step is to minimize the GL free energy. $f$ is free energy density, so we must minimize the integral of $f$,

$$F = \iiint f(\mathbf{r}) d^3 \mathbf{r}$$  \hspace{1cm} (5)

with respect to $\psi$. Let $\psi_0$ be the order parameter for which $F = F_0$ is minimum. We replace $\psi$ with $\psi_0 + \delta \psi$, $F$ by $F_0 + \delta F$. $\delta F$ should be 0 for all small $\delta \psi^*$. To first order in $\delta \psi^*$, $F_0 + \delta F = \frac{\hbar^2}{8\pi} + \alpha \int d^3 r (|\psi_0|^2 + \psi_0^* \delta \psi + \psi_0 \delta \psi^*)$. 

4
\[ + \frac{\beta}{2} \int d^3r (|\psi|^4 + 2|\psi|^2 \psi \delta \psi^* + 2|\psi|^2 \psi^* \delta \psi) \]
\[ - \frac{1}{2m^*} \int d^3r \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) (\psi \delta \psi) \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) (\psi^* + \delta \psi^*) \]

Integrating the third term by parts yields,
\[ \delta F = \int d^3r (\alpha \psi \delta \psi^* + \beta |\psi|^2 \psi \delta \psi^*) \]
\[ - \frac{1}{2m^*} \int d^3r \delta \psi^* \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right) \psi_0 \]
\[ \equiv 0 \quad \forall \text{ small } \delta \psi^* . \]

\( \psi_0 \) is \( \psi \) at minimum free energy, so
\[ \alpha \psi + \beta |\psi|^2 \psi + \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right)^2 \psi \equiv 0 . \]

This equation then determines the order parameter, \( \psi \), and hence both the number density of BCS pairs and the superconductors, as will be shown later. However, as \( T \to T_c \), \( \psi \to 0 \), so the \( \beta |\psi|^2 \psi \) term is negligible compared to the \( \alpha \psi \) term. Neglecting this term yields the linearized GL equation
\[ \alpha \psi + \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla - \frac{e^*}{c} A \right)^2 \psi = 0 . \quad (6) \]

The linearized GL equation bears a striking resemblance to the Schrödinger equation for a single electron with energy \(-\alpha\). In fact, GL theory and the time independent Schrödinger equation on the same geometric pattern turn out to be equivalent [1]. Furthermore, the order parameter on adjacent nodes effectively determines the supercurrents. Thus, the GL problem should reduce to solving for \( \psi \) on the nodes.

With the introduction of the order parameter, \( \psi \), another key property of superconductivity, fluxoid quantization, can be discussed [11, Ch. 4]. Consider a superconducting wire in a closed loop enclosing a non-superconducting hole. If we write \( \psi = |\psi|e^{i\theta} \), then for \( \psi \) to be single valued, we mandate that
\[ \oint \nabla \theta \cdot ds = 2\pi n, \text{ for } n \in \mathbb{Z} . \quad (7) \]

But if we also make the association \( \mathbf{p} = m^* \mathbf{v} + \frac{e^*}{e} \mathbf{A} \) with \( \mathbf{v} \) the velocity of the electron pairs, then we see [11]
\[ n \frac{\hbar c}{e^*} = \frac{e}{e^*} \oint \left( m^* \mathbf{v} + \frac{e^*}{c} \mathbf{A} \right) ds \equiv \phi . \quad (8) \]

We take this to be the definition of fluxoid, \( \phi \) and see that it is quantized in units of \( \frac{\hbar c}{e^*} = \phi_0 \) — what we will call a flux quantum. BCS theory tells us that
$e^* = 2e$ and so this flux quanta differs from our previous definition by a factor of $\frac{1}{2}$. To avoid any confusion, future discussions will use the filling ratio—flux quanta per area of interest—instead of the flux itself.

Quantization of fluxoid does however impose certain restrictions on the superconducting state. For all temperatures greater than $T_c$, the material is in the normal state and so $|\psi|^2 = 0$. We are interested in the regime in which $T \approx T_c$, and the free energy expansion is dominated by the lowest order terms in $|\psi|^2$. But then, because any $A$ caused by the supercurrents are of order $|\psi|^2$, in linearized GL theory, $A \approx A_{\text{external}}$ [12, Ch. 2]. Therefore, the actual flux, $\oint dA \cdot d\ell$ is considered given and there arises a quantization condition on the integral of the velocity of the BCS pairs.

5 Periodic potentials with magnetic fields

In a magnetic field with vector potential $A$, $(-i\hbar \nabla - \frac{e}{c}A)$ replaces $p$ [8]. This replacement is known as a Peierls substitution. For our Hamiltonian, we take [4, Ch. 4]

$$H = \frac{1}{2m}(-i\hbar \nabla - \frac{e}{c}A)^2 - V_0(c) \cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a} - 2),$$

(9)

a periodic potential with period $a$. The tight-binding limit is the limit $V_0 \gg 1$. First looking at the $B = 0, A = 0$ case, we see that the lowest energy wave functions will be localized near the vertices of the square lattice with lattice constant $a$. Near $(ma, na)$ for $m, n \in \mathbb{Z}$, the potential term in the Hamiltonian, to second order, is $V_0(2\pi^2 (\frac{a}{y})^2 + 2\pi^2 (\frac{b}{x})^2)$ and is very large away from $(ma, na)$. Therefore, for $(x, y)$ near a lattice point, $H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2} m \frac{4\omega^2}{m^2 c^2} (x^2 + y^2)$, a simple harmonic oscillator potential with $\omega^2 = \frac{4\mu_0}{m \pi^2}$. Thus, the ground state solutions are approximately two-dimensional Gaussians centered at the $(ma, na)$. The energy of a state localized at one vertex is, therefore, approximately, $\hbar \omega$ plus the energy of mixing with the nearest neighbor sites [4].

Because these states are localized and uniform, one complex number per vertex suffices to describe the state of the system. In the tight binding approximation, we only concern ourselves with the values $\psi$ assumes on this discrete lattice. We now can turn our attention to finding a “discretized” Schrodinger’s—one defined only on the lattice points.

Two neighbor states on the lattice interact with one another. Let $t$ be the “hopping matrix element” between two neighbor states. This $t$ is the interaction energy between states at adjacent vertices [4, Ch. 4]. We are assuming an extended plane, so, for the ease of computation, we will set the zero of energy to be the zero point energy—i.e. $\hbar \omega$ per site. The energy of the state localized at $(x, y)$, $\psi(x, y)$, is, therefore, $-t(\psi(x, y + a) + \psi(x, y - a) + \psi(x + a, y) + \psi(x - a, y))$—the sum of the interaction energies with the nearest neighbor sites. (For non-nearest neighbors, the interaction is exponentially smaller and, hence, neglected.) This formula for energy seems almost like an operator, so
maybe we should treat this expression as the Hamiltonian acting upon $\psi(x,y)$. We, thus, get a discretized Schrödinger’s Equation:

$$H\psi(x,y) = -t(\psi(x,y+a) + \psi(x,y-a) + \psi(x+a,y) + \psi(x-a,y)) = E\psi(x,y)$$  

(10)

By Bloch’s Theorem [2, Ch. 8], we can create eigenstates of the form

$$\psi(x,y) = e^{ik_x x} e^{ik_y y} \phi(x,y)$$

where $\phi$ is periodic with period $a$. Because we are only interested in the values $\psi$ assumes on lattice points, we set $\psi(x,y) = e^{ik_x x} e^{ik_y y}$. Plugging this ansatz into equation (10), we find $E = -2t(\cos(k_y a) + \cos(k_x a))$. $E$ is periodic in $k_i$, so we can take $-\frac{\pi}{a} \leq k_i \leq \frac{\pi}{a}$ getting the standard Brillouin zones.

But what about when $B \neq \bar{B}$. A convenient gauge to work in is $A = (-By, 0)$ [4, Ch. 4]. In one dimension, $\psi_A(x+k) = e^{\frac{i\epsilon}{t} \int_{x}^{x+k} A \cdot d\mathbf{l}} \psi_0(x+k)$. Schrödinger’s equation becomes

$$-t(\psi(x,y+a) + \psi(x,y-a) + e^{i\epsilon \frac{By}{a}} \psi(x+a,y) + e^{i\epsilon \frac{By}{a}} \psi(x-a,y)) = E\psi(x,y).$$

Recalling a “free-particle” in a uniform B-field with no potential $V_0$ and the resulting Landau levels and plane wave/harmonic oscillator eigenstates [4], we try the ansatz of

$$\psi(x,y) = e^{i k_x a} g_k(y/a)$$

and let $x = ma$, $y = na$. The Schrödinger equation becomes

$$g_k(n+1) + g_k(n-1) + 2g_k(n) \cos(k - 2\pi \alpha n) = -\epsilon g_k(n),$$

where we have let $\alpha = \frac{eB}{2\pi \hbar}$ and $\epsilon = E/t$ [5].

This equation is called Harper’s Equation (for the square lattice) and it is a simpler version of the Harper’s Equation that we will get for the kagomé lattice later. But first, the solutions for this Harper’s equation are in order...

To solve (11), we will want to choose a value for $\alpha$. Here we will set $\alpha = \frac{1}{2}$.

What does this mean? Imagine a magnetic field $B$ applied in the $z$ direction, normal to the plane of the two dimensional wire lattice. Each square of the lattice would have a flux of $\Phi = Ba^2$ through it. $\alpha$ is the filling ratio of the system. In other words, $\alpha = \frac{\phi_0}{\Phi}$, where $\phi_0$ is the flux quanta.

Harper’s equation now reads

$$g_k(n+1) + g_k(n-1) + 2(-1)^n \cos(k)g_k(n) = -\epsilon g_k(n).$$

(12)

The $(-1)^n$ in Harper’s equation suggests a periodic ansatz with period 2 [4].

$$g_k(n) = \begin{cases} Ae^{i\lambda n}, & n \text{ even} \\ Be^{i\lambda n}, & n \text{ odd} \end{cases}$$

(13)

Allowing $n$ to be first even and then odd, and simplifying, yields

$$\frac{A}{B} = \frac{2\cos(\lambda)}{-\epsilon - 2\cos(k)}.$$  

(14)

This statement implies,

$$\epsilon^2 = 4(\cos^2 k + \cos^2 \lambda).$$  

(15)
Extremum for \( \epsilon \) are \( \pm 2 \sqrt{2} \) and occur at \( k = j \pi, \lambda = l \pi \) for \( j, l \in \mathbb{Z} \). For \( \alpha = \frac{1}{2} \), \( \epsilon \) varies continuously between these extremum. Interestingly, away from \( \alpha = \frac{1}{2} \), the spectrum reveals fractal-like properties [5].

We now have the spectrum of this tight binding model. The next question to ask is “What are the ground state wave functions?” \( k = j \pi \) and \( \lambda = l \pi \) suggests trying the four combinations between \( j, l = 0, 1 \).

\[
\frac{A}{B} = \frac{2 \cos \lambda}{-\epsilon - 2 \cos k} = \frac{\pm 1}{\sqrt{2} \pm 1} \quad (16)
\]

For an infinitely extended plane wave solution, as is the case here, normalizations are somewhat arbitrary, so take \( B = 1 \). We then have
\[
\begin{array}{ccc}
k & \lambda & \psi(ma, na(even)) \quad \psi(ma, na(odd)) \\
0 & 0 & \frac{1}{\sqrt{2-1}} e^{i \pi n} \quad 1 \\
0 & \pi & \frac{1}{\sqrt{2+1}} e^{i \pi m} \quad e^{i \pi m} \\
\pi & 0 & \frac{1}{\sqrt{2+1}} e^{i \pi m} \quad e^{i \pi m} \\
\pi & \pi & \frac{1}{\sqrt{2+1}} e^{i \pi m} \quad e^{i \pi m} e^{i \pi n}
\end{array}
\]

But the exponential terms all drop to either \( 1 \) or \(-1\) and, even better, the second and fourth terms are not independent of the first and third. We are left with the following two independent solutions:
\[
\psi(ma, na) = \begin{cases} 
\frac{1}{\sqrt{2-1}} & n \text{ even} \\
1 & n \text{ odd}
\end{cases} \quad \text{and} \quad \psi(ma, na) = \begin{cases} 
\frac{(-1)^m}{\sqrt{2+1}} & n \text{ even} \\
\frac{(-1)^m}{(1)^m} & n \text{ odd}
\end{cases}
\quad (17)
\]

These two states are the two independent, orthogonal ground state eigenvectors for the square lattice tight binding model at filling ratio \( \alpha = \frac{1}{2} \).

6 Ginzberg Landau and tight binding

The linearized GL equation, (6), will provide the connection between the order parameter of the superconducting lattice and the tight binding model of a single electron [1]. In order to arrive at the tight binding model, I will base my analysis on the approach taken by Yi Xiao [12, Ch. 2].

We define the “coherence length” \( \xi = \frac{\hbar}{\sqrt{2m} \alpha} \), the characteristic length scale of changes in the order parameter. Recall the definition of one flux quanta (for the super conductor), \( \phi_0 = \frac{\hbar}{e} \). The linearized GL equation becomes
\[
(\nabla - \frac{2\pi A}{\phi_0})^2 \psi + \frac{1}{\xi^2} \psi = 0. \quad \text{It should be noted that this coherence length} \ \xi \ \text{is basically a function of} \ \alpha \ \text{which in turn is a function of the critical temperature.}
\]

The goal is for this equation to reduce to the tight binding model. But the tight binding model only concerns itself with the wave function/order parameter on a discrete lattice, so we now hope to show that if \( \psi \) is specified on adjacent
lattice points, the order parameter is then well determined on the connecting wire.

As before with the electron in a vector potential, we perform the gauge transformation $\eta = \psi e^{-\frac{i}{\hbar} \oint A \cdot dl}$. The integral is taken along the path starting at one endpoint, $A$ is the component of $A$ tangent to the wire, and $dl$ is along the wire. The GL equation is $\frac{\partial^2}{\partial l^2} \psi + \frac{1}{2} \frac{1}{\xi} \psi \cos(l/\xi) = 0$. $\eta$ is sinusoidal, $C \sin(l/\xi) + D \cos(l/\xi)$, and so $\psi = e^{\frac{i}{\hbar} \oint A \cdot dl} (C \sin(l/\xi) + D \cos(l/\xi))$.

Now boundary conditions will fix $C$ and $D$. Let $\psi_i$ be the order parameter at the lattice point from where we start integrating and let $\psi_f$ be the order parameter at the other vertex. Let $L$ be the length of the wire.

$$D = \psi_i$$
$$C = \frac{1}{\sin(L/\xi)} (e^{\frac{i}{\hbar} \oint A \cdot dl} \psi_f - \psi_i \cos \frac{L}{\xi})$$

Therefore, we now have $\psi$ on the wire between adjacent lattice points. ($l$ varies from 0 to $L$.)

$$\psi(l) = e^{\frac{i}{\hbar} \oint_0^l A \cdot dl} \sin \frac{l}{\xi} \left( \psi_f e^{\frac{i}{\hbar} \oint_0^l A \cdot dl} \sin \frac{l}{\xi} + \psi_i \sin \frac{L - l}{\xi} \right)$$

Superconducting current is given by

$$J_{if} = \frac{e^*}{m^*} \hbar \Im(\psi^*(\nabla - \frac{i 2\pi A}{\phi_0})\psi).$$

But in the actual lattice, for a point to be a vertex, at least two wires must intersect at the point. “A ‘Kirchoff’ matching condition” [1], implies

$$\sum_j \left( \psi_j^* (\nabla - \frac{i 2\pi A}{\phi_0}) \psi_j \right) \equiv 0,$$

where the sum in $j$ is taken over all neighboring vertices and the derivatives are taken along the wire path at the vertex. Using equation (19), this expression becomes to

$$\sum_j \frac{1}{\xi \sin(a/\xi)} (-\psi_i \cos(a/\xi) + \psi_j e^{\frac{i}{\hbar} \oint_j A \cdot dl} ) \equiv 0,$$

where $j$ represents neighboring vertices and $a$ is the length between vertex points. Note that it is assumed that the width of each wire is the same and that $a$ is the lattice constant, the length between vertices.

This equation can be rearranged:

$$\sum_j \psi_j e^{\frac{i}{\hbar} \oint_j A \cdot dl} = 4 \cos \frac{a}{\xi} \psi_i$$
This equation is the tight binding model with the substitution \( E = 4\cos \frac{a}{2} \). And the variable of interest, \( T_c \), the critical temperature of the phase transition at which superconducting begins, is related to \( \xi \) through [12, Ch. 2]

\[
\delta T_c = T_{c0} \left( \frac{\xi(0)}{\xi(T)} \right)^2
\]

(24)

\[
\delta T_c = T_{c0} \left( \frac{\xi(0)}{a} \arccos \left( \frac{E}{2(1 + \tau)} \right) \right)^2.
\]

(25)

Therefore, the problem of finding the critical temperature of a superconducting lattice reduces to finding the highest eigenenergy of the geometrically equivalent single-electron tight binding model. Thus we adopt the convention of calling the state with the highest energy, the ground state. Thus, the ground state goes superconducting at the highest temperature. Finally, we see that the single electron \( \psi \) is identical to the order parameter \( \psi \).

7 Kagomé lattice

The kagomé lattice consists of triangles and hexagons arranged to form a grid of six pointed stars (See Fig. 1). The first challenge is interpreting equation (23) to arrive at an effective tight binding model Schrodinger’s equation for this geometry.

Label one of the wires in the lattice the x-axis. Choose as an origin the intersection of the x-axis and a line cutting it at \( \frac{\pi}{3} \). The y-axis extends as expected perpendicular from the x. Later on, it will be useful to count rows along the line intersecting the x-axis at the origin. Label that line the “z-axis.”

For a convenient unit of length, let \( a \) be the length of one edge of a triangle. It will also be convenient to adopt a numbering convention for the three geometrically distinct types of vertices (with respect to the axis labellings). The origin is a type 1 vertex. The vertex at \( x = 0, y = \frac{\sqrt{3}}{2}a \), or alternatively \( x = 0, z = a \) is a type 2 vertex. And the remaining \( x = a, y = 0 \) is a type 3 vertex (See Fig. 1). These three vertices form what will be called the up pointing triangle at (0,0).

8 Vector potential

The next issue is that of the vector potential, \( \mathbf{A} \). \( \nabla \times \mathbf{A} \) must equal \( B \), the applied field. But the theory allows any gauge transformation of \( A \).

In the square lattice, \( A = (-By, 0, 0) \) was convenient. This gauge eliminated the effect of the magnetic field on all wires travelling in the y direction. Along all vertical wires, \( \mathbf{A} \cdot d\mathbf{l} \equiv 0 \). This gauge choice simplified the calculations immensely because it removed the problem of the magnetic field from consideration along an entire dimension.

I attempted to use an analogous gauge choice here. The key point above was that \( \mathbf{A} \cdot d\mathbf{l} = 0 \). The vanishing of the dot product between the potential
and the direction of one wire suggests the following gauge:
\[
\vec{A} = By(-1\hat{x} + \frac{1}{\sqrt{3}}\hat{y}) = (-1, \frac{1}{\sqrt{3}}, 0)By.
\]  
(26)

This vector potential represents a magnetic field in the direction perpendicular to the plane—call it the \(\zeta\) direction: \(\vec{\nabla} \times \vec{A} = B\zeta\). (I would say the z-direction, but remember that I am reserving z to lie along a wire in the plane.)

\(\vec{A}\) no longer has zero divergence but, along the z-axis, \(\vec{A} \cdot d\vec{l} = 0\). This gauge choice will, therefore, simplify the Hamiltonian by eliminating field dependent factors from one third of the terms. Furthermore, by having the potential’s magnitude depend only upon \(y\), along all wires parallel to the x-axis, \(\vec{A} \cdot d\vec{l}\) = a constant = \((-By)d\vec{l}\). This gauge choice is different from the gauge chosen by Xiao [12]. This gauge, along with the use of x and z instead of x and y axes, simplifies the calculations leading to Harper’s equation.

The potential enters into equation (23) via factors of the form
\[
e^{-\frac{4\pi i}{\hbar\gamma} \int A \cdot dl}.
\]  
(27)

For all \(y\) values, \(A\) is perpendicular to the z-axis. So, factor (27) = 1 along the z-axis and between any type 1 and type 2 vertices. Between 1 and 3 vertices in the same upward pointing triangle, factor (27) = \(e^{-\frac{4\pi i}{\hbar\gamma}(-By)(a)}\). Because the integral is taken along the wire from one vertex to another, factor (27) is conjugated when taken in the opposite direction as the integral is reversed and changes sign.
The final factor involving the vector potential is the most complicated. Let us try to evaluate (27) acting from a 3 vertex at (2a)n above the origin along the z axis to the 2 above it.

\[
|\vec{A}| = \frac{2}{\sqrt{3}} y B \text{ and the length along the wire, } l = \frac{2}{\sqrt{3}} y
\]

\[
\vec{A} \cdot d\vec{l} = \frac{\sqrt{3}}{2} |\vec{A}| dl \text{ along the path }
\]

\[
\int \vec{A} \cdot d\vec{l} = \frac{B}{\sqrt{3}} (y_2^2 - y_3^2) \text{ with } y_j = \frac{\sqrt{3}}{2} z_j
\]

the filling ratio, \( f = \frac{B \sqrt{3} A^2}{\phi_0} \)

\[
e^{-\frac{2\pi}{3} n} \int A \cdot dl = e^{-i8\pi f (n + \frac{1}{3})}
\]

If we always take the starting point as being at a height 2a above the origin, similar calculations will yield the following values for (27).

From a 2 down to a 3: \( e^{i8\pi f (n + \frac{1}{3})} \)

From a 2 up to a 3: \( e^{-i8\pi f (n + \frac{1}{3})} \)

From a 3 down to a 2: \( e^{i8\pi f (n - \frac{1}{3})} \)

A brief notational convention used in the rest of this paper is to label lattice points and the argument of a wave function by the location of the upward pointing triangle the point is in and with an additional 1, 2, or 3 to indicate the vertex. \( \psi = \psi(x, z, j) = \psi(MA, NA, j) \), where \( A = 2a \), M indicates the mth triangle along the x-axis, N indicates the nth triangle along the z-axis, and \( j = 1, 2, 3 \) the vertex in that triangle. Note that \( (MA, NA, 1) \) is quite literally at \( (2Ma, 2Na) \) but that \( (MA, NA, 2) \) is at \( (2a, (2N + 1)a) \) and \( (MA, NA, 3) \) is at \( ((2M + 1)a, 2Na) \).

9 Schrodinger equation and Harper’s equation for kagomé lattice

We now apply equation (23) to the three vertices in the triangle at \( (MA, NA) \) to arrive at the Schrodinger equation for the kagomé lattice.

\[
E\psi(MA, NA, 1) = \psi(MA, (N - 1)A, 2) + \psi(MA, NA, 2) + e^{-i8\pi N \frac{1}{3}} \psi((M - 1)A, NA, 3) + e^{i8\pi N \frac{1}{3}} \psi(MA, NA, 3)
\]

\[
E\psi(MA, NA, 2) = \psi(MA, NA, 1) + \psi(MA, (N + 1)A, 2) + e^{-i8\pi f (N + 3/4)} \psi((M + 1)A, (N + 1)A, 3) + e^{i8\pi f (N + 1/4)} \psi(MA, NA, 3)
\]

\[
E\psi(MA, NA, 3) = e^{-i8\pi f (N + 1/4)} \psi(MA, NA, 1) + e^{i8\pi f (N + 1/4)} \psi((M + 1)A, NA, 1) + e^{-i8\pi f (N - 1/4)} \psi((M + 1)A, (N - 1)A, 2) + e^{i8\pi f (N - 1/4)} \psi((M + 1)A, (N - 1)A, 2)
\]
The right hand sides of these equations can be treated as $H\psi$, a Hamiltonian acting upon the order parameter.

To arrive at a one dimensional Harper’s equation, employ the ansatz

$$\psi(MA,NA,j) = e^{ikM}g_k(N,j).$$

And the three Schrödinger equations become the Harper’s equations:

$$g_k(N,1) = g_k(N-1,2) + g_k(N,2) + e^{-i\theta N}e^{-ik}g_k(N,3) + e^{i\theta N}g_k(N,3)$$

$$g_k(N,2) = g_k(N,1) + g_k(N+1,1) + e^{-i\theta N/2}e^{-ik}g_k(N+1,3) + e^{i\theta N/2}g_k(N,3)$$

$$g_k(N,3) = e^{-i\theta N}g_k(N,1) + e^{i\theta N}e^{ik}g_k(N,1) + e^{-i\theta N}e^{-ik}g_k(N,2) + e^{i\theta N}e^{ik}g_k(N-1,2)$$

### 10 One half filling ratio

The case for $f = \frac{1}{2}$ can be solved exactly. For this half filling scenario, equations (28) through (30) simplify to the following:

$$E\psi(MA,NA,1) = \psi(MA,(N-1)A,2) + \psi(MA,NA,2) + \psi((M-1)A,NA,3) + \psi(MA,NA,3)$$

$$E\psi(MA,NA,2) = \psi(MA,NA,1) + \psi(MA,(N+1)A,2) - \psi((M-1)A,(N+1)A,3) - \psi(MA,NA,3)$$

$$E\psi(MA,NA,3) = \psi(MA,NA,1) + \psi((M+1)A,NA,1) - \psi(MA,NA,2) - \psi((M+1)A,(N-1)A,2)$$

Even better, at $f = \frac{1}{2}$, the above Schrödinger equations are periodic with a period of $A = 2a$. Therefore, by Bloch’s theorem, solutions should be linear combinations of the ansatz $\psi(MA,NA,j) = e^{ikM}e^{ilN}\psi_j$. With this ansatz, Harper’s equation has a simple matrix form

$$E\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 + e^{-il} & 1 + e^{-ik} \\ 1 + e^{il} & 0 & -1 - e^{-ik}e^{il} \\ 1 + e^{ik} & -1 - e^{ik}e^{-il} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

Solving the characteristic equation for this matrix equation, yields

$$(-2 + E)(2 - E(2 + E) + 2\cos(k) + 2\cos(k-l) + 2\cos(l)) = 0$$

$$E = \begin{cases} 2 \\ -1 \pm \sqrt{3 + 2\cos(k) + 2\cos(k-l) + 2\cos(l)} \end{cases}$$

The maximum value of $E$—the value that corresponds to $T_c$—is 2. The second term has $E = 2$ only for $k = l = 0$ which represents one eigenvector. We will solve for the more general eigenvector, for which we will assume $k \neq 0, l \neq 0,$
by finding the null space of the Hamiltonian matrix minus the constant \( E = 2 \)
times the identity.

\[
\begin{pmatrix}
-2 & 1 + e^{-i\theta} & 1 + e^{-ik} \\
1 + e^{i\theta} & -2 & -1 - e^{-i\theta}e^{i\phi} \\
1 + e^{i\phi} & -1 - e^{i\theta}e^{-i\phi} & -2
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & \frac{e^{-i\theta} - e^{-ik}}{1 - e^{-i\phi}} \\
0 & 1 & \frac{e^{-i\theta} - e^{-ik}}{1 - e^{i\phi}} \\
0 & 0 & 0
\end{pmatrix}
\tag{41}
\]

And the general eigenvector for \( E = 2 \),

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix}
= \begin{pmatrix}
\frac{-e^{-i\theta} - e^{-ik}}{1 - e^{i\phi}} \\
-\frac{e^{i\theta} - e^{ik}}{1 - e^{i\phi}} \\
1
\end{pmatrix}
\tag{42}
\]

### 11 Localized states

Any linear combination of the above general solutions,

\[
\psi(MA, NA, j) = e^{ikM} e^{iN} \psi_j
\]

with

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix}
= \begin{pmatrix}
\frac{-e^{-i\theta} - e^{-ik}}{1 - e^{i\phi}} \\
-\frac{e^{i\theta} - e^{ik}}{1 - e^{i\phi}} \\
1
\end{pmatrix}
\tag{43}
\]

will satisfy the \( f = \frac{1}{2} \) Schrödinger equation. However, through simple inspection
of the interaction between nearest neighbor sites, it is possible to construct
simple grounds states that are localized.

For example, notice that with this choice of potential, there is no potential
effect along any wire parallel to the \( z \)-axis. Thus, the hopping term is 1. Take
\( \psi \) to be constant along one such wire,

\[
\psi(MA, NA, j) = \begin{cases} 
1, & \text{if } M = C \text{ (a constant) and } j = 1 \text{ or } 2 \\
0, & \text{all others}
\end{cases}
\tag{44}
\]

This \( \psi \) satisfies equations (28)-(30) provided \( E = 2 \). Any point on the wire
must have an order parameter equal to half the sum of its nearest neighbors.

But, for \( f = \frac{1}{2} \), the hopping terms along wires parallel to the \( x \) axis are
identical to the hopping terms along wires parallel to the \( z \) axis. In each case,
the terms due to the vector potential, the factor (27), reduce to 1. Therefore,
there is the analogous eigenstate localized in the \( x \) direction.

\[
\psi(MA, NA, j) = \begin{cases} 
1, & \text{if } N = C \text{ (a constant) and } j = 1 \text{ or } 3 \\
0, & \text{all others}
\end{cases}
\tag{45}
\]

State (44) is a constant \( l = 0 \) state while (45) is a constant \( k = 0 \) state.

So a logical question would be: does there exist a corresponding state along
the third direction? Yes. The wires parallel to neither axis pass through type 2
and 3 vertices and thus have hopping terms that reduce to -1 at \( f = \frac{1}{2} \). Nearest
neighbor sites must have opposite signs. Therefore, we may take alternating 1’s and (-1)’s.

\[ \psi(MA, NA, j) = \begin{cases} 
1 & \text{if } M + N = C \text{ (a constant) and } j = 2 \\
-1 & \text{if } M + N = C \text{ (a constant) and } j = 3 \\
0 & \text{all others} 
\end{cases} \] (46)

All three states may be multiplied by complex phases and added together while still remaining \( f = \frac{1}{2} \) eigenstates.

The goal is to find an eigenstate at one half filling ratio whose energy changes linearly with small perturbations in \( f - \frac{1}{2} \). The energies can be calculated as

\[ <H> = \frac{<\psi|H|\psi>}{<\psi|\psi>} \] (47)

where \( <H> \) is the expectation value of the Hamiltonian—the energy of the state—\( <\psi|\psi> \) is a normalizing factor, and \( H|\psi> \) can be calculated using the Hamiltonian side of the Schrodinger equations (28)-(30) to calculate the energy of the hopping terms. For all three states (44)-(46), we should only need to calculate \( <H> \) on one triangle to determine the effect of \( f \) on \( H \).

State (44) presents a trivial calculation:

\[ <H> = \frac{<\psi|H|\psi>}{<\psi|\psi>} = \frac{1}{1 + 1}((1 + 1) + (1 + 1)) = 2 \] (48)

Thus, these states have \( E = 2 \) for all \( f \) and are, in fact, eigenstates at all filling ratios. This result should not be surprising though, as the vector potential was chosen specifically to eliminate filling ratio dependent hopping along wires parallel to the \( z \) axis (27) \( \equiv 1 \).

Let us now look at the energy of the states parallel to the \( x \) axis, (45)

\[ <H> = \frac{<\psi|H|\psi>}{<\psi|\psi>} = \frac{1}{1 + 1}((e^{-i8\pi NF} + e^{i8\pi NF}) + (e^{-8\pi NF} + e^{8\pi NF})) \\
= 2\cos(8\pi NF) \text{, for a constant N} \]

The energy of these states decreases quadratically in \( f \) for all \( f \). (Well, there is the \( N = 0 \) state of constant energy, but this state still does not have energy its vary linearly in \( f - \frac{1}{2} \). It is, however, analogous to the previous states running in the \( z \) direction.)

Similarly, for the states given by equation (46)

\[ <H> = \frac{<\psi|H|\psi>}{<\psi|\psi>} \\
= \frac{1}{2}((-e^{-i8\pi f(n+\frac{1}{4})} - e^{8\pi f(n+\frac{1}{4})}) - (e^{-i8\pi f(n+\frac{1}{4})} + e^{8\pi f(n-\frac{1}{4})})) \\
= -\cos(8\pi f(n + \frac{1}{4})) - e^{-8\pi f(n+3/4)} - e^{8\pi f(n-1/4)} \]
This energy might not seem reasonable at first. It is not even real. However, this result is for one N value—two sites in one triangle. When this “energy” is summed up over all N, the two exponential terms in this sum will be simplified into cosines, similar to the cosine already present. This state, too, is then strictly quadratic in f. Furthermore, these energies appear to just be sums of the energy of different N states from the state (45). One word of caution: when summing, the \(< \psi|\psi >\) term in the denominator must equal two times the number of triangles being summed over in order to maintain a consistent normalization procedure.

Because the energy of each of these states is strictly quadratic and never linear in \(f - \frac{1}{2}\), any linear combination of these states is still nonlinear in \(f - \frac{1}{2}\).

But maybe there are other states that are localized along one wire but are neither constant nor merely alternating sign along the wire. Let us look for \(f = \frac{1}{2}\) eigenstates along the x-axis that increase by a phase factor of k between successive triangles. Start with arbitrary coefficients on one triangle.

\[
\psi(MA, NA, j) = \begin{cases} 
C e^{ikM} & N = 0, j = 1 \\
D e^{ikM} & N = 0, j = 3 \\
0 & \text{all others}
\end{cases} \tag{49}
\]

The Schrödinger equation demands, for \(E = 2, f = \frac{1}{2}\)

\[
2C = e^{-ik}D + D \quad \text{and} \quad 2D = C + e^{ik}C \tag{30}
\]

\[
\Rightarrow \cos(k) = 1 \Rightarrow k = 0. \tag{51}
\]

So, by Bloch’s theorem, the state given by (45) is the only eigenstate localized on a single wire running in the x direction. The value of N does not enter into the \(f = \frac{1}{2}\) Schrödinger equation, so the specific wire is irrelevant.

## 12 Localized k-momentum states

If there are no one wire states with either \(k \neq 0\) or energy linear in \(f - \frac{1}{2}\), let us look for a solution localized on and between two parallel wires running in the x direction.

Assume the state has momentum k and each lattice point picks up a phase factor of \(e^{ik}\) between adjacent triangles. To ease the calculation, set \(N = 0\), i.e. let the lower wire be the x axis.

The basic cell in this formation will then consist of five lattice points—two along the wire on the x-axis, one in between the two wires and two along the \(z=A\) wire—forming five of the six corners of a hexagon (See Fig. 2).

We now need to setup and solve the resulting Schrödinger equations (28)-(30) for this order parameter pattern.

Let us look at one basic cell of five vertices. Let

\[
\psi(0, 0, 1) = B \quad \psi(0, 0, 3) = C \quad \psi(0, 0, 2) = D
\]
In other words, let the vertices starting in the lower left and moving clockwise have order parameters $B, C, D, E, F$ respectively. These five values and $k$ then uniquely determines the state along the whole of the two wires because $\psi((M + r)A, NA, j) = e^{ikr}\psi(MA, NA, j)$. We can always divide through by one of the values without changing the physics, so we can assign an arbitrary value to one vertex. Because we already know the solution localized on one wire, we want to take $\psi(MA, 0, 2) = D \neq 0$. Let us choose $D = -1 + e^{-ik}$. The reason for this choice will become apparent later on when the other four values are found. This choice also has the advantage that setting $k = 0$ isolates the two wires and should return a sum of one wire states.

![Diagram](image)

**Figure 2:** Two rows of the lattice with arbitrary order parameter values on one five vertex cell.

Then, by the Schrödinger equations, (28)-(30), taken with $E = 2, f = \frac{1}{2}$,

\[
\begin{align*}
2B &= -e^{ik} + 1 + C(1 + e^{ik}) \\
2C &= -(-1 + e^{-ik}) + B(e^{-ik} + 1) \\
2(1 - e^{-ik}) &= Be^{-ik} + E - C - Fe^{-ik} \\
2E &= -1 + e^{ik} + F(1 + e^{-ik}) \\
2F &= -1 + e^{ik} + E(1 + e^{ik}).
\end{align*}
\]

This systems solves uniquely to

\[
B = 1 \ ; \ C = 1 \ ; \ D = -1 + e^{-ik} \ ; \ E = -1 \ ; \ F = -1.
\]

which defines the order parameter on the $M$th hexagon as follows:

\[
\psi(MA, NA, j) = \begin{cases} 
    e^{ik(M-1)} & N = 0, j = 1 \\
    e^{ik} & N = 0, j = 3 \\
    e^{ik(M-1)} - e^{ikM} & N = 0, j = 2 \\
    -e^{ikM} & N = 1, j = 1 \\
    -e^{ik(M-1)} & N = 1, j = 3 
\end{cases}
\]

The first and third $(M-1)$'s occur because the particular point is in the $M$th hexagon but has $x$ value, $x = (M + 1)A$. The middle $(M-1)$ term is there because of the choice $D = -1 + e^{-ik}$. Now we see the value in our guess of $D = -1 + e^{-ik}$—four of the five vertices per hexagon agree up to sign differences.
Before going on any further and calculating the energy of this state, let us digress and examine what this solution tells us in terms of highly localized states.

This state is the unique, up to a phase factor, k momentum state on these two wires. So, by Bloch’s theorem, we should now have all of the $f = \frac{1}{2}$ two wire localized highest energy eigenstates in the x direction. Later, after we calculate some energies, we will see that none of the states have energy linear in $f - \frac{1}{2}$, and there are no off-diagonal mixing terms either. We will thus have shown that no states localized strictly on these two wires have energy linear in $f - \frac{1}{2}$. But before that, let us examine this particular solution.

Let us examine one hexagon. Dividing out the factor of $e^{ikM}$ leaves only one reference to the momentum—the $e^{-ik}$ term in our original guess. We can interpret this term as an overlap with the preceding hexagon. In other words, dropping that one $e^{-ik}$ term should yield us a solution localized on just one hexagon—if we manage to replace the lost $e^{-ik}$ with the $+1$ on the other middle vertex from the next adjacent hexagon.

![Figure 3: One hexagon localized $f = \frac{1}{2}$ eigenstate. On all other vertices, $\psi \equiv 0$.]

So, therefore, taking the trial solution of $-1$ on the upper left three vertices and $+1$ on the lower right vertices in this one hexagon yields a solution to the Schrodinger equation at $f = \frac{1}{2}$ (See Fig. 3). Foreshadowing events to come, we will now find the energy of such a one hexagon/six vertex localized state to be strictly quadratic in $f - \frac{1}{2}$. Note that we will still be taking $N = 0, 1$ to simplify the calculation and that the calculation will start with $<\psi|H|\psi>$ in the middle right type 2 vertex and proceed clockwise.

$$<H> = \frac{<\psi|H|\psi>}{<\psi|\psi>}$$

$$= \frac{1}{6} \left( + (1 - e^{-\pi f}) + (2) + (1 - e^{-2\pi f}) \right.$$

$$- (e^{i2\pi f} - 1) - (-1 - e^{i8\pi f}) - (-e^{-i8\pi f} + e^{-\pi f}) \right)$$

$$= \frac{1}{6} (6 - 2 \cos(2\pi f) - 2 \cos(6\pi f) + 2 \cos(8\pi f))$$

This energy, first of all, is $2$ at $f = \frac{1}{2}$ as it should be by construction. However, it is also strictly quadratic in $f$. One final point worth mentioning is that these highly localized states may be added together linearly to gain other eigenstates. (This addition is, in effect, what the two wire states are.) But that then means we have independent, orthogonal eigenstates on disjoint hexagons. The number
of such states scales with the area of the lattice and we begin to see one possible cause of the large entropy of the half filling ratio ground state.

Now we go back to the extended state. We will perform the calculation first for one “hexagon”—or the five lattice sites comprising the unit block that is repeated given by equation (58). Because of the bra in $\langle \psi | H | \psi \rangle$ which conjugates the lattice values of the ket, the specific M (x coordinate) dependence drops out. Therefore, the energy is the same for each five vertex unit. The calculation will start from the lower right vertex and proceed clockwise.

$$\langle H \rangle = \frac{1}{6 - 2 \cos(k)} \left( \begin{array}{c} + (1 - e^{ik} + e^{ik} + 1) \\
+ (1 + e^{-ik} + e^{-i2\pi k} (e^{-ik} - 1)) \\
+ (e^{i+k} - 1) (e^{-ik} + e^{i2\pi k} - 1 - e^{-i2\pi k} e^{-ik}) \\
- (e^{-ik} - 1 - e^{-i2\pi k} e^{-ik} - e^{i2\pi k}) \\
- (-e^{-i8\pi k} - e^{i8\pi k} e^{ik} + e^{i8\pi k} (1 - e^{ik})) \end{array} \right)$$

$$= \frac{1}{6 - 2 \cos(k)} \left( \begin{array}{c} 6 + 2 \cos(2\pi f + k) - 2 \cos(2\pi f) \\
- 2 \cos(6\pi f) - 2 \cos(k) + 2 \cos(6\pi f + k) \\
+ 2 \cos(8\pi f) + 2 \cos(8\pi f + k) \end{array} \right)$$

$$= 2 + \frac{4\pi^2 (13 + 3\cos(k))}{\cos(k) - 3} (f - \frac{1}{2})^2 + O[(f - \frac{1}{2})^3].$$

This state, too, has energy strictly quadratic in $f - \frac{1}{2}$. One would correctly guess that the setting of $N = 0$ loses no generality; the same procedure for a state at an arbitrary $\bar{N}$ (of the bottom wire) yields

$$\langle H \rangle = 2 + \frac{4\pi^2 (13 + 32N(N + 1) + 3\cos(k))}{\cos(k) - 3} (f - \frac{1}{2})^2 + O[(f - \frac{1}{2})^3].$$

### 13 Perturbation theory

So we now have the set of all states localized on these two wires—Bloch’s theorem says that all eigenstates on these two wires can be built up from states with the form of equation (58). Moreover, each state has its energy quadratic in $f - \frac{1}{2}$. If we imagine moving $f$ from $\frac{1}{2}$ as a small perturbation in the Hamiltonian described by the right hand sides of (28)-(30), then we can try to use perturbation theory to determine if any combination of states on these two wires will give us the desired energy linear in $f - \frac{1}{2}$.

To find the perturbing operator, we take the previous Hamiltonian and isolate the effects due to changes in $f$ around $f = \frac{1}{2}$. To this effect, let $g = f - \frac{1}{2}$ so $f = \frac{1}{2} + g$. Then, $V$, the perturbation to the Hamiltonian takes the form

$$V \psi(MA, NA, 1) = - e^{-i8\pi Ng} \psi((M - 1)A, NA, 3) - e^{i8\pi Ng} \psi(MA, NA, 3)$$

$$V \psi(MA, NA, 2) = - e^{-i8\pi g(N+\frac{3}{4})} \psi((M - 1)A, (N + 1)A, 3) - e^{i8\pi g(N+\frac{1}{4})} \psi(MA, NA, 3)$$

$$V \psi(MA, NA, 3) = - e^{-i8\pi Ng} \psi(MA, NA, 1) - e^{i8\pi Ng} \psi((M + 1)A, NA, 1)$$

$$V \psi(MA, NA, 2) = - e^{-i8\pi g(N+\frac{3}{4})} \psi(MA, NA, 3) - e^{i8\pi g(N+\frac{1}{4})} \psi((M + 1)A, (N - 1)A, 2)$$

$$V \psi(MA, NA, 3) = - e^{-i8\pi g(N+\frac{3}{4})} \psi(MA, NA, 2) - e^{i8\pi g(N+\frac{1}{4})} \psi((M + 1)A, (N - 1)A, 2)$$

$$= e^{i8\pi g(N+\frac{1}{4})} \psi(MA, NA, 3).$$
First we need to decide upon specific states to use. We have the two obvious $k = 0$ states: constant 1 along the lower wire, and constant 1 along the upper wire. Let us denote these states by $|0_a\rangle$ along the lower ($N = 0$) and $|0_b\rangle$ along the upper ($N = 1$). These states are orthogonal to one another—they have disjoint support—and $<0_a|V|0_b\rangle \equiv 0$ because both $H$ and $V$ include hopping terms only between nearest neighbor sites.

Let us denote the general state defined by (58) as $|k\rangle$. We will impose a wrap around condition on a basic block of $L$ five vertex sites, $|k(L)\rangle = |k(0)\rangle$. We therefore can take $k = \frac{2\pi j}{L}$ for $j \in [1, L - 1]$. The $j = L$ state would be equivalent to $k = 0$ which has already been handled above.

We now want to check for orthogonality.

$$<0|k_j\rangle = 2 \sum_{m=0}^{L-1} e^{i2\pi jm/L} \equiv 0,$$

the roots of unity sum to zero. And the zero momentum states are thus orthogonal to the non zero momentum states.

$$< k_j | k_j > = \sum_{m=0}^{L-1} \left( \frac{2e^{-i2\pi \frac{j}{L} m/L} e^{i2\pi jm/L}}{e^{i2\pi \frac{j}{L} (m-1)/L} e^{i2\pi jm/L}} + (e^{-i2\pi \frac{j}{L} (m-1)/L} - e^{-i2\pi \frac{j}{L} m/L}) ight)$$

$$= \begin{cases} 
0 & \text{if } k_{j'} \neq k_j \\
C & \text{a constant} & \text{if } k_{j'} = k_j 
\end{cases},$$

a result standard in finite Fourier analysis and in the theory of finite abelian groups. Thus, all the different $k$ states are mutually orthogonal. We, therefore, have an orthogonal, complete basis of the $f = \frac{1}{2}$, $E = 2$ eigenstates localized along these two wires.

The remaining calculation is to determine any states whose energy is linear in $g$. Because any linear combination of terms quadratic in $g$ is still quadratic in $g$, the following calculations will only keep terms that are $O(g)$.

$$< k_j | V | 0_a \rangle = 2 \sum_{m=0}^{L-1} -e^{-i2\pi jm/L} = 0$$

$$< k_j | V | 0_b \rangle = 2 \sum_{m=0}^{L-1} \cos(8\pi g)e^{-i2\pi jm/L} = 0$$

(63)  

(64)
Because we have already checked the diagonal elements of $<V>$ by noting that the energy of each eigenstate is quadratic in $g$, $<i|V|j>$ is not $O[g]$ for any states, $|i>,|j>$. Degenerate perturbation theory says to diagonalize $V$ and take its eigenvalues as perturbed energies. But $V$ has no terms linear in $f$. Therefore, there are no eigenstates of $f = \frac{1}{3}$ localized upon these two wires with energy that varies linearly in small perturbations of the filling ratio.

14 Conclusion

But might such states exist at other $N$ and other orientations? Although this paper only presents proof that no such linear-in-$g$ state exists along two specific wires, the geometry would suggest that this result should not be unique to these two wires. Looking at the eigenenergies of states parallel to the $x$ axis, we notice that shifting in $z$ merely changes the argument of the cosine term, $\cos(8\pi N f)$, but does not add any substantially new linear terms. Furthermore, we examined the state highly localized on only one hexagon. Its energy was also quadratic in $g$. The two wire extended solutions are really nothing more than strings of these solutions pasted together with an increasing phase factor. No linear dependence should be introduced. As for other orientations, we should realize that the choice of gauge presented in this paper was not unique. $A$ could be rotated by $\pm \frac{\pi}{4}$ to have the same effect: eliminating $\vec{A} \cdot \vec{d}$ along particular wires. Thus, we can surmise that no $f = \frac{1}{3}, E = 2$ eigenstates exists localized along just two wires with our desired energy-linear-in-$g$.

Nevertheless, according to the numeric work of Lin and Nori [9] and the experimental evidence of Higgins, et al. [7] such states should exist.

They are unlikely to exist localized on three parallel wires. The energy of one such state—effectively another row of hexagons with an increasing phase factor stacked above the one presented here—was calculated and found to also be strictly quadratic in $g$.

It is likely that these eigenstates are not localized states, but, in fact, extend fully in two dimensions. One suggested approach for future study would be to look at the Landau levels of a free electron in a magnetic field and adapt those eigenstates to the geometry of the kagomé lattice.
15 References


