Nonnegative Polynomials and Dynamical Systems

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Optimization over nonnegative polynomials

Defn. A polynomial $p(x) := p(x_1, \ldots, x_n)$ is nonnegative if $p(x) \geq 0, \forall x \in \mathbb{R}^n$.

Example: When is

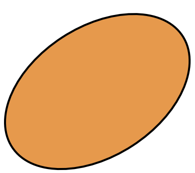
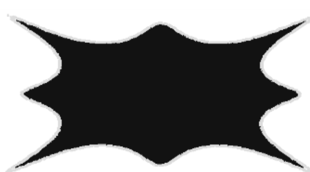
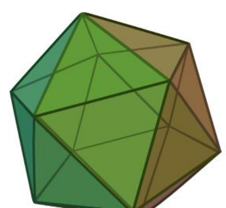
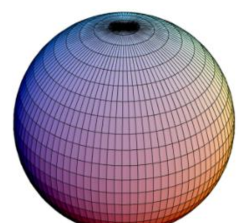
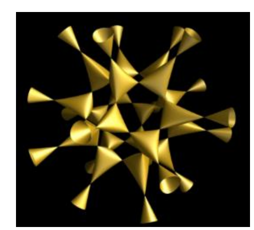
$$p(x_1, x_2) = c_1 x_1^4 - 6x_1^3 x_2 - 4x_1^3 + c_2 x_1^2 x_2^2 + 10x_1^2 + 12x_1 x_2^2 + c_3 x_2^4$$

nonnegative?
nonnegative over a given basic semialgebraic set?

Basic semialgebraic set: $\{x \in \mathbb{R}^n |\ g_i(x) \geq 0\}$

Ex:

$$x_1^3 - 2x_1 x_2^4 \geq 0$$
$$x_1^4 + 3x_1 x_2 - x_2^6 \geq 0$$

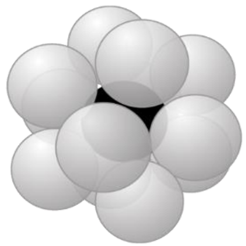
Why do this?!

Optimization over nonnegative polynomials

Is $p(x) \geq 0$ on $\{g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$?

Optimization

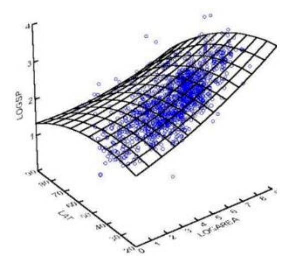
- Lower bounds on polynomial optimization problems
- Proving infeasibility of a system of polynomial inequalities



$$x_i^2 + y_i^2 + z_i^2 = 4,\ \ i = 1, \dots, 13$$
$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \geq 4,$$
$$i, j \in \{1, \dots, 13\}^2$$

Statistics

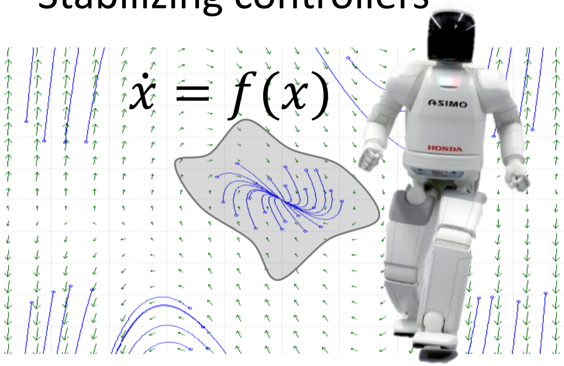
- Fitting a polynomial to data subject to shape constraints (e.g., convexity, or monotonicity)



$$\frac{\partial p(x)}{\partial x_j} \geq 0, \forall x \in B$$

Control

- Stabilizing controllers



$$V(x) > 0,$$
$$V(x) \leq \beta \Rightarrow \nabla V(x)^T f(x) < 0$$

Implies that
$\{x \mid V(x) \leq \beta\}$
is in the region of attraction

How to prove nonnegativity? The SOS approach

- A polynomial p is a **sum of squares (sos)** if it can be written as

$$p(x) = \sum_i q_i^2(x),$$

where q_i are polynomials.

Ex:

$$\begin{aligned} p(x) &= x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ &\quad -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \\ &= (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2 \end{aligned}$$

- A polynomial p of degree $2d$ is **sos** if and only if $\exists Q \succcurlyeq 0$ such that

$$p(x) = z(x)^T Q z(x)$$

where $z = [1, x_1, \dots, x_n, x_1x_2, \dots, x_n^d]^T$ is the vector of monomials of degree up to d.

Optimizing over set of sos polynomials is an SDP!

How to prove nonnegativity over a basic semialgebraic set?

Positivstellensatz: Certifies that

$$p(x) > 0 \text{ on } \{g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$$

Putinar's Psatz: (1993)

$$p(x) > 0 \text{ on } \{g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$$

under Archimedean condition

$$\Downarrow$$

$$p(x) = \sigma_0(x) + \textstyle\sum_i \sigma_i(x) g_i(x),$$

where $\sigma_i, i = 0, \dots, m$ are sos

Search for σ_i is an SDP when we bound the degree.

Stengle's Psatz (1974)

Schmudgen's Psatz (1991)

…

All use sos polynomials…

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Outline

Part I:

Avoiding SDP in optimization over nonnegative polynomials

- LP, SOCP
- An “optimization-free” Positivstellensatz

Part II:

Asymptotic stability of polynomial vector fields

- Complexity
- Computational converse Lyapunov questions

Practical limitations of SOS

- **Scalability** is a nontrivial challenge!

Thm: $\boldsymbol{p(x)}$ of degree $\boldsymbol{2d}$ is sos if and only if

$$p(x) = z^T Q z \quad Q \succeq 0$$

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]^T$$

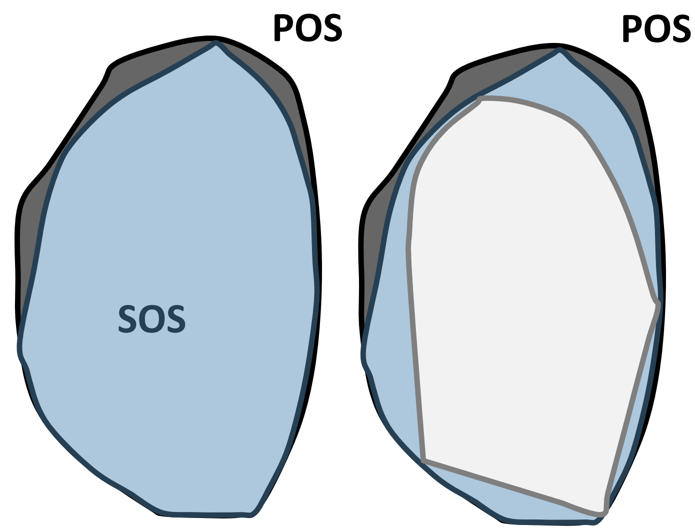
- The size of the Gram matrix is:

$$\binom{n+d}{d} \times \binom{n+d}{d}$$

- Polynomial in n for fixed d, but grows quickly
  - **The semidefinite constraint is expensive**
- E.g., local stability analysis of a 20-state cubic vector field is typically an SDP with ~1.2M decision variables and ~200k constraints

Simple idea…

- Let's not work with SOS…
- Give other sufficient conditions for nonnegativity that are **perhaps stronger than SOS, but hopefully cheaper**



Not any set inside SOS would work!

1) **sums of 4th powers of polynomials**
2) **sums of 3 squares of polynomials**

Both sets are clearly inside the SOS cone,
but linear optimization over them is **intractable**.

dsos and sdsos polynomials (1/3)

Defn. A polynomial p is *diagonally-dominant-sum-of-squares (dsos)* if it can be written as:

$$p(x) = \sum_i \alpha_i m_i^2(x) + \sum_{i,j} \beta_{ij}^+ (m_i(x) + m_j(x))^2 + \sum_{i,j} \beta_{ij}^- (m_i(x) - m_j(x))^2,$$

for some monomials m_i, m_j
and some nonnegative constants $\alpha_i, \beta_{ij}^+, \beta_{ij}^-$.

Defn. A polynomial p is *scaled-diagonally-dominant-sum-of-squares (sdsos)* if it can be written as:

$$p(x) = \sum_i \alpha_i m_i^2(x) + \sum_{i,j} (\hat{\beta}_{ij}^+ m_i(x) + \tilde{\beta}_{ij}^+ m_j(x))^2 + \sum_{i,j} (\hat{\beta}_{ij}^- m_i(x) - \tilde{\beta}_{ij}^- m_j(x))^2,$$

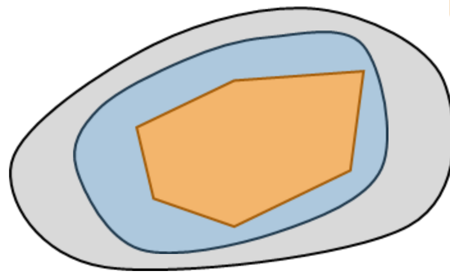
for some monomials m_i, m_j
and some constants $\alpha_i, \hat{\beta}_{ij}^+, \tilde{\beta}_{ij}^+, \hat{\beta}_{ij}^-, \tilde{\beta}_{ij}^-$ $\mathit{with}\ \alpha_i \geq 0.$

Note: $DSOS_{n,d} \subseteq SDSOS_{n,d} \subseteq SOS_{n,d} \subseteq POS_{n,d}$

dsos and sdsos polynomials (2/3)

Sum of squares **(sos)**	$p(x) = z(x)^T Q z(x), Q \succcurlyeq 0$	**SDP**

PSD cone$:= \{Q \mid Q \succcurlyeq 0\}$



DD cone $:= \{Q \mid Q_{ii} \geq \sum_{j \neq i} |Q_{ij}|, \forall i\}$

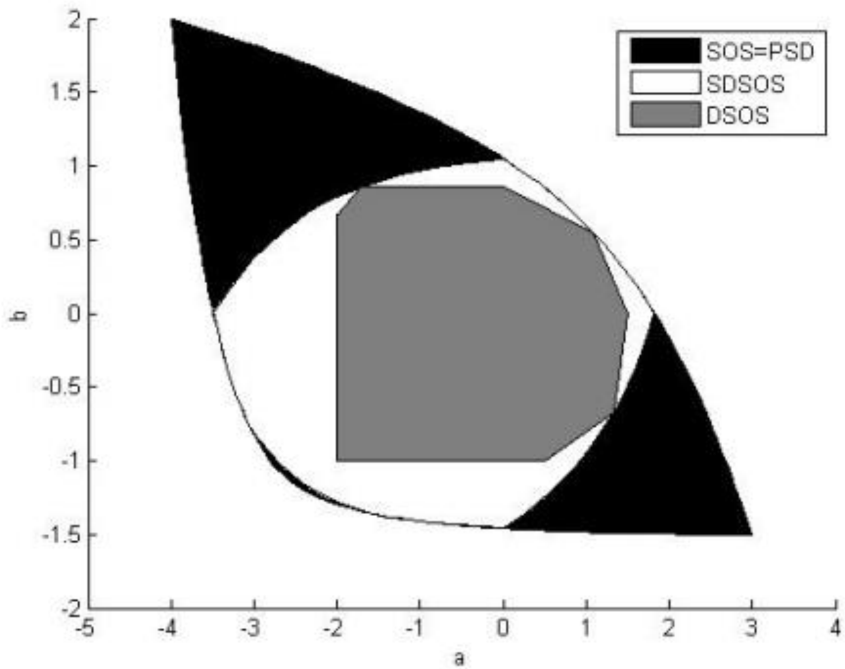
SDD cone $:= \{Q \mid \exists$ **diagonal** D **with** $D_{ii} > 0$ **s.t.** DQD $dd\}$

Diagonally dominant sum of squares **(dsos)**	$p(x) = z(x)^T Q z(x), Q$ *diagonally dominant* (dd)	**LP**

Scaled diagonally dominant sum of squares **(sdsos)**	$p(x) = z(x)^T Q z(x), Q$ *scaled diagonally dominant* (sdd)	**SOCP**

[AAA, Majumdar, *SIAM J. on Applied Algebra and Geometry*]

dsos and sdsos polynomials (3/3)



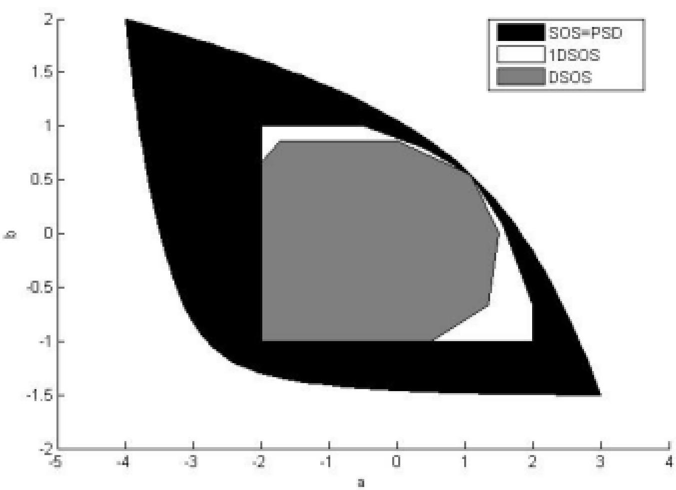
$$p(x_1, x_2) = x_1^4 + x_2^4 + a x_1^3 x_2 + (1 - \frac{1}{2}a - \frac{1}{2}b) x_1^2 x_2^2 + 2b x_1 x_2^3$$

How to do better?

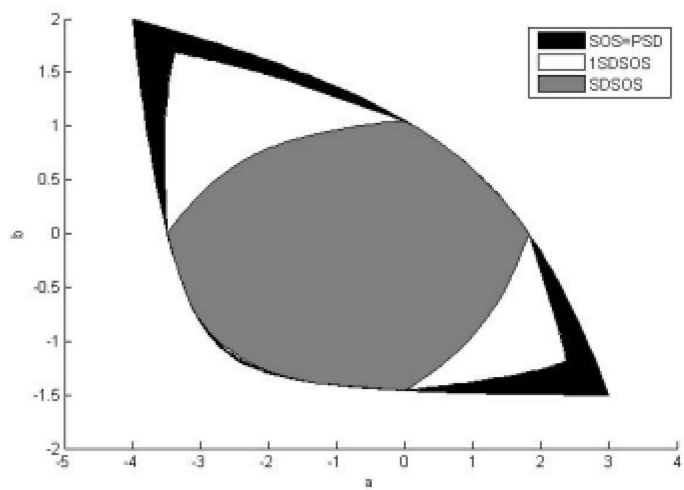
Method #1: r-dsos and r-sdsos polynomials (1/2)

Defn.

- A polynomial p is **r-dsos** if $p(x) \cdot \left(\sum_i x_i^2\right)^r$ is dsos.
- A polynomial p is **r-sdsos** if $p(x) \cdot \left(\sum_i x_i^2\right)^r$ is sdsos.



(a) The LP-based r-dsos hierarchy.



(b) The SOCP-based r-sdsos hierarchy.

$$p(x_1, x_2) = x_1^4 + x_2^4 + a x_1^3 x_2 + (1 - \frac{1}{2}a - \frac{1}{2}b) x_1^2 x_2^2 + 2b x_1 x_2^3$$

Method #1: r-dsos and r-sdsos polynomials (2/2)

- r-dsos can outperform sos!

$$p(x) = x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2$$

is 1-dsos but not sos.

Theorem: Any even positive definite form is r-dsos for some r.

- Even forms include *copositive programming (and all problems in NP)*.
- Shows that LP-based proofs of nonnegativity always possible.

Method #2: dsos/sdsos + change of basis (1/2)

$$p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4$$

$$p(x) = z^T(x)Qz(x)$$

$$Q = \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ -3 & 9 & 0 & -3 & 0 & -6 \\ 0 & 0 & 16 & 0 & 0 & -4 \\ 1 & -3 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 2 & -6 & 4 & 2 & 0 & 5 \end{pmatrix}$$

$$z(x) = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2)^T$$

$$p(x) = \tilde{z}^T(x) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4. \end{pmatrix} \tilde{z}(x)$$

$$\tilde{z}(x) = \begin{pmatrix} 2x_1^2 - 6x_1x_2 + 2x_1x_3 + 2x_3^2 \\ x_1x_3 - x_2x_3 \\ x_2^2 - \frac{1}{4}x_3^2 \end{pmatrix}$$

Goal: iteratively improve $z(x)$

Method #2: dsos/sdsos + change of basis (2/2)

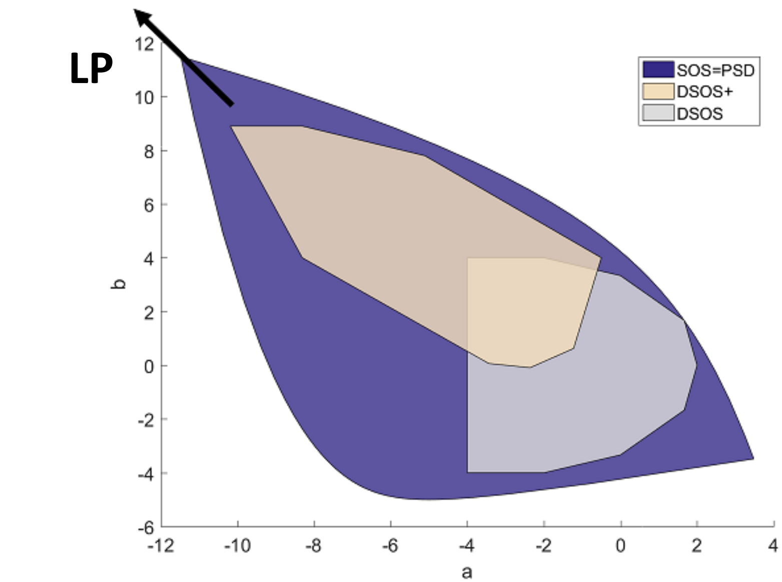
LP
$$\max_{\vec{P},Q} \ \ell(\vec{P}) \quad \text{s.t. } p(x) = z^T(x) Q z(x) \ \forall x, \quad Q \text{ dd}$$

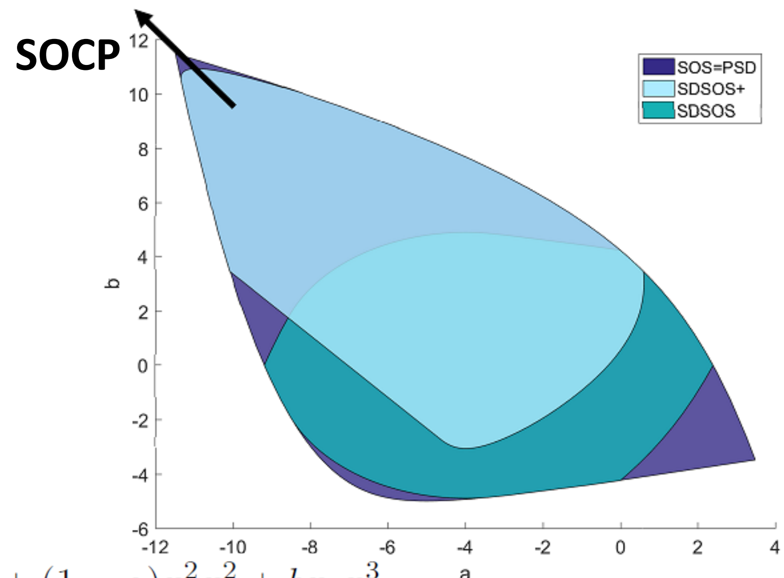
→ Optimal soln. Q^*

→ Cholesky: $Q^* = U^T U$

LP_+
$$\max_{\vec{P},Q} \ \ell(\vec{P}) \quad \text{s.t. } p(x) = z^T(x) U^T Q U z(x) \ \forall x, \quad Q \text{ dd}$$

Works beautifully!



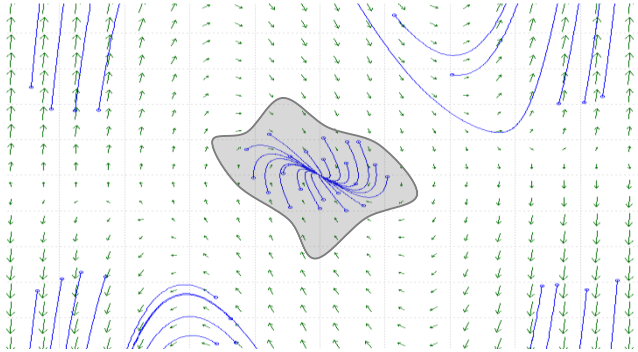


$$p_{a,b}(x_1, x_2) = 2x_1^4 + 2x_2^4 + a x_1^3 x_2 + (1-a) x_2^2 x_2^2 + b x_1 x_2^3$$

Reminder

$\dot{x} = f(x, u)$

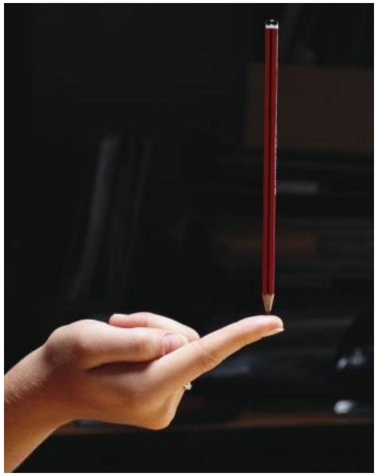
Stability of equilibrium points



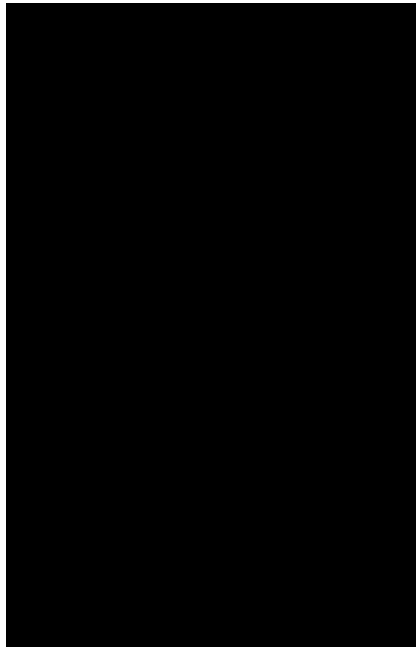
$$V(x) > 0,$$
$$V(x) \le \beta \Rightarrow \dot{V}(x) < 0$$

implies $\{x \mid V(x) \le \beta\}$ is in the region of attraction (ROA)

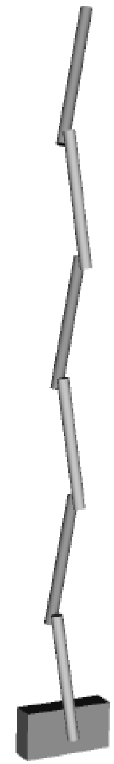
Stabilizing the inverted N-link pendulum (2N states)



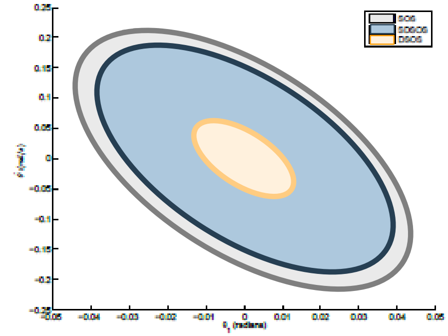
N=1



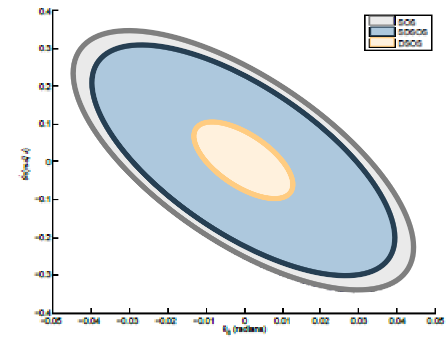
N=2



N=6



(a) θ_1-$\dot{\theta}_1$ subspace.



(b) θ_6-$\dot{\theta}_6$ subspace.

Runtime:

2N (# states)	4	6	8	10	12	14	16	18	20	22
DSOS	< 1	0.44	2.04	3.08	9.67	25.1	74.2	200.5	492.0	823.2
SDSOS	< 1	0.72	6.72	7.78	25.9	92.4	189.0	424.74	846.9	1275.6
SOS (SeDuMi)	< 1	3.97	156.9	1697.5	23676.5	∞	∞	∞	∞	∞
SOS (MOSEK)	< 1	0.84	16.2	149.1	1526.5	∞	∞	∞	∞	∞

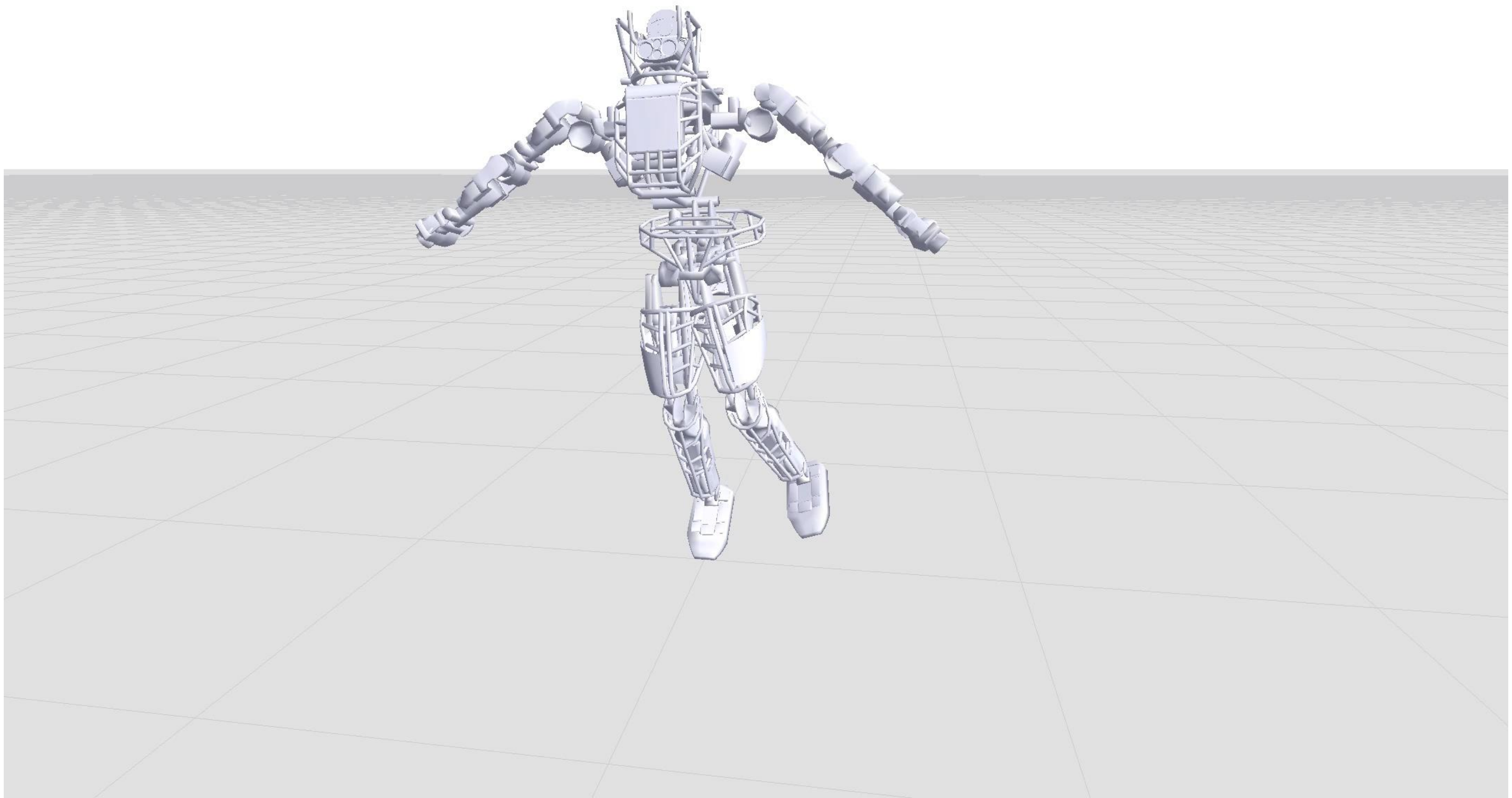
ROA volume ratio:

2N (states)	4	6	8	10	12
ρ_{dsos}/ρ_{sos}	0.38	0.45	0.13	0.12	0.09
ρ_{sdsos}/ρ_{sos}	0.88	0.84	0.81	0.79	0.79

[Majumdar, AAA, Tedrake, CDC]

Stabilizing ATLAS

- 30 states 14 control inputs Cubic dynamics



Done by **SDSOS Optimization**

[Majumdar, AAA, Tedrake, CDC]

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https://github.com/spot-toolbox/spotless

What can DSOS/SDSOS do in theory?

$$p(x) > 0, \forall x \in \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

- Is there always an SOS proof?

Yes, e.g. based on Putinar's Psatz.
(under a compactness assumption)

Putinar

If $p(x) > 0, \forall x \in S$,
then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x) g_i(x)$,
where σ_0, σ_i are sos

- Is there always an SDSOS proof?

- Is there always an DSOS proof?

Yes! In fact, a much stronger statement is true.

An optimization-free Positivstellensatz (1/2)

$$p(x) > 0, \forall x \in \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

$2d$ =maximum degree of p, g_i

$\Updownarrow$ Under compactness assumptions, i.e., $\{x \mid g_i(x) \geq 0\} \subseteq B(0, R)$

$\exists\, r \in \mathbb{N}$ such that

$$\left(f(v^2 - w^2) - \frac{1}{r}\left(\sum_i (v_i^2 - w_i^2)^2\right)^d + \frac{1}{2r}\left(\sum_i (v_i^4 + w_i^4)\right)^d\right) \cdot \left(\sum_i v_i^2 + \sum_i w_i^2\right)^{r^2}$$

has **nonnegative coefficients**,

where f is a form in $n + m + 3$ variables and of degree $4d$, which can be explicitly written from p, g_i and R.

[AAA, Hall, *Math of OR* '18] (2018 INFORMS Young Researchers Prize)

An optimization-free Positivstellensatz (2/2)

$$p(x) > 0 \text{ on } \{x \mid g_i(x) \geq 0\} \Leftrightarrow$$
$$\exists r \in \mathbb{N} \text{ s.t.} \left(f(v^2 - w^2) - \frac{1}{r}\left(\sum_i (v_i^2 - w_i^2)^2\right)^d + \frac{1}{2r}\left(\sum_i (v_i^4 + w_i^4)\right)^d \right) \cdot \left(\sum_i v_i^2 + \sum_i w_i^2\right)^{r^2}$$
has ≥ 0 coefficients

- $p(x) > 0$ on $\{x \mid g_i(x) \geq 0\} \Leftrightarrow f$ is pd
- **Result by Polya (1928):**

f **even** and pd $\Rightarrow \exists r \in \mathbb{N}$ such that $f(z) \cdot \left(\sum_i z_i^2\right)^r$ has nonnegative coefficients.

- Make $f(z)$ even by considering $f(v^2 - w^2)$. We lose positive definiteness of f with this transformation.
- Add the positive definite term $\frac{1}{2r}\left(\sum_i (v_i^4 + w_i^4)\right)^d$ to $f(v^2 - w^2)$ to make it positive definite. **Apply Polya's result.**
- The term $-\frac{1}{r}\left(\sum_i (v_i^2 - w_i^2)^2\right)^d$ ensures that the converse holds as well.

As a corollary, gives LP/SOCP-based converging hierarchies…
(Even forms with nonnegative coefficients are trivially dsos.)

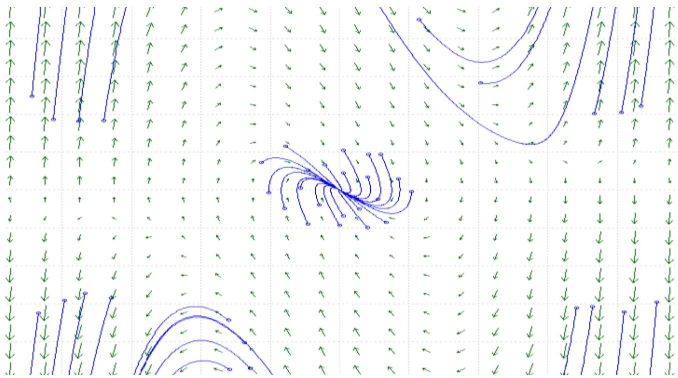
Part 2:
Asymptotic Stability of Polynomial Vector Fields

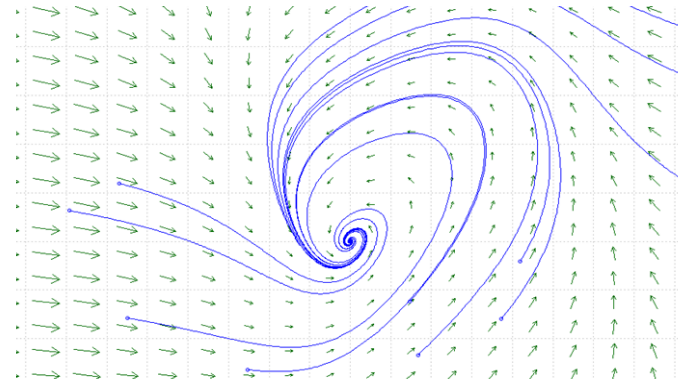
Asymptotic stability

$$\dot{x} = f(x) \qquad f : \mathbb{R}^n \to \mathbb{R}^n$$ polynomial with rational coefficients $$f(0) = 0$$

Example $\dot{x}_1 = -x_2 + \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 x_2$

$\dot{x}_2 = 3x_1 - x_1 x_2$



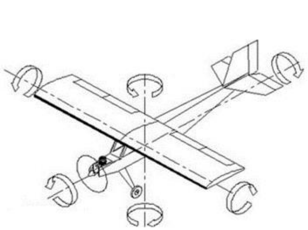


Locally Asymp. Stable (LAS) if

- $\forall \epsilon > 0, \exists \delta > 0$, s.t.
  $x(0) \in B_\delta \Rightarrow x(t) \in B_\epsilon \ \forall t$
- $\exists \alpha > 0$ s.t.
  $x(0) \in B_\alpha \Rightarrow \lim_{t \to \infty} x(t) = 0.$

Globally Asymp. Stable (GAS) if

- $\forall \epsilon > 0, \exists \delta > 0$, s.t.
  $x(0) \in B_\delta \Rightarrow x(t) \in B_\epsilon \ \forall t$
- $\forall x_0 \in \mathbb{R}^n$, $\lim_{t \to \infty} x(t) = 0.$



Complexity of deciding asymptotic stability?

$$\dot{x} = Ax$$

- d=1 (linear systems): decidable, and polynomial time
  - **Iff *A* is Hurwitz** (i.e., eigenvalues of *A* have negative real part)
  - **Quadratic Lyapunov functions** always exist:
    - $V(x) = x^T P x, \dot{V}(x) = x^T(A^T P + PA)x$
      $(P \succ 0, A^T P + PA \prec 0).$
  - A polynomial time algorithm is the following:
    - Solve $A^T P + PA = -I$
    - Check if P is positive definite

What if deg(*f*)>1? …

Complexity of deciding asymptotic stability?

What if deg(f)>1? ...

- **Conjecture of Arnol'd (1976):** undecidable (still open)

Fact: Existence of **polynomial Lyapunov functions**, together with a **computable upper bound** on the degree would imply decidability (e.g., by quantifier elimination)

Thm: Deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard.

[AAA]

(In particular, this rules out tests based on polynomially-sized convex programs.)

Thm: Deciding asymptotic stability of cubic *homogeneous* vector fields is strongly NP-hard.

Homogeneous means:

$$\dot{x} = f(x)$$

$$f(\lambda x) = \lambda^d f(x)$$

- All monomials in f have the same degree
- Local Asymptotic Stability = Global Asymptotic Stability

Proof

Thm: Deciding asymptotic stability of cubic homogeneous vector fields is strongly NP-hard.

Reduction from: **ONE-IN-THREE 3SAT**

$$\overset{1}{x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}$$

$$(\overset{1}{x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})$$

$$x_1 = 1, x_2 = 1, x_3 = 0$$

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

Goal: Design a cubic differential equation which is a.s. iff ONE-IN-THREE 3SAT has no solution

Proof (cont'd)

ONE-IN-THREE 3SAT

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4)$$

↓

Positivity of quartic forms

$$\begin{aligned} p(x) \;=\; & \textstyle\sum_{i=1}^{5} x_i^2(1-x_i)^2 \\ & +(x_1+(1-x_2)+x_4-1)^2+((1-x_2)+(1-x_3)+x_5-1)^2 \\ & +((1-x_1)+x_3+(1-x_5)-1)^2+(x_1+x_3+x_4-1)^2 \end{aligned}$$

$$p_h(x,y) = y^4 p\left(\tfrac{x}{y}\right)$$

↓

Asymptotic stability of cubic homogeneous vector fields

$$z := (x, y)$$

$$\dot{z} = -\nabla p_h(z)$$

Proof (cont'd)

Thm: Let $V(x)$ be a homogeneous polynomial. Then,
$V(x)$ is positive definite $\Leftrightarrow$ $\dot{x} = -\nabla V(x)$ is GAS

Proof: $\Rightarrow$

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle = -||\nabla V(x)||^2 \leq 0$$

$$V(x) = \frac{1}{4} x^T \nabla V(x)$$ implies strict decrease…

Apply Lyapunov's theorem.

$\Leftarrow$

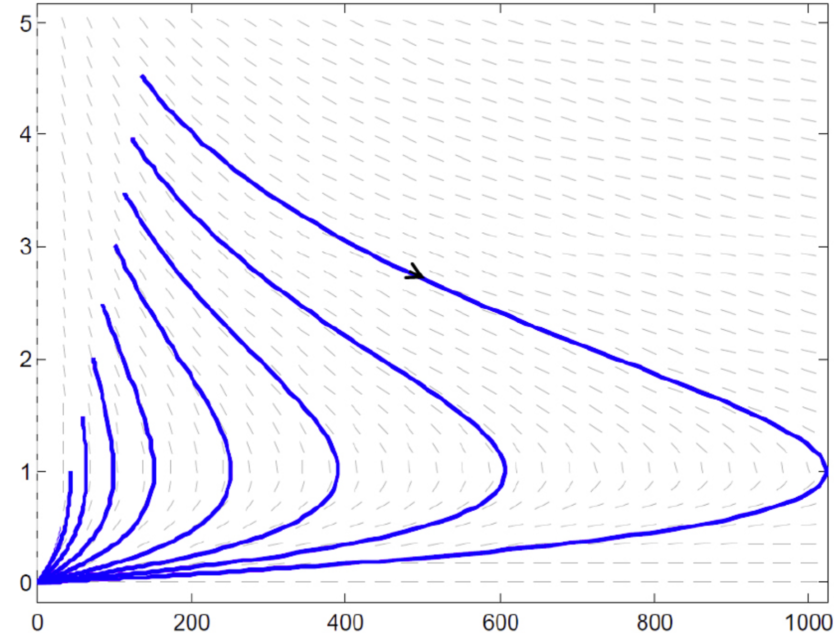
- $V(x)$ must be nonnegative because…
- If $V(x)$ were to vanish, its gradient would vanish also…

Nonexistence of polynomial Lyapunov functions (1/4)

$$\begin{aligned}\dot{x} &= -x + xy \\ \dot{y} &= -y\end{aligned}$$

Claim 1: System is GAS.

Claim 2: No polynomial Lyapunov function (of any degree) exists!



Proof:

$$\boxed{V(x,y) = \ln(1+x^2) + y^2}$$

$$\begin{aligned}\dot{V}(x,y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= -\frac{x^2+2y^2+x^2y^2+(x-xy)^2}{1+x^2}\end{aligned}$$

[AAA, Krstic, Parrilo, CDC'11]

Nonexistence of polynomial Lyapunov functions (2/4)

$$\begin{aligned}\dot{x} &= -x + xy \\ \dot{y} &= -y\end{aligned}$$

Claim 2: No polynomial Lyapunov function (of any degree) exists!

Proof:

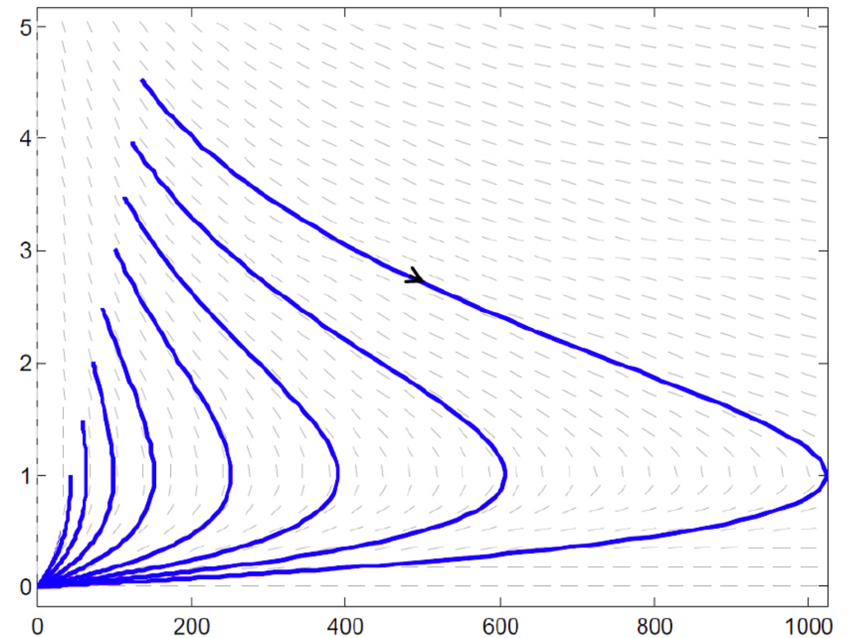
$$\begin{aligned}x(t) &= x(0)e^{[y(0)-y(0)e^{-t}-t]} \\ y(t) &= y(0)e^{-t}\end{aligned}$$

$$t^* = \ln(k)$$

$$(k, \alpha k) \longrightarrow (e^{\alpha(k-1)}, \alpha)$$

$$V(e^{\alpha(k-1)}, \alpha) < V(k, \alpha k)$$

Impossible. ∎



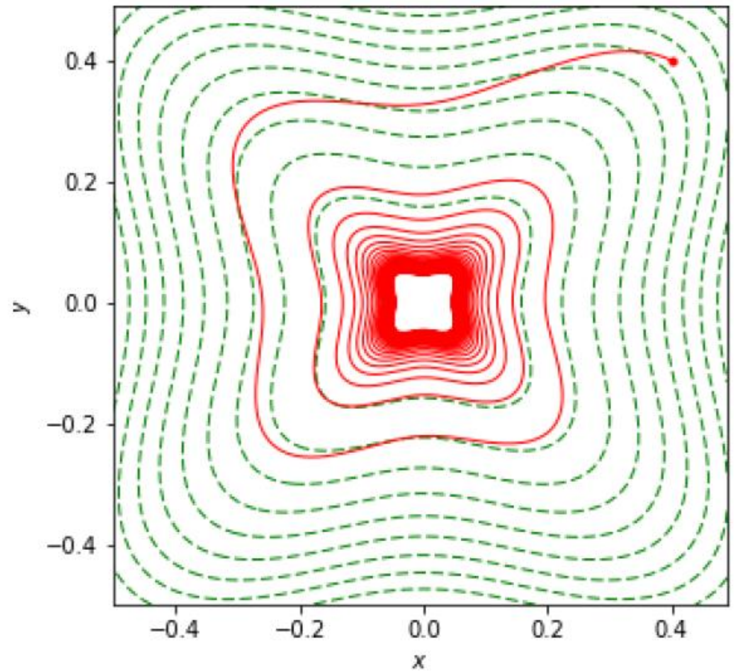
- No rational Lyapunov function either [AAA, El Khadir '18].
- But a quadratic Lyapunov function locally.

Nonexistence of polynomial Lyapunov functions (3/4)

$$f(x,y) = \begin{pmatrix} -2y(-x^4+2x^2y^2+y^4) \\ 2x(x^4+2x^2y^2-y^4) \end{pmatrix} - (x^2+y^2)\begin{pmatrix} 2x(x^4+2x^2y^2-y^4) \\ 2y(-x^4+2x^2y^2+y^4) \end{pmatrix}$$

Claim 1: System is GAS.

Claim 2: No polynomial Lyapunov function (of any degree) **even locally**!



Proof:

$$W(x,y) = \frac{x^4+y^4}{x^2+y^2}$$

Nonexistence of polynomial Lyapunov functions (4/4)

$$f(x,y) = \overbrace{\begin{pmatrix} -2y(-x^4 + 2x^2y^2 + y^4) \\ 2x(x^4 + 2x^2y^2 - y^4) \end{pmatrix}}^{f_0(x,y)} \overbrace{- (x^2+y^2) \begin{pmatrix} 2x(x^4 + 2x^2y^2 - y^4) \\ 2y(-x^4 + 2x^2y^2 + y^4) \end{pmatrix}}^{f_1(x,y)}$$

Claim 2: No polynomial Lyapunov function (of any degree) **even locally**!

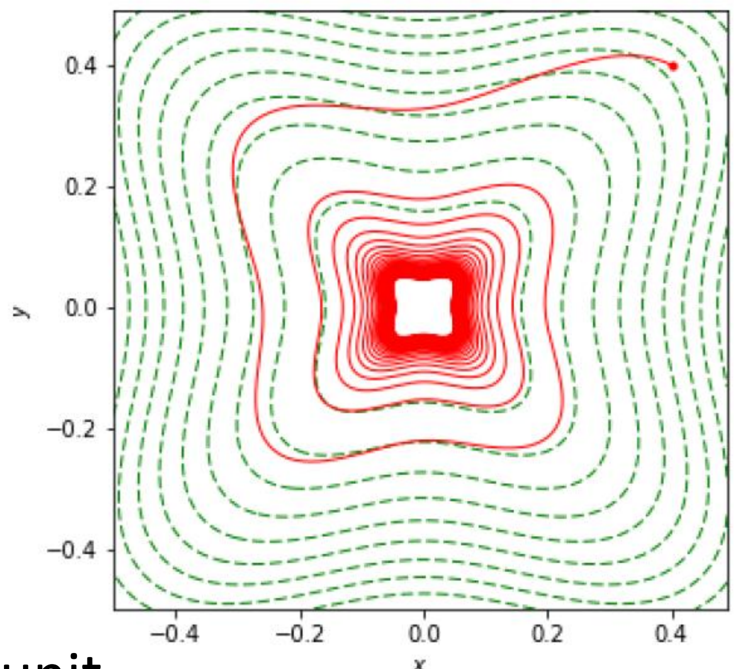
Proof idea:

Suppose we had one: $p = \sum_{k=0}^{\infty} p_k$

➔ $\langle \nabla p_{k_0}(x,y), f_0(x,y) \rangle \leq 0$

➔ $\langle \nabla p_{k_0}(x,y), f_0(x,y) \rangle = 0.$

➔ A polynomial must be constant on the unit level set of $W(x,y) = (x^4 + y^4)/(x^2 + y^2)$



Let's end on a positive note!

Thm. A **homogeneous** polynomial vector field is asymptotically stable iff it admits a rational Lyapunov function of the type

$$V(x) = \frac{p(x)}{\left(\sum_{i=1}^n x_i^2\right)^r}$$

where p is a homogeneous polynomial.

$$f(cx) = c^d f(x)$$

Linear case, $d = 1$

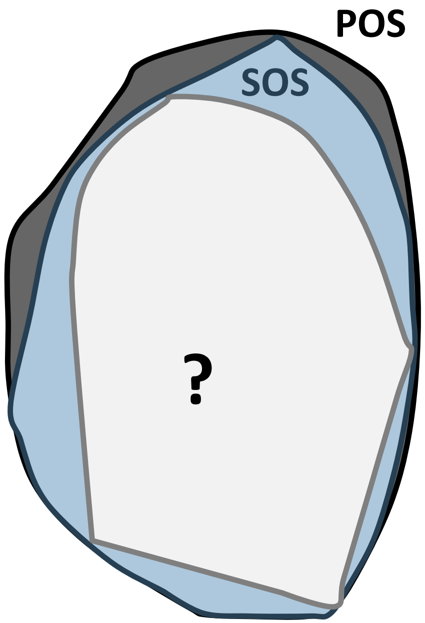
i.e. $f(x) = Ax$

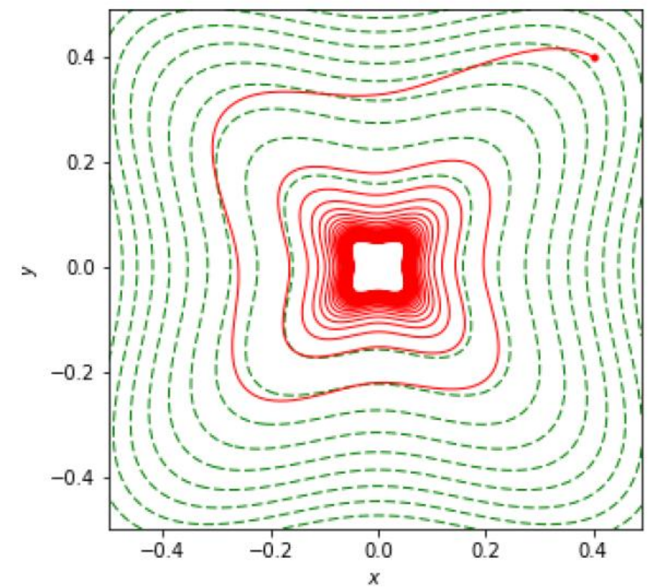
$r = 0, p(x) = x^T P x$

- We show that V and $-\dot{V}$ both have "strict SOS certificates."
  → V can be found by SDP!
- Useful also for local asym. stability of non-homogeneous systems.
- We show that unlike the linear case, the degree of V cannot be bounded as a function of the dimension and degree of f.

[AAA, El Khadir, *TAC, accepted with minor revision*]

Main messages





- SDP-free alternatives to SOS
  - DSOS/SDSOS (LP and SOCP)
  - Infeasibility certificates based on poly-poly multiplication

- No pseudo-poly-time algorithm for asymptotic stability of poly vector fields
- Polynomail Lyapunov function can fail even locally
- Rational Lyapunov functions deserve more attention

Want to know more? aaa.princeton.edu