Nonnegative Polynomials in Optimization and Control

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What is this talk about?

"Optimization over <u>nonnegative</u> polynomials"

1. **Introduction to sum of squares (sos) programming**
   Underlying numerical engine: SDP

2. **"dsos and sdsos" programming**
   Underlying numerical engine: LP/SOCP

Joint work with Anirudha Majumdar (MIT)

Optimization over Nonnegative Polynomials

Defn. A polynomial $p(x) := p(x_1, \ldots, x_n)$ is nonnegative if $p(x) \geq 0, \forall x \in \mathbb{R}^n$.

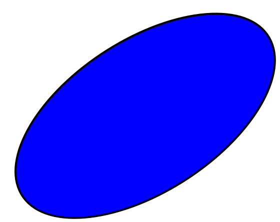
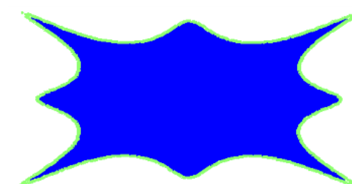
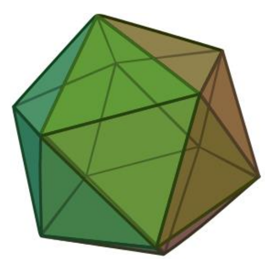
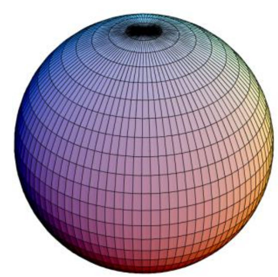
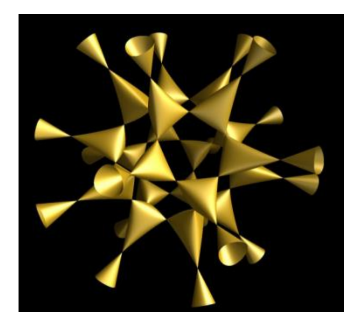
Ex. Decide if the following polynomial is nonnegative:

$$\begin{aligned} p(x) \;=\; & x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ & -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \end{aligned}$$

Basic semialgebraic set:

$$\{x \in \mathbb{R}^n |\; f_i(x) \geq 0, h_i(x) = 0\}$$

Ex.
$$2x_1 + 5x_1^2x_2 - x_3 \geq 0$$
$$5 - x_1^3 + 2x_1x_3 = 0$$

Ubiquitous in computational mathematics!

Why would you want to do this?!

- Let’s start with four application areas…

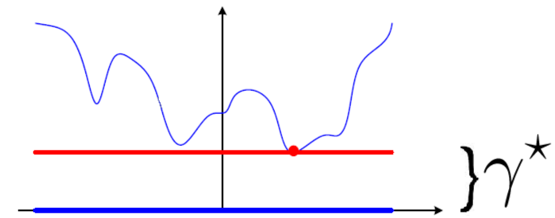
1. Polynomial Optimization

$$\min_x p(x)$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

Decidable, but intractable
(includes your favorite NP-complete problem)

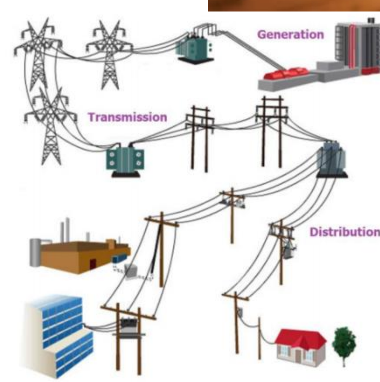
Equivalent formulation:

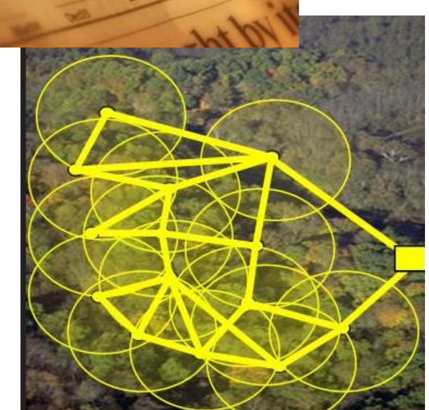
$$\max_\gamma \gamma$$
$$p(x) - \gamma \geq 0$$
$$\forall x \in \{f_i(x) \leq 0,\ h_i(x) = 0\}$$



- **Many applications:**

- Combinatorial optimization
- Option pricing with moment information
- The optimal power flow (OPF) problem
- Sensor network localization
- ...





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2. Infeasibility Certificates in Discrete Optimization

- **PARTITION**

- **Input:** A list of positive integers $a_1, \dots, a_n$.

- **Question:** Can you split them into to bags such that the sum in one equals the sum in the other?

$\{5,2,1,6,3,8,5,4,1,1,10\}$

- Note that the YES answer is easy to certify.

- How would you certify a NO answer?

2. Infeasibility Certificates in Discrete Optimization

- **PARTITION**

- **Input:** A list of positive integers $a_1, \ldots, a_n$.

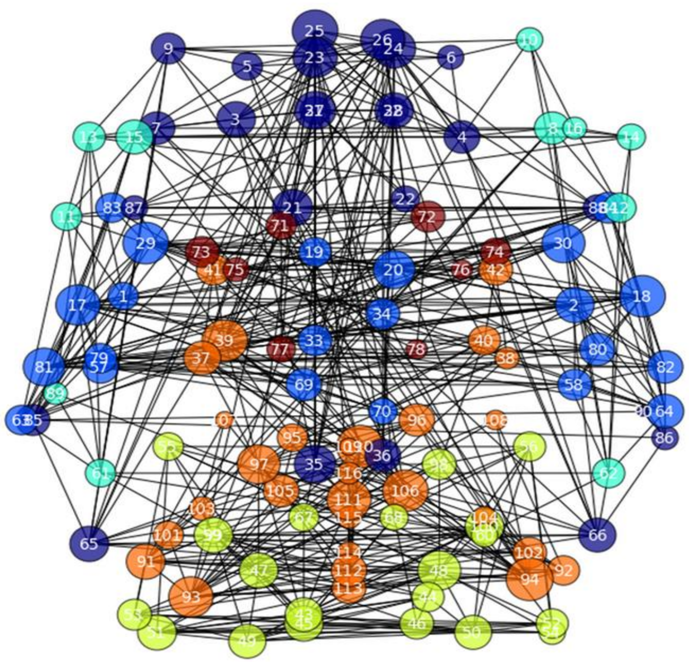
- **Question:** Can you split them into to bags such that the sum in one equals the sum in the other?

$$[a_1, a_2, \ldots, a_n]$$

$$\exists ?\ x_i \in \{-1, 1\} \text{ s.t. } \sum_{i=1}^{n} x_i a_i = 0$$

Infeasible iff $\displaystyle \sum_{i=1}^{n} (x_i^2 - 1)^2 + (\sum_{i=1}^{n} x_i a_i)^2 > 0$

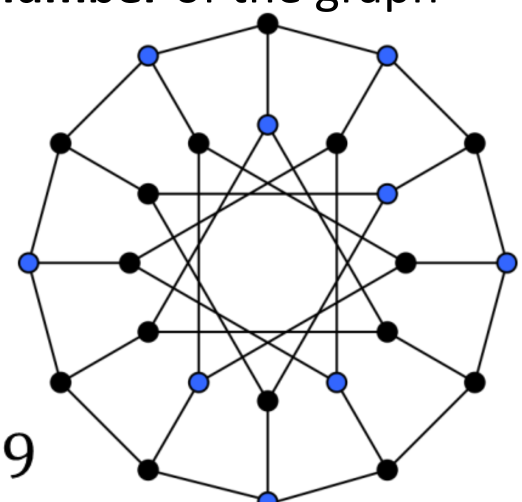
2. Discrete Optimization (Cont'd.)



- **How many final exams** can the President schedule on the same day at UCL, such that no student has to take more than one?

- **Nodes:** course numbers
- **Edges:** iff there is at least one student who is taking both courses

- Need the **independent set number** of the graph



$$\alpha(G) = 9$$

How to certify optimality?

- **A theorem of Motzkin & Straus (1965):**

$$\alpha(G) \leq k$$

if and only if

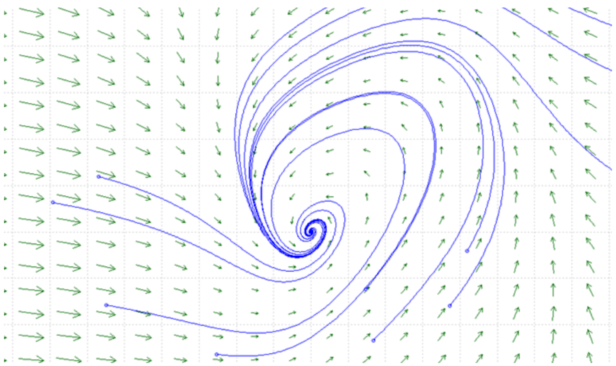
$$-2k \sum_{(i,j)\in\overline{E}} x_i x_j y_i y_j - (1-k)\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

is nonnegative.

- Similar algebraic formulations for other combinatorial optimization problems…

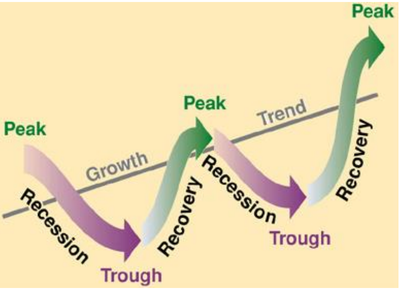
3. Dynamical Systems & Trajectory Optimization

$$\dot{x} = f(x)$$
$$x_{k+1} = f(x_k)$$

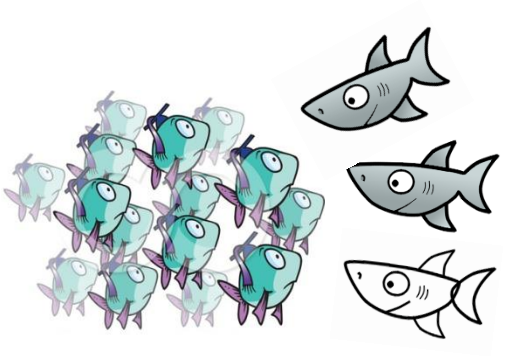


Properties of interest:

- Stability of equilibrium points
- Boundedness of trajectories
- Invariance
- Safety, collision avoidance
- …



Dynamics of prices



Equilibrium populations

Spread of epidemics

Robotics

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What does this have to do with optimization?

Questions about dynamical systems (e.g. stability, safety)

Lyapunov Theory →

Search for functions satisfying certain properties (e.g. nonnegativity,convexity)

Ex. Lyapunov's stability theorem.

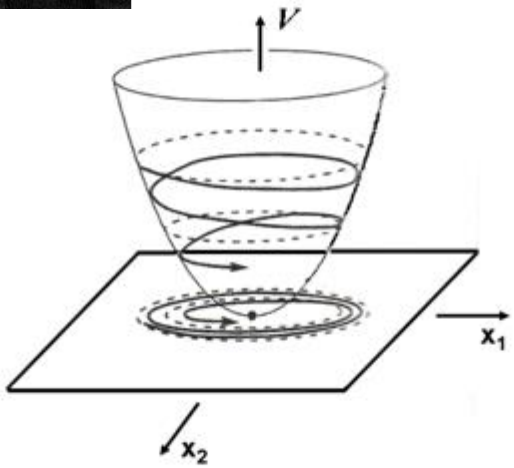
$$\dot{x} = f(x)$$

Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$

$$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$$

$$V(x) > 0$$

$$-\dot{V}(x) > 0 \quad \Rightarrow \text{GAS}$$

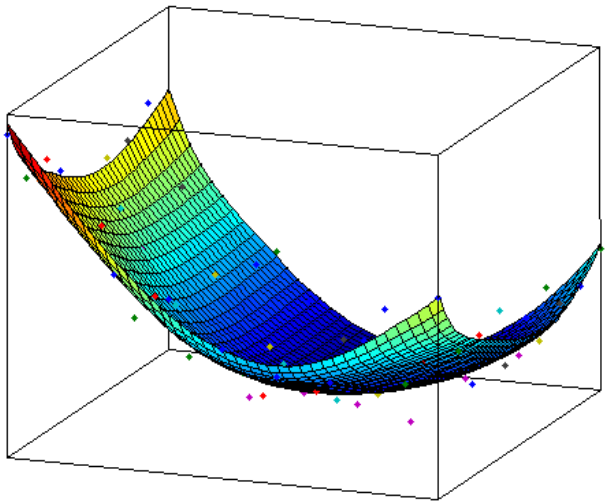


(similar local version)

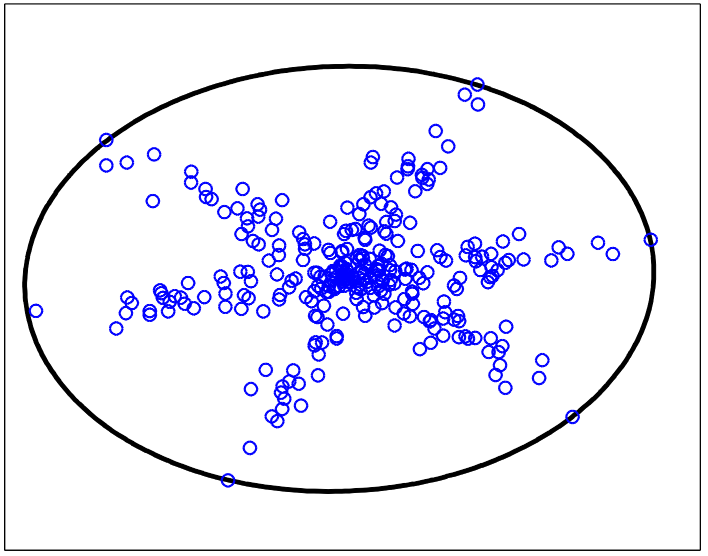
4: Statistics and Machine Learning

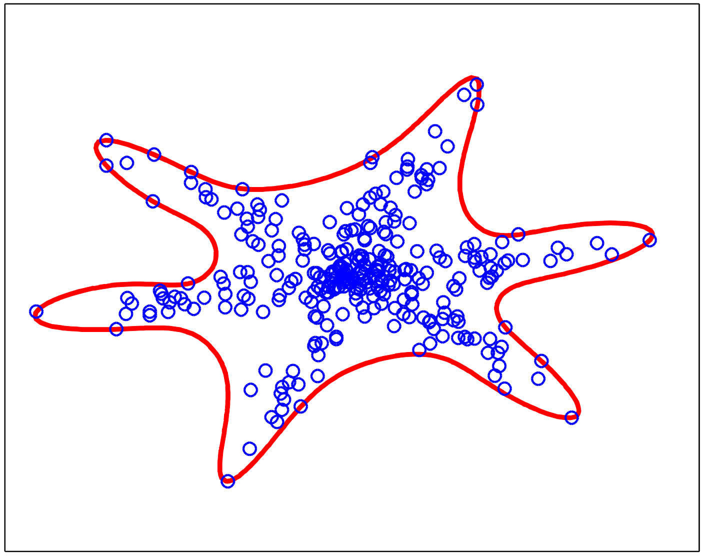
- **Shape-constrained regression**
  - **e.g., convex regression**

$$p(x) \text{ convex} \Leftrightarrow y^T \nabla^2 p(x) y \text{ psd}$$



- **Clustering with semialgebraic sets**





[AAA, Luss, Malioutov, '14]

How would you prove nonnegativity?

Ex. Decide if the following polynomial is nonnegative:

$$\begin{aligned} p(x) \;=\;& x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ & -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \end{aligned}$$

- Not so easy! (In fact, **NP-hard for degree ≥ 4**)
- But what if I told you:

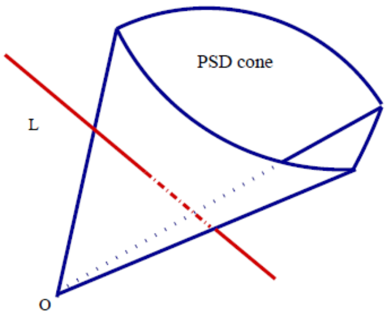
$$\begin{aligned} p(x) \;=\;& (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 \\ & +(4x_2^2 - x_3^2)^2. \end{aligned}$$

Natural questions:

- Is it any easier to test for a sum of squares (SOS) decomposition?
- Is every nonnegative polynomial SOS?

Sum of Squares and Semidefinite Programming

[Lasserre], [Nesterov], [Parrilo]



Q. Is it any easier to decide sos?

- Yes! Can be reduced to a **semidefinite program (SDP)**
  - –A broad generalization of linear programs
  - –Can be solved efficiently (e.g., using interior point algorithms)

[Nesterov, Nemirovski], [Alizadeh]

- Can also efficiently **search and optimize** over sos polynomials
- Numerous applications…

SOS→SDP

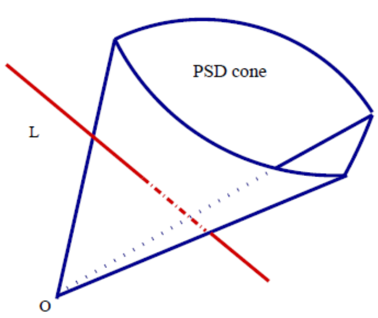
Thm: A polynomial ***p(x)*** of degree ***2d*** is sos if and only if there exists a matrix Q such that

$$Q \succcurlyeq 0,$$
$$p(x) = z(x)^T Q z(x),$$

where z is the vector of monomials of degree up to ***d***

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]^T$$

The set of such matrices Q forms the feasible set of an SDP.



Example

$$p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4$$

$$p(x) = z^TQz \qquad z = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2)^T$$

$$Q = \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ -3 & 9 & 0 & -3 & 0 & -6 \\ 0 & 0 & 16 & 0 & 0 & -4 \\ 1 & -3 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 2 & -6 & 4 & 2 & 0 & 5 \end{pmatrix}$$

$$Q = \sum_{i=1}^{3} a_i a_i^T$$

$$a_1 = (1, -3, 0, 1, 0, 2)^T, \quad a_2 = (0, 0, 0, 1, -1, 0)^T, \quad a_3 = (0, 0, 4, 0, 0, -1)^T$$

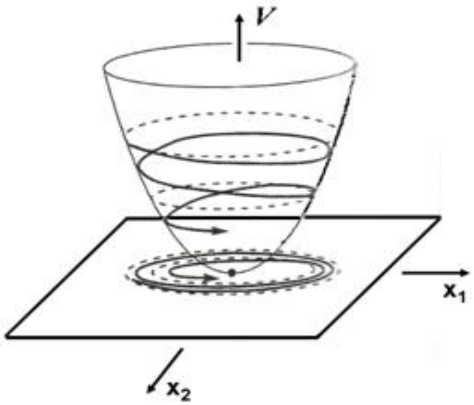
$$p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2.$$

Lyapunov theory with sum of squares (sos) techniques

Questions about dynamical systems (e.g., stability, safety) → (Lyapunov Theory) → Search for sign-definite functions → Polynomial inequalities → (SOS) → Semidefinite programming

Ex. Lyapunov's stability theorem.

$$\dot{x} = f(x)$$



Lyapunov function

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

$$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$$

$$\begin{matrix} V(x) & \text{sos} \\ -\dot{V}(x) & \text{sos} \end{matrix} \Rightarrow \begin{matrix} V(x) > 0 \\ -\dot{V}(x) > 0 \end{matrix} \Rightarrow \text{GAS}$$

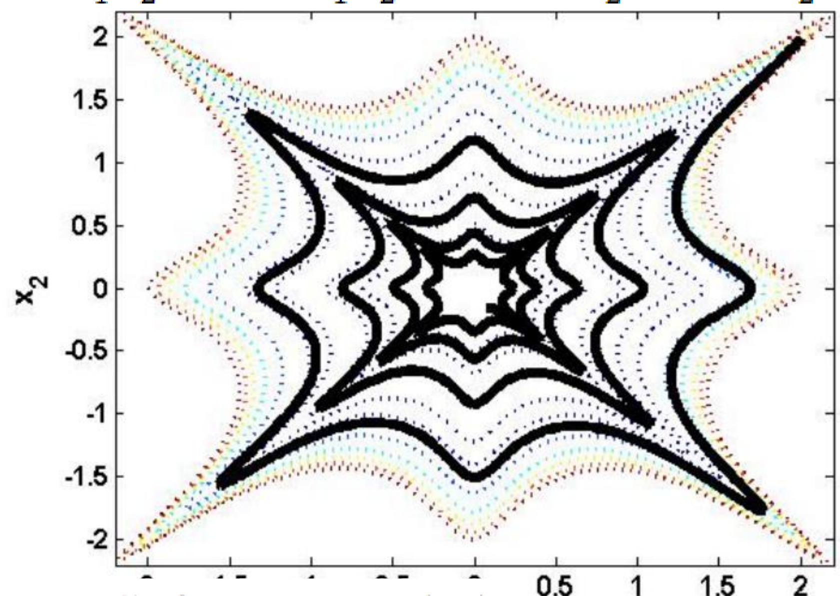
(similar local version)

Global stability

$$\left.\begin{array}{r} V(x) \ \text{sos} \\ -\dot{V}(x) \ \text{sos} \end{array}\right\} \Rightarrow \begin{array}{r} V(x) > 0 \\ -\dot{V}(x) > 0 \end{array} \Rightarrow \text{GAS}$$

Example.

$$\begin{aligned} \dot{x}_1 &= -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7 \\ \dot{x}_2 &= -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7 \end{aligned}$$



Output of SDP solver:

$$\begin{aligned} V &= 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4 \\ &\quad + 0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8. \end{aligned}$$

Hilbert's 1888 Paper

Q. SOS $\overset{?}{\Leftarrow}$ Nonnegativity

Polynomials

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	no
3	yes	no	no
≥4	yes	no	no

Forms (homog. polynomials)

n,d	2	4	≥6
1	yes	yes	yes
2	yes	yes	yes
3	yes	yes	no
≥4	yes	no	no

From Logicomix

Motzkin (1967):

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6$$

Robinson (1973):

$$\begin{aligned} R(x_1, x_2, x_3, x_4) \quad = \quad & x_1^2(x_1 - x_4)^2 + x_2^2(x_2 - x_4)^2 + x_3^2(x_3 - x_4)^2 \\ & + 2x_1 x_2 x_3 (x_1 + x_2 + x_3 - 2x_4) \end{aligned}$$

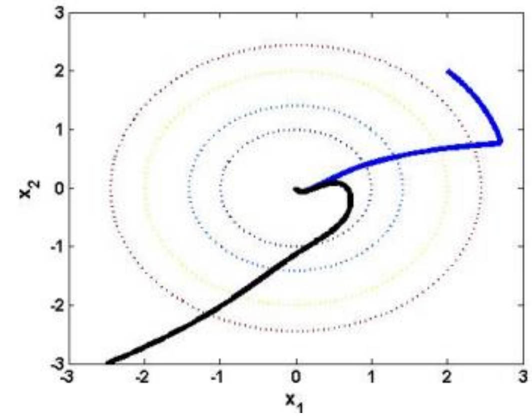
Failure of converse implications for Lyapunov analysis

$$\dot{x}_1 = -x_1^3 x_2^2 + 2x_1^3 x_2 - x_1^3 + 4x_1^2 x_2^2 - 8x_1^2 x_2 + 4x_1^2 - x_1 x_2^4 + 4x_1 x_2^3 - 4x_1 + 10x_2^2$$

$$\dot{x}_2 = -9x_1^2 x_2 + 10x_1^2 + 2x_1 x_2^3 - 8x_1 x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2$$

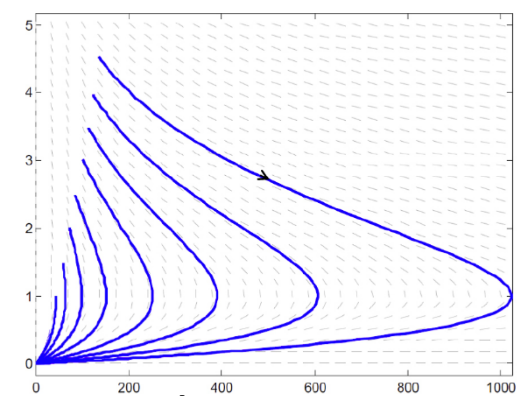
[AAA,Parrilo]

- $V(x) = x_1^2 + x_2^2$ proves GAS.
- SOS fails to find *any* quadratic Lyapunov function.



$$\dot{x} = -x + xy$$
$$\dot{y} = -y$$

[AAA, Krstic, Parrilo]



- Globally asymptotically stable.
- But no polynomial Lyapunov function of any degree exists!

These examples are to be expected for complexity reasons

Thm: Deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard.

[AAA]

Implication:

- Unless P=NP, there cannot be *any* polynomial time (or even pseudo-polynomial time) algorithm.

- In particular, the size of SOS certificates must be at least exponential.

Similar NP-hardness results for other problems

1. Inclusion of the unit ball in region of attraction (d=3)
2. Invariance of the unit ball (d=3)
3. Invariance of a quartic semialgebraic set (d=1)
4. Boundedness of trajectories (d=3)
5. Stability in the sense of Lyapunov (d=4)
6. Local attractivity (d=3)
7. Local collision avoidance (d=4)
8. Existence of a quadratic Lyapunov function (d=3)
9. Existence of a stabilizing control law (d=3)
10. Local asymptotic stability for trigonometric vector fields (d=4)

[AAA, Majumdar, Tedrake]

The good news

- In relatively small dimensions and degrees, it seems difficult to construct nonnegative polynomials that are not sos

- Especially true if additional structure is required

- For example, the following is OPEN:

Construct a *convex*, nonnegative polynomial that is not sos

(known to exist in high dimensions via a non-constructive proof of Blekherman)

- Empirical evidence from various domains over the last decade:

SOS is a very powerful relaxation.

Hilbert's 17th Problem (1900)

Q. p **nonnegative** $\overset{?}{\Rightarrow}$ $$p = \sum_i \left(\frac{g_i}{q_i}\right)^2$$

- Artin (1927): **Yes!**

- Implications:
  - $p \geq 0 \Rightarrow \exists \boldsymbol{h}$ **sos** such that $\boldsymbol{p}.\boldsymbol{h}$ **sos**
  - **Reznick:** (under mild conditions) can take $\boldsymbol{h} = \left(\sum_i \boldsymbol{x}_i^2\right)^{\boldsymbol{r}}$
  - Certificates of nonnegativity can *always* be given with sos (i.e., with semidefinite programming)!
  - We'll see how the Positivstellensatz generalizes this even further…

Positivstellensatz: a complete algebraic proof system

- Let's motivate it with a toy example:

Consider the task of proving the statement:

$$\forall a, b, c, x,\ \ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0$$

Short algebraic proof (certificate):

$$b^2 - 4ac = (2ax + b)^2 - 4a(ax^2 + bx + c)$$

- The Positivstellensatz vastly generalizes what happened here:
  - Algebraic certificates of infeasibility of any system of polynomial inequalities (or algebraic implications among them)
  - **Automated** proof system (via semidefinite programming)

Positivstellensatz: a generalization of Farkas lemma

Farkas lemma (1902):

$$Ax = b \text{ and } x \geq 0 \quad \text{is infeasible}$$

$$\Updownarrow$$

There exists a y such that $y^T A \geq 0$ and $y^T b < 0$.

(The S-lemma is also a theorem of this type for quadratics)

Positivstellensatz

Stengle (1974):

The basic semialgebraic set

$$K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, k\}$$

is empty

$$\Updownarrow$$

there exist polynomials $t_1, \ldots, t_k$ *and sum of squares polynomials* $s_0, s_1, \ldots, s_m, s_{12}, s_{13}, \ldots, s_{m-1m}, s_{123}, \ldots, s_{m-2m-1m}, \ldots, s_{12\ldots m}$ *such that*

$$\begin{aligned} -1 \quad = \quad & \sum_{i=1}^{k} t_i(x)h_i(x) + s_0(x) + \sum_{\{i\}} s_i(x)g_i(x) \\ + \quad & \sum_{\{i,j\}} s_{ij}(x)g_i(x)g_j(x) + \sum_{\{i,j,k\}} s_{ijk}(x)g_i(x)g_j(x)g_k(x) \\ + \quad & \cdots + s_{ijk\ldots m}(x)g_i(x)g_j(x)g_k(x)\ldots g_m(x). \end{aligned}$$

- Comments:
  - Hilbert's 17th problem is a straightforward corollary
  - Other versions due to Shmudgen and Putinar (can look simpler)

Parrilo/Lasserre SDP hierarchies

Recall POP:

$$\begin{array}{ll} \text{minimize} & p(x) \\ \text{subject to} & x \in K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_i(x) = 0\} \end{array}$$

Idea:

obtain the largest lower bound by finding the largest γ for which the set $\{x \in K, p(x) \leq \gamma\}$ is empty.

certify this emptiness by finding Positivstellensatz certificates.

$$\begin{aligned} -1 \;=\; & \sum_{i=1}^{k} t_i(x)h_i(x) + s_0(x) + \sum_{\{i\}} s_i(x)g_i(x) \\ + \; & \sum_{\{i,j\}} s_{ij}(x)g_i(x)g_j(x) + \sum_{\{i,j,k\}} s_{ijk}(x)g_i(x)g_j(x)g_k(x) \\ + \; & \cdots + s_{ijk\ldots m}(x)g_i(x)g_j(x)g_k(x)\ldots g_m(x) \end{aligned}$$

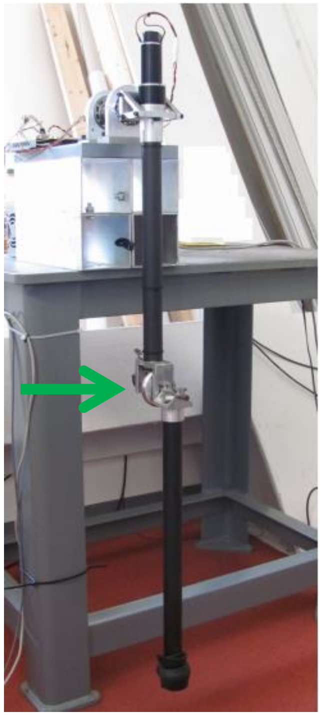
In level l of the hierarchy, degree of the polynomials t_i and the sos polynomails s_i is bounded by l.

Comments:

- Each fixed level of the hierarchy is an SDP of polynomial size
- Originally, Parrilo's hierarchy is based on Stengle's Psatz, whereas Lasserre's is based on Putinar's Psatz

Local stability – SOS on the Acrobot

https://www.youtube.com/watch?v=FeCwtvrD76I

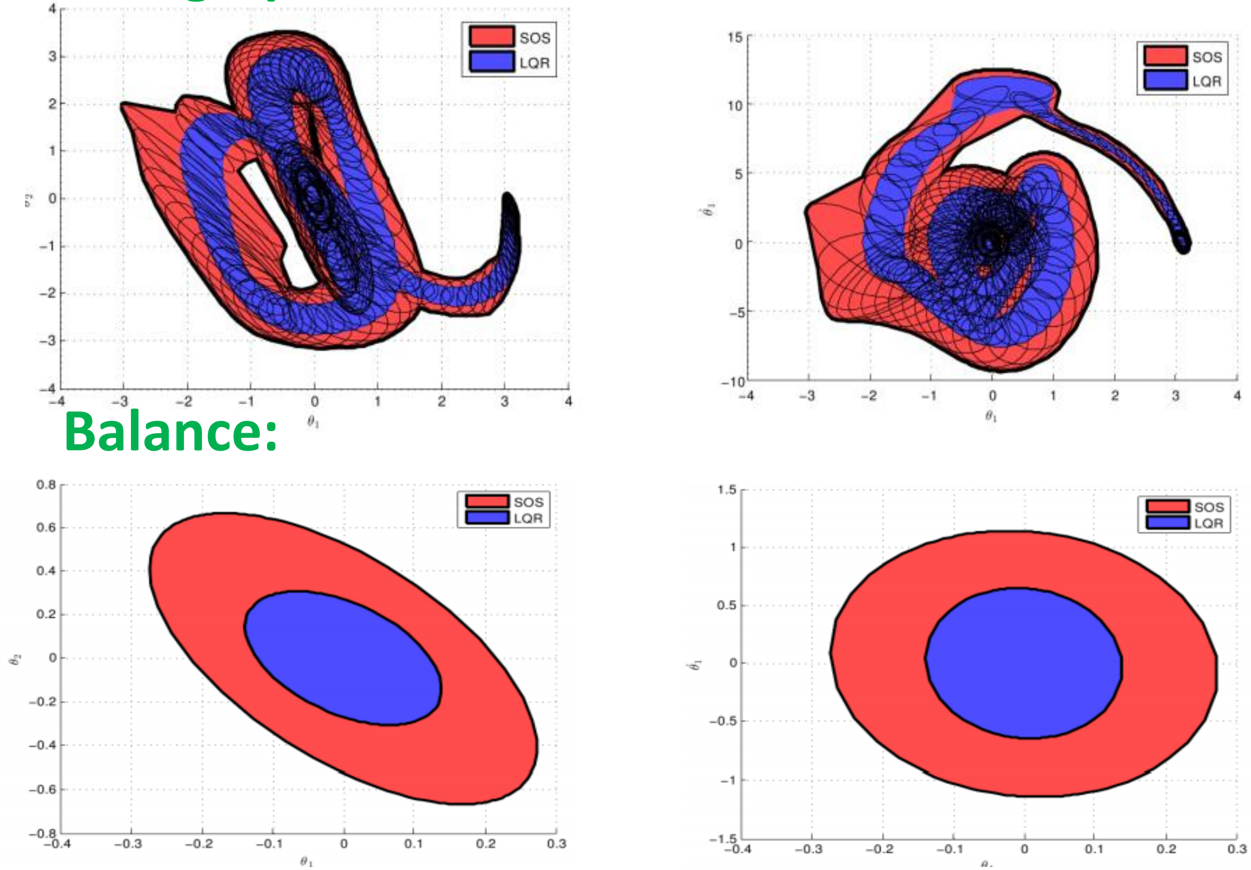


(4-state system)

Controller designed by SOS

Swing-up:

Balance:



[Majumdar, AAA, Tedrake]

(Best paper award - *IEEE Conf. on Robotics and Automation*, ’13)

DSOS and SDSOS Optimization

Practical limitations of SOS

- **Scalability** is often a real challenge!!

Thm: ***p(x)*** of degree ***2d*** is sos if and only if

$$p(x) = z^T Q z \quad Q \succeq 0$$

$$z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]^T$$

- The size of the Gram matrix is:

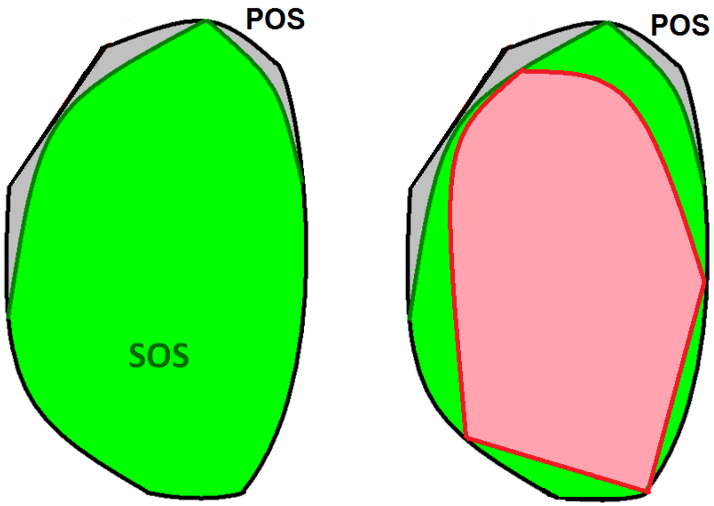
$$\binom{n+d}{d} \times \binom{n+d}{d}$$

- Polynomial in n for fixed d, but grows quickly
  - **The semidefinite constraint is expensive**
- E.g., local stability analysis of a 20-state cubic vector field is typically an SDP with ~1.2M decision variables and ~200k constraints

Many interesting approaches to tackle this issue…

- Techniques for exploiting structure (e.g., symmetry and sparsity)
  - [Gatermann, Parrilo], [Vallentin], [de Klerk, Sotirov], [Papachristodoulou et al.], …
- Customized algorithms (e.g., first order or parallel methods)
  - [Bertsimas, Freund, Sun], [Nie, Wang], [Peet et al.], …

Our approach [AAA, Majumdar]:



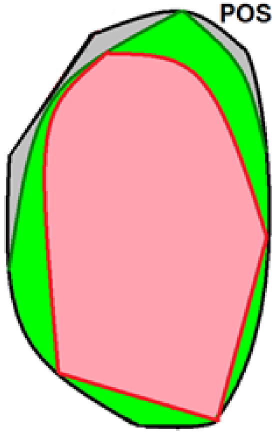
- Let's not work with SOS to begin with…
- Give other sufficient conditions for non
  perhaps stronger than SOS, but hopefully cheaper

Not totally clear a priori how to do this…

Consider, e.g., the following two sets:

1) All polynomials that are **sums of 4th powers of polynomials**
2) All polynomials that are **sums of 3 squares of polynomials**

Both sets are clearly inside the SOS cone



- But linear optimization over either set is intractable!
- So set inclusion doesn't mean anything in terms of complexity
- We have to work a bit harder…

dsos and sdsos

Defn. A polynomial ***p*** is *diagonally-dominant-sum-of-squares (dsos)* if it can be written as:

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} \beta_{ij}^{+}(m_i + m_j)^2 + \beta_{ij}^{-}(m_i - m_j)^2,$$

for some monomials m_i, m_j

and some nonnegative constants $\alpha_i, \beta_{i,j}$.

Defn. A polynomial ***p*** is *scaled-diagonally-dominant-sum-of-squares (sdsos)* if it can be written as:

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} (\beta_i^{+} m_i + \gamma_j^{+} m_j)^2 + (\beta_i^{-} m_i - \gamma_j^{-} m_j)^2,$$

for some monomials m_i, m_j

and some constants $\alpha_i \geq 0, \beta_i, \gamma_i$.

Obvious: $DSOS_{n,d} \subseteq SDSOS_{n,d} \subseteq SOS_{n,d} \subseteq POS_{n,d}$

r-dsos and r-sdsos

Defn. A polynomial $\boldsymbol{p}$ is *r-diagonally-dominant-sum-of-squares* ***(r-dsos)*** if

$$p \cdot \left(\sum_i x_i^2\right)^r$$

is dsos.

Defn. A polynomial $\boldsymbol{p}$ is *r-scaled-diagonally-dominant-sum-of-squares* ***(r-sdsos)*** if

$$p \cdot \left(\sum_i x_i^2\right)^r$$

is sdsos.

Allows us to develop a ***hierarchy*** of relaxations…

dd and sdd matrices

Defn. A symmetric matrix A is *diagonally dominant* ***(dd)*** if

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}| \ \ for \ \ all \ \ i.$$

Defn*. A symmetric matrix A is *scaled diagonally dominant* ***(sdd)*** if there exists a diagonal matrix $D>0$ s.t.

DAD is dd.

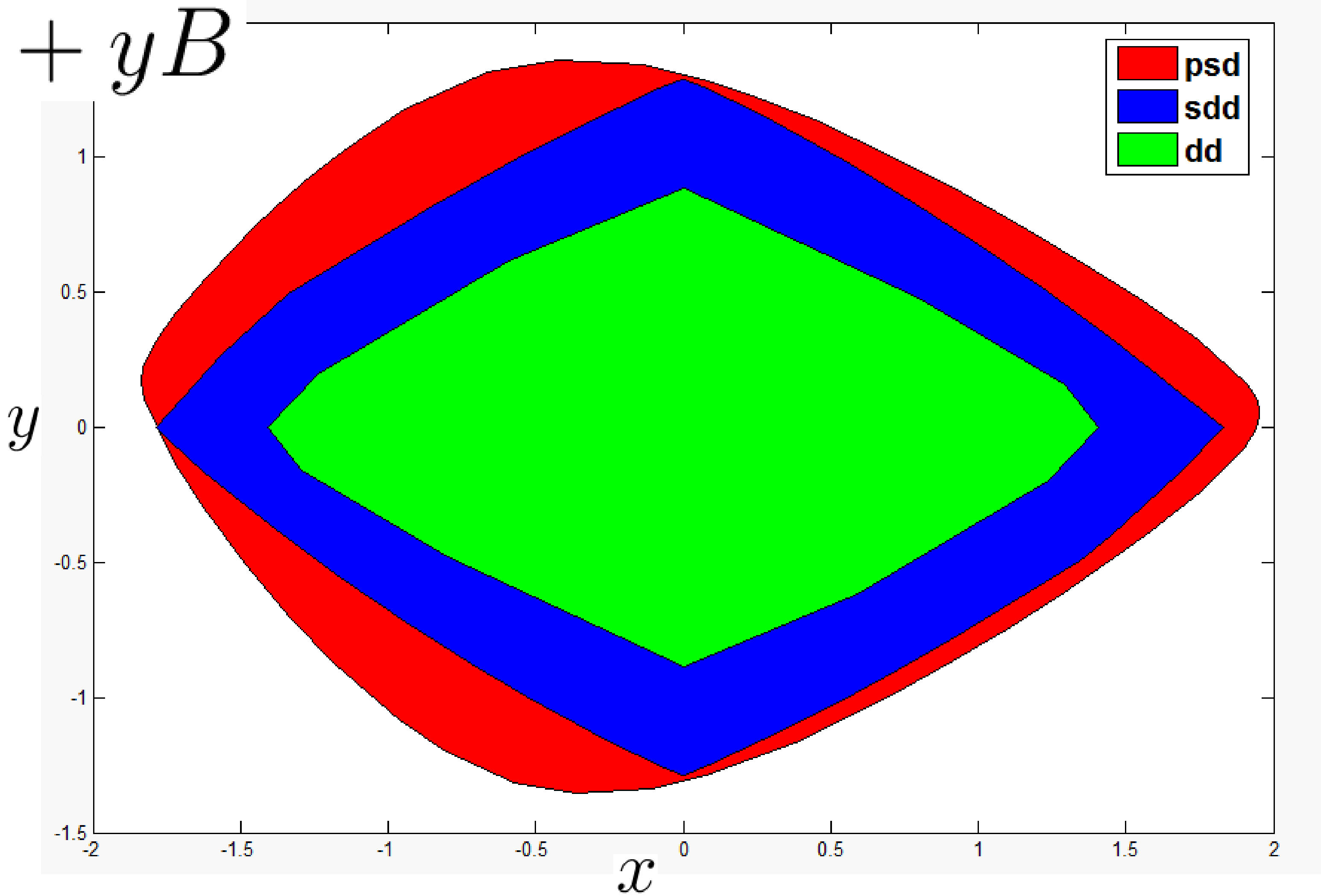
$$dd \Rightarrow sdd \Rightarrow psd$$

Greshgorin's circle theorem

*Thanks to Pablo Parrilo for telling us about sdd matrices.

$I + xA + yB$

A, B
10×10
random



Optimization over these sets is an SDP, SOCP, LP !!

Two natural matrix programs: DDP and SDPP

LP:

$$\min \langle C, X \rangle$$
$$A(X) = b$$
$$X \quad \text{diagonal\&nonnegative}$$

DDP:

$$\min \langle C, X \rangle$$
$$A(X) = b$$
$$X \quad \text{dd}$$

SDDP:

$$\min \langle C, X \rangle$$
$$A(X) = b$$
$$X \quad \text{sdd}$$

SDP:

$$\min \langle C, X \rangle$$
$$A(X) = b$$
$$X \succeq 0$$

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From matrices to polynomials

Thm. A polynomial ***p*** is ***dsos***

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} \beta_{ij}^+ (m_i + m_j)^2 + \beta_{ij}^- (m_i - m_j)^2,$$

if and only if

$$p(x) = z^T(x) Q z(x)$$

$$Q \quad dd$$

Thm. A polynomial ***p*** is ***sdsos***

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} (\beta_i^+ m_i + \gamma_j^+ m_j)^2 + (\beta_i^- m_i - \gamma_j^- m_j)^2,$$

if and only if

$$p(x) = z^T(x) Q z(x)$$

$$Q \quad sdd$$

Optimization over r-dsos and r-dsos polynomials

- Can be done by LP and SOCP respectively!
- Commercial solvers such as CPLEX and GUROBI are very mature (very fast, deal with numerical issues)
- iSOS: add-on to SPOTless (package by Megretski, Tobenkin, Permenter –MIT)

https://github.com/spot-toolbox/spotless

How well does it do?!

- We show encouraging experiments from:
  Control, polynomial optimization, statistics, copositive programming, cominatorial optimization, options pricing, sparse PCA, etc.
- And we'll give Positivstellensatz results (converse results)

First observation: r-dsos can outperform sos

The Motzkin polynomial:

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6$$

psd but *not* sos!

…but it's 2-dsos.
(certificate of nonnegativity using LP)

Another ternary sextic:

$$p(x_1, x_2, x_3) = x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2$$

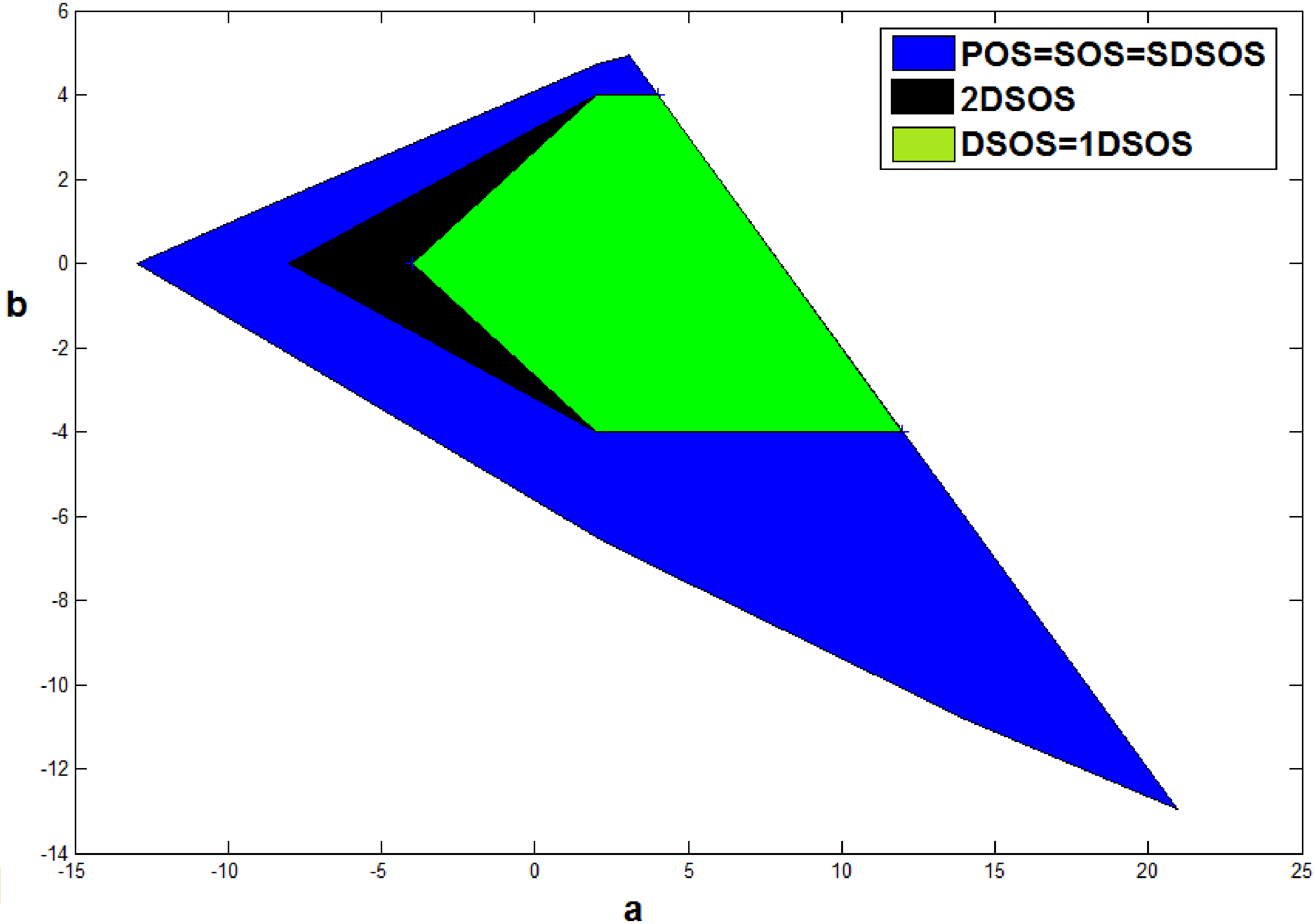
***not* sos but 1-dsos (hence psd)**

A parametric family of polynomials

$$p(x) = 2x_1^4 + cx_2^4 + ax_1^2x_2^2 + bx_1^3x_2$$

Compactify:

$$p(x) = 2x_1^4 + (8 - a - b)x_2^4 + ax_1^2x_2^2 + bx_1^3x_2$$



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Minimizing a form on the sphere

$$\min_{x \in \mathcal{S}^{n-1}} p(x)$$

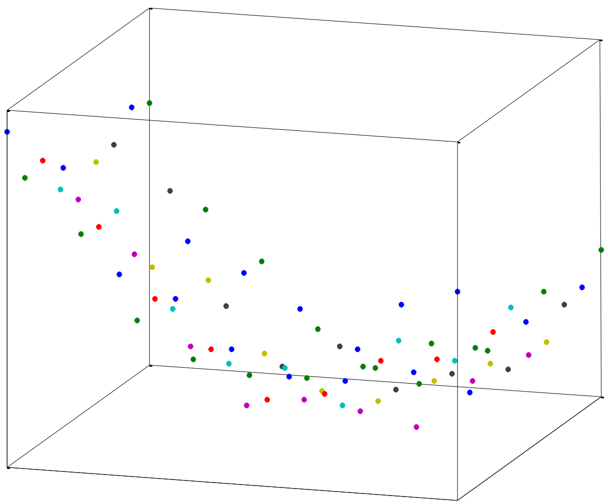
- degree=4; all coefficients present – generated randomly
- PC: 3.4 GHz, 16 Gb RAM

n=10	Lower bound	Run time (secs)	n=15	Lower bound	Run time (secs)	n=20	Lower bound	Run time (secs)
sos (sedumi)	-1.920	1.01	sos (sedumi)	-3.263	165.3	sos (sedumi)	-3.579	5749
sos (mosek)	-1.920	0.184	sos (mosek)	-3.263	5.537	sos (mosek)	-3.579	79.06
sdsos	-5.046	0.152	sdsos	-10.433	0.444	sdsos	-17.333	1.935
dsos	-5.312	0.067	dsos	-10.957	0.370	dsos	-18.015	1.301
BARON	-175.4	0.35	BARON	-1079.9	0.62	BARON	-5287.9	3.69

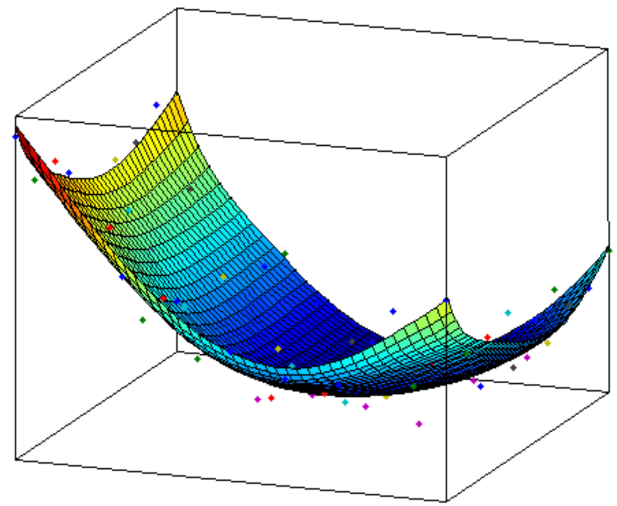
n=30	Lower bound	Run time (secs)	n=40	Lower bound	Run time (secs)	n=50	Lower bound	Run time (secs)
sos (sedumi)	---------	∞	sos (sedumi)	---------	∞	sos (sedumi)	---------	∞
sos (mosek)	---------	∞	sos (mosek)	---------	∞	sos (mosek)	---------	∞
sdsos	-36.038	9.431	sdsos	-61.248	53.95	sdsos	-93.22	100.5
dsos	-36.850	8.256	dsos	-62.2954	26.02	dsos	-94.25	72.79
BARON	-28546.1							

Convex regression

300 points in $\mathbb{R}^{20}$



Observation: $e^{||x||} + \text{noise}$



Best convex polynomial fit of degree d (sd)sos constraint in 40 variables:

$$y^T H(x) y \ \text{(sd)sos}$$

d=2	Max Error	Run time (secs)
sos (mosek)	21.282	~1
sdsos	33.918	~1
dsos	35.108	~1

d=4	Max Error	Run time (secs)
sos (mosek)	---------	∞
sdsos	12.936	231
dsos	14.859	150

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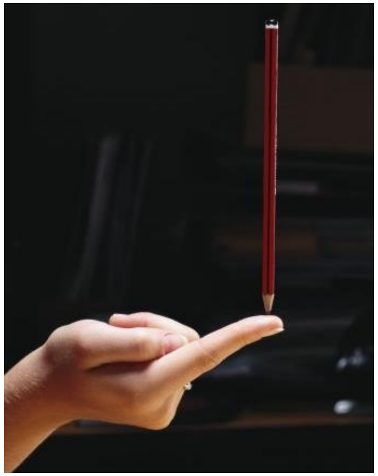
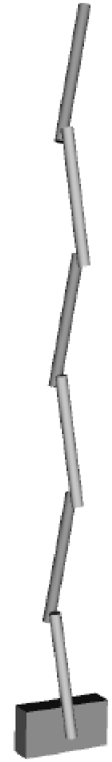
Some control applications

Stabilizing the inverted N-link pendulum (2N states)

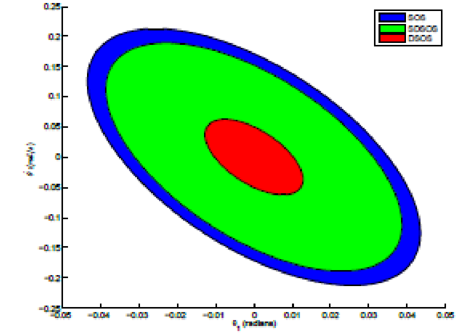


N=1

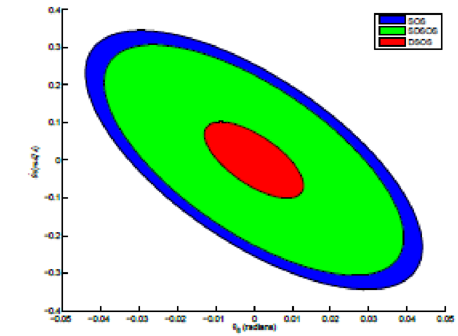
N=2



N=6



(a) θ_1-$\dot{\theta}_1$ subspace.



(b) θ_6-$\dot{\theta}_6$ subspace.

Runtime:

2N (# states)	4	6	8	10	12	14	16	18	20	22
DSOS	< 1	0.44	2.04	3.08	9.67	25.1	74.2	200.5	492.0	823.2
SDSOS	< 1	0.72	6.72	7.78	25.9	92.4	189.0	424.74	846.9	1275.6
SOS (SeDuMi)	< 1	3.97	156.9	1697.5	23676.5	∞	∞	∞	∞	∞
SOS (MOSEK)	< 1	0.84	16.2	149.1	1526.5	∞	∞	∞	∞	∞

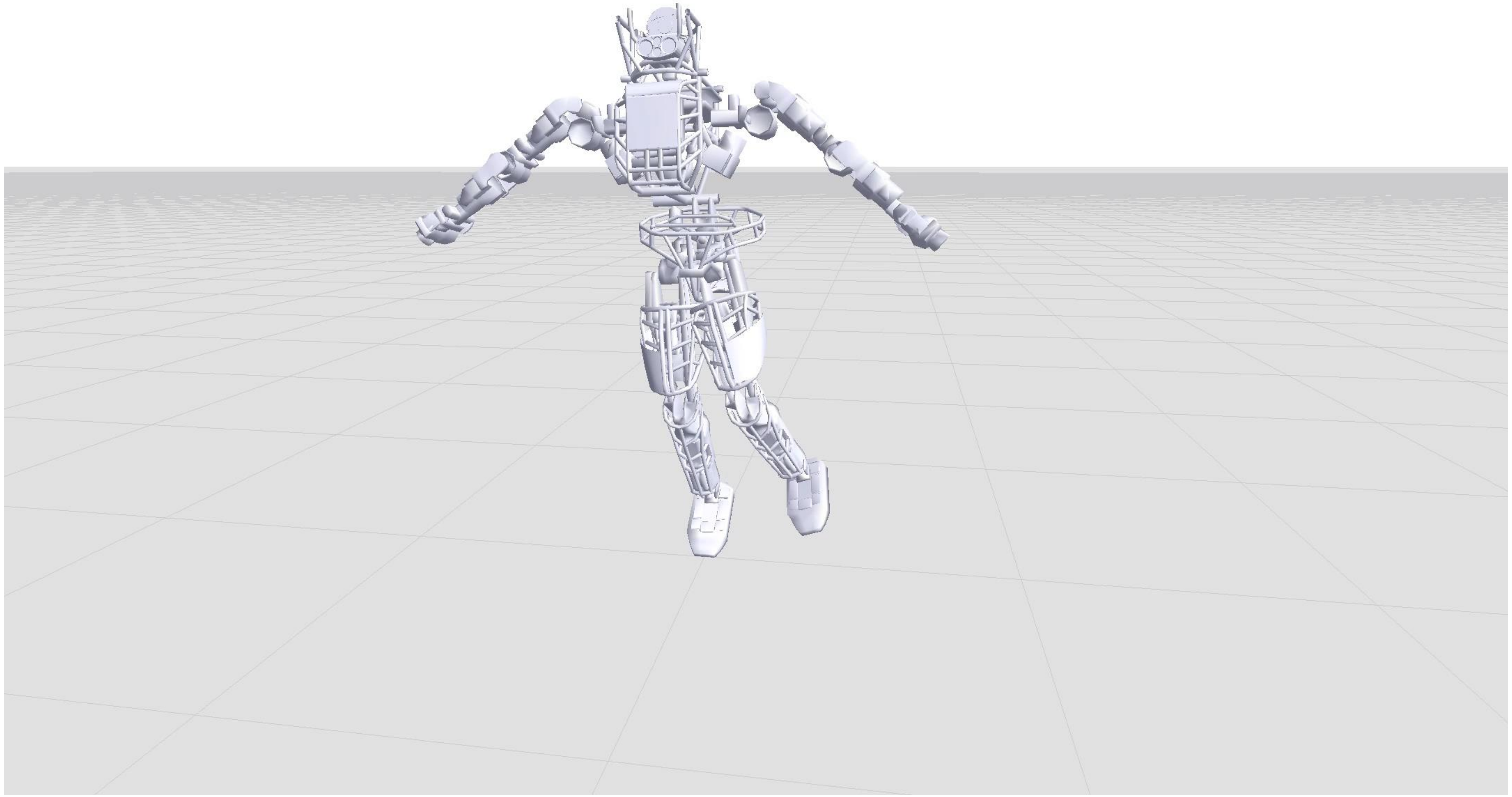
ROA volume ratio:

2N (states)	4	6	8	10	12
ρ_{dsos}/ρ_{sos}	0.38	0.45	0.13	0.12	0.09
ρ_{sdsos}/ρ_{sos}	0.88	0.84	0.81	0.79	0.79

(w/ Majumdar, Tedrake)

Stabilizing ATLAS

- 30 states   14 control inputs   Cubic dynamics



https://www.youtube.com/watch?v=lmAT556Ar5c

Done by **SDSOS Optimization**

[Majumdar, AAA, Tedrake]

Lyapunov Barrier Certificates

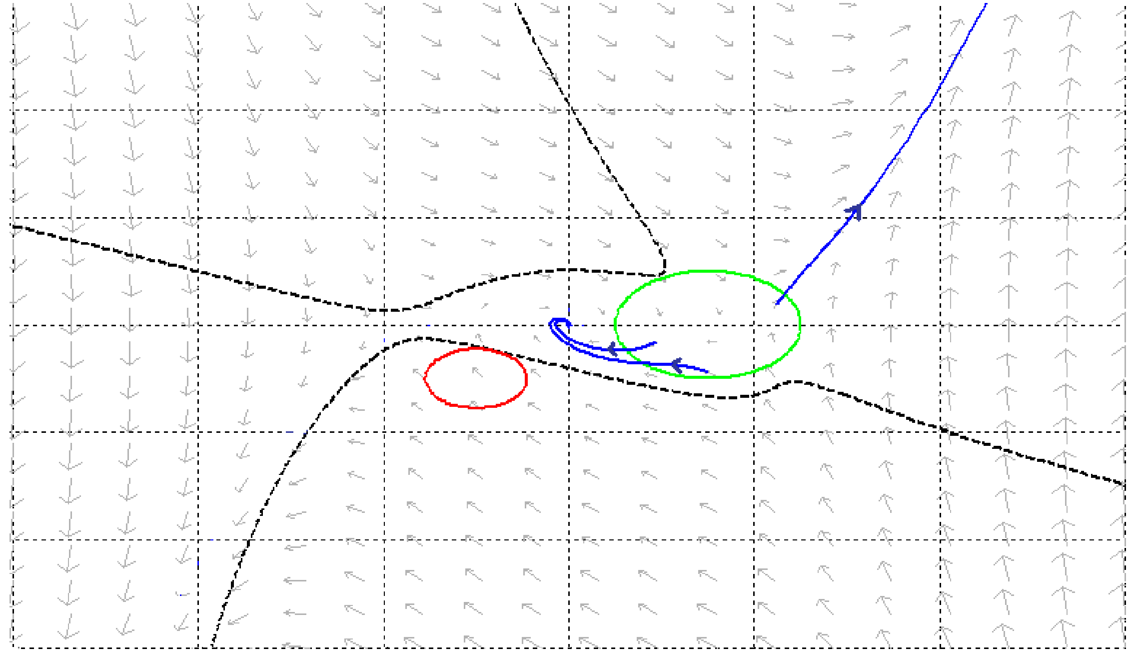
$$\dot{x} = f(x)$$

(vector valued polynomial)

$\mathcal{S}$: needs safety verification

$\mathcal{U}$: unsafe (or forbidden) set

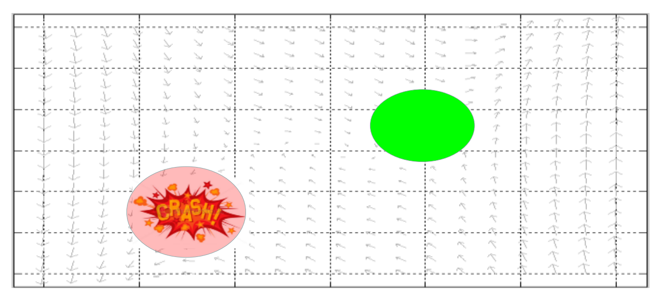
(both sets semialgebraic)



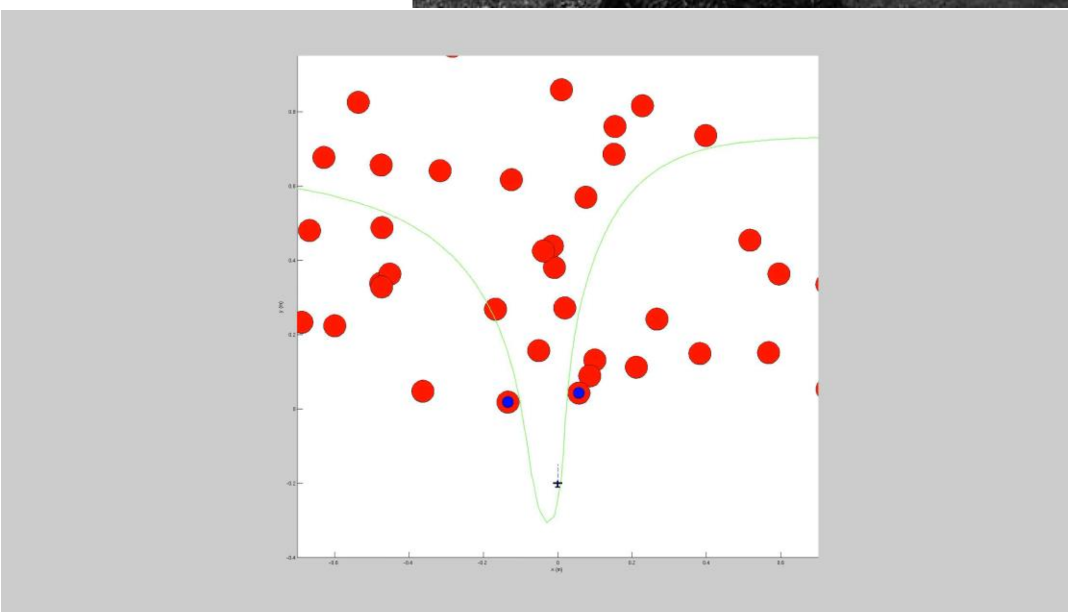
Safety assured if we find a "Lyapunov function" such that:

$$B(\mathcal{S}) < 0$$
$$B(\mathcal{U}) > 0$$

$$\dot{B} = \langle \nabla B(x), f(x) \rangle \leq 0$$



Real-time collision avoidance



https://www.youtube.com/watch?v=J3a6v0tlsD4

Done by
SDSOS Optimization

Dubins car model

Run-time: 20 ms

Converse results 1&2

Thm. Any **even** positive definite form ***p*** **is r-dsos** for some ***r*****.**

- Hence proof of positivity can always be found with LP
- Proof follows from a result of **Polya (1928)** on Hilbert's 17th problem
- Even forms include, e.g., **copositive programming**!
- $r \leq \alpha^2(G)$ for finding independent sets in graphs (corollary of a result of de Klerk & Pasechnik)

Thm. Any positive definite **bivariate** form ***p*** **is r-sdsos** for some ***r*****.**

- Proof follows from a result of **Reznick (1995)**
  - $p.||x||^r$ will always become a sum of powers of linear forms for sufficiently large r.

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Converse result 3 & Polynomial Optimization

Thm. For any positive definite form ***p***, there exists an integer r and a **polynomial *q* of degree *r*** such that

q is dsos and **pq is dsos.**

- Search for q is an LP
- Such a q is a certificate of nonnegativity of p
- Proof follows from a result of **Habicht (1940)** on Hilbert's 17th problem

$$\min_x p(x)$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

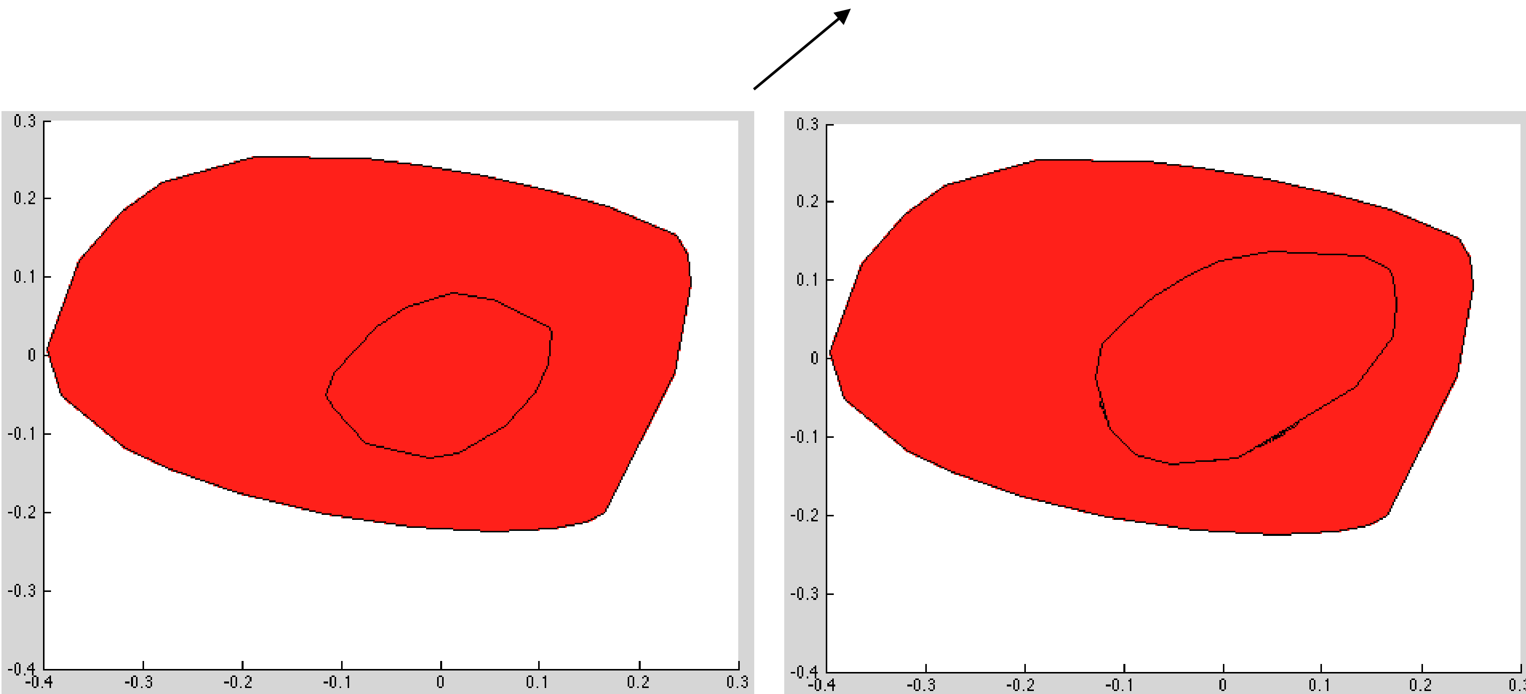
- Similar to the **Lasserre/Parrilo SDP hierarchies, polynomial optimization can be solved to global optimality using hierarchies of LP and SOCP coming from dsos and sdsos.**

Ongoing directions…

Iterative DSOS via

- Column generation

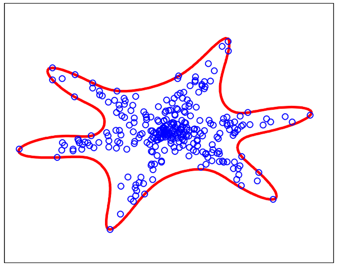
- Cholesky change of basis

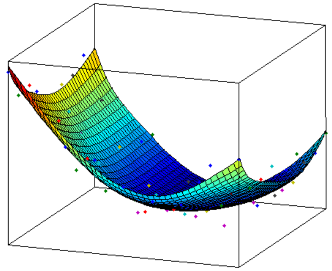


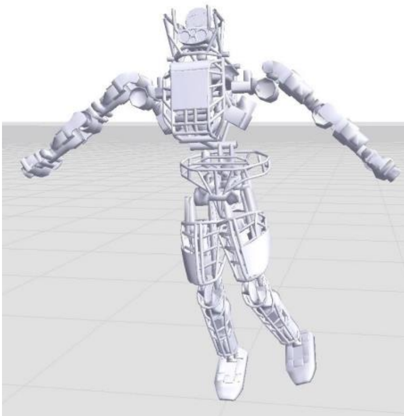
(w/ G. Hall,
S. Dash, IBM)

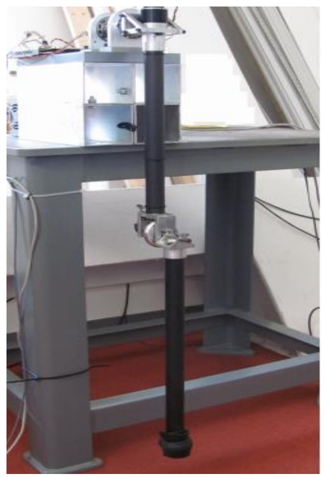
Main messages…

Many applications





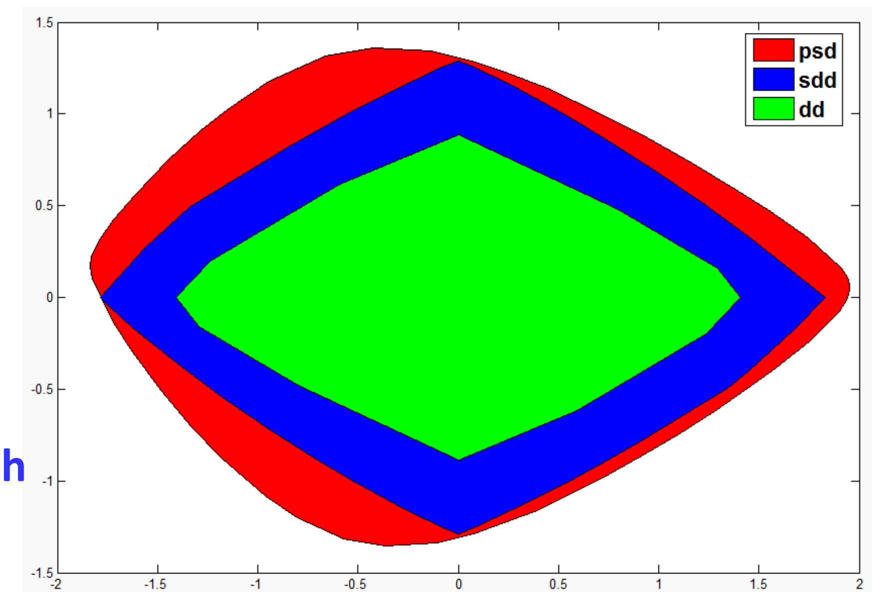




Cool and classical theory

Rejuvenated thanks to modern interface with computation and optimization

From Logicomix



Move from SDP to LP and SOCP
New possibilities likely to become within reach

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