# Complexity of Testing Existence of Solutions in Polynomial Optimization
# +
# A New Positivstellensatz

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# Existence of solutions

Consider a polynomial optimization problem (POP):

$$\inf_{x \in \mathbb{R}^n} p(x)$$
$$\text{s.t. } g_i(x) \geq 0, i = 1, \dots, m.$$

Suppose the optimal value is finite (i.e., POP is feasible and bounded below).
We would like to **test if there exists an optimal solution**, i.e.,

a feasible point $x^*$ such that $p(x^*) \leq p(y), \forall y$ feasible.

Informally: *"Can we replace the `inf' with a `min'?"*

**Remarks:**

- If feasible set is bounded, a solution always exists.
- If $n = 1$, a solution always exists.
- Finiteness of optimal value comes as a "promise".

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# Motivation

- An exact algorithm cannot return a solution if there is none!
- Existence of solutions essential for algorithms that exploit optimality conditions.

*[Nie, Demmel, Sturmfels, "Minimizing polynomials via sum of squares over the gradient ideal", Math. Prog. 2005]:*

*"This assumption [existence of minimizers] is nontrivial, and we do not address the (important and difficult) question of how to verify that a given polynomial $p(x)$ has this property."*

There are algothims that check existence of solutions:

- Greuet, Safey El Din, "Deciding reachability of the infimum of a multivariate polynomial", *International Symposium on Symbolic and Algebraic Computation*, 2011
- Boucero, Mourrain, "Border basis relaxation for polynomial optimization", *Journal of Symbolic Computation*, 2016
- Greuet, Safey El Din, "Probabilistic algorithm for polynomial optimization over a real algebraic set", *SIAM J. on Optimization*, 2014
- Quantifier elimination
- …

***All have running time at least exponential in dimension…***
***Can there be a faster algorithm?***

# Existence of a solution guaranteed?

$$\min_{x \in \mathbb{R}^n} p(x)$$
$$\text{s.t. } g_i(x) \geq 0, i = 1, \ldots, m$$

Degree of constraints

Degree of objective

| | 0 | 1 | 2 |
|---|---|---|---|
| **1** | Yes | Yes<br>Linear Programming | NP-hard to test<br>(This work) |
| **2** | Yes<br>Linear Algebra | Yes<br>Frank, Wolfe (1956) | |
| **3** | Yes | Yes<br>Andronov, Belousov, Shironin (1982) | |
| **4** | NP-hard to test<br>(This work) | | |

# Outline

1) NP-hardness of testing existence of solutions

2) Sufficient conditions for existence of solutions
   a. Review of SOS and Positivstellensatze
   b. An SOS hierarchy for coercivity

3) An optimization-free Positivstellensatz
   (brief and independent)

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# Main hardness results

**Theorem (AAA, Zhang)**

6 (for this presentation)

*Testing whether a degree-~~4~~ polynomial attains its unconstrained infimum is strongly NP-hard.*

**Theorem (AAA, Zhang)**

*Testing whether a degree-1 polynomial attains its infimum on a set defined by degree-2 inequalities is strongly NP-hard.*

Proof: Reduction from 1-in-3 3SAT.

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# 1-in-3 3SAT

- Input: A CNF formula with three literals per clause.
- Goal: Find a Boolean assignment so that each clause has exactly one true literal.

$$1 \quad -1 \quad -1 \qquad -1 \quad 1 \quad -1 \qquad -1 \quad -1 \quad 1$$

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

$$x_1 = 1, x_2 = 1, x_3 = -1$$

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

Not satisfiable.

This problem is NP-hard.

# NP-hardness of checking attainment (1/2)

**Goal:** Given any instance of 1-in-3 3SAT, construct a polynomial that attains its infimum if and only if the instance is satisfiable

**Step 1:**

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

$$p_\phi(x) \coloneqq \sum_{i=1}^{n} \left(1 - x_i^2\right)^2 + (x_1 + x_2 - x_3 + 1)^2 + (-x_1 - x_2 + x_3 + 1)^2$$

Important Property: $p_\phi(x)$ has a zero if and only if $\phi$ is satisfiable

But $\boldsymbol{p_\phi(x)}$ **always attains its infimum** (independent of whether $\phi$ is satisfiable) as its highest order component is $\sum_{i=1}^{n} x_i^4$.

# NP-hardness of checking attainment (2/2)

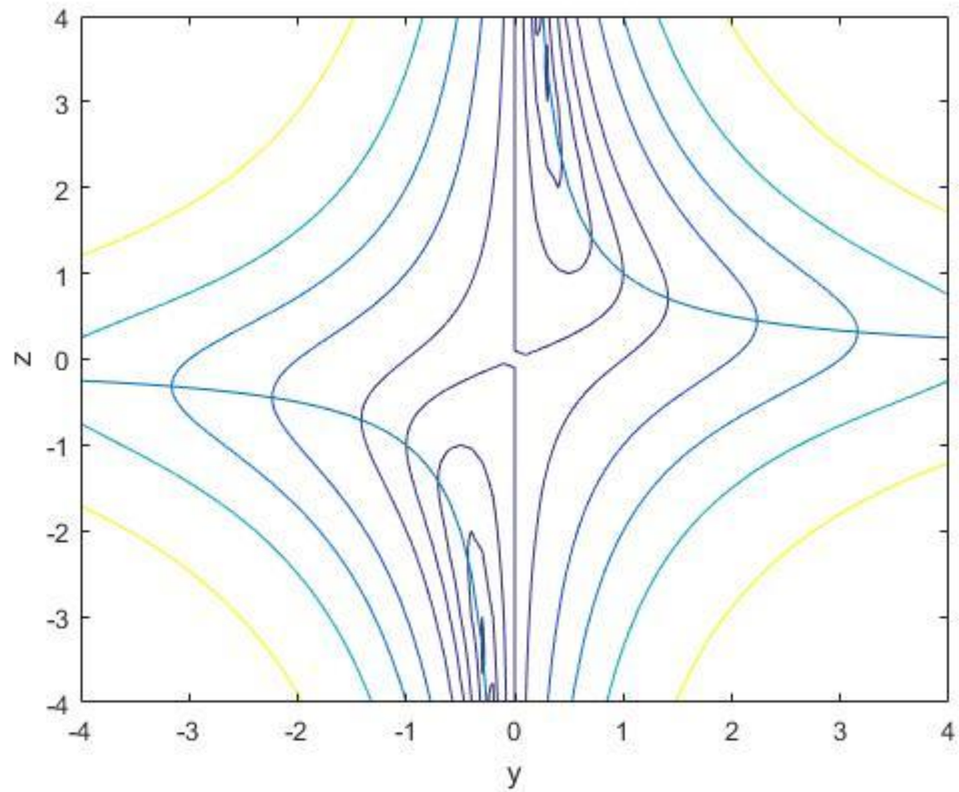
**Step 2:**

$$\boldsymbol{q_\phi(x_1, \ldots, x_n, y, z, \lambda) =}$$

$$(\boldsymbol{1-\lambda})^{\boldsymbol{2}}(y^2 + (1-yz)^2)$$

$$+$$

$$\boldsymbol{\lambda}^{\boldsymbol{2}} p_\phi(x)$$




- Infimum is always zero ($q_\phi$ is a sum of squares; take $\lambda = 0, y \to 0, z = \frac{1}{y}$).
- If $\phi$ satisfiable, take $\lambda = 1$, and $x$ the satisfying assignment → Infimum attained.
- If $\phi$ not satisfiable, $q_\phi$ does not vanish → Infimum not attained.

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**2) Sufficient conditions for existence of solutions**

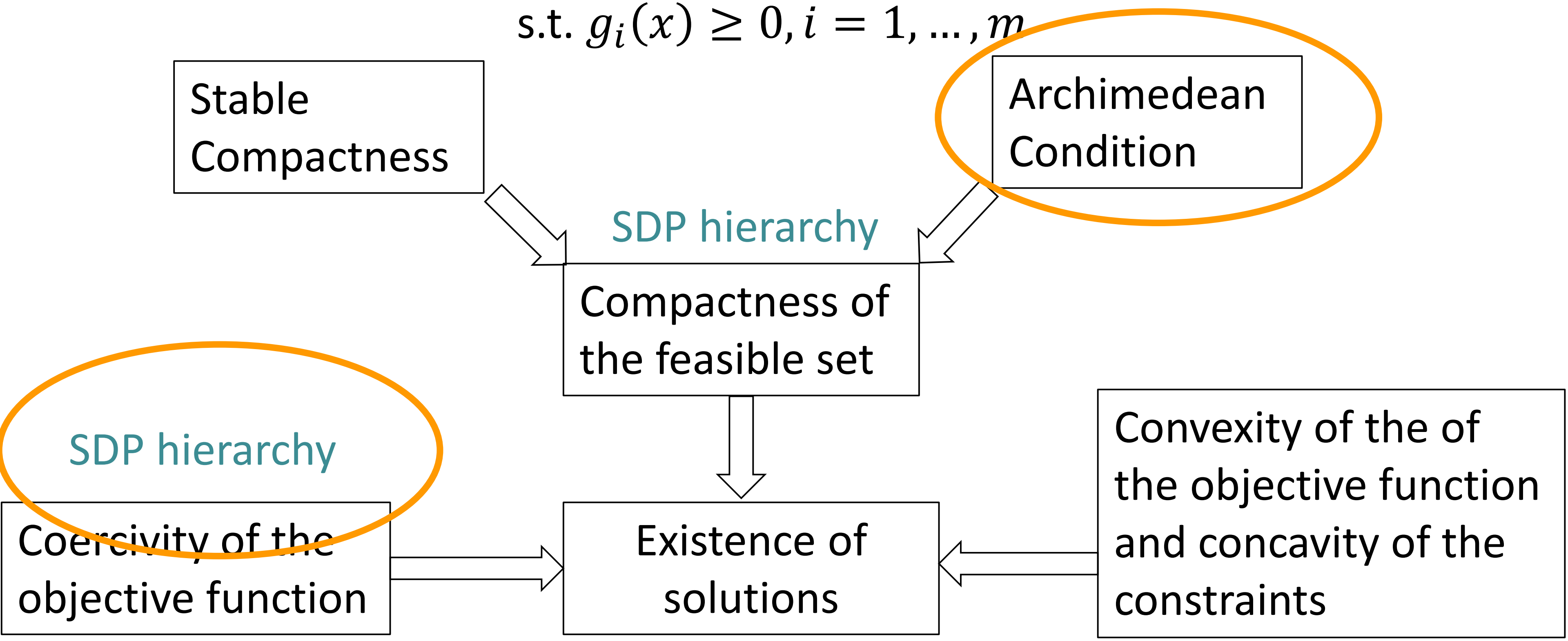
   a. Review of SOS and Positivstellensatze

   b. An SOS hierarchy for coercivity

3) An optimization-free Positivstellensatz
   (brief and independent)

# Sufficient conditions for existence of solutions

$$\min_{x \in \mathbb{R}^n} p(x)$$
$$\text{s.t. } g_i(x) \geq 0, i = 1, \ldots, m$$




**Theorem**

*- Testing whether a polynomial optimization problem satisfies any of these conditions is strongly NP-hard.*

*- Our results are minimal in the degree.*


# Review of sum of squares and Positivstellensatze

# How to prove positivity?

Is $p(x) > 0$ on $\{g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$?

## Why prove positivity?

- Infeasibility certificates for systems of polynomial inequalities

$$\{g_1(x) \geq 0, g_2(x) \geq 0, \dots, g_m(x) \geq 0\} \quad \text{empty}$$
$$\Leftrightarrow$$
$$-g_1(x) > 0 \text{ on } \{g_2(x) \geq 0, \dots, g_m(x) \geq 0\}$$

- (Tight) lower bounds for polynomial minimization problems
- Dynamics and control (Lyapunov functions)
- Stats/ML (shape-constrained regression),…

# Sum of squares and SDP

- A polynomial $p$ is a **sum of squares (sos)** if it can be written as

$$p(x) = \sum_i q_i^2(x)\,,$$

where $q_i$ are polynomials.

**Ex:**
$$\begin{aligned} p(x) &= x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 \\ &\quad -14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \\ &= (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2 \end{aligned}$$

- A polynomial $p$ of degree $2d$ is **sos** if and only if $\exists Q \succcurlyeq 0$ such that

$$p(x) = z(x)^T Q z(x)$$

where $z = [1, x_1, \ldots, x_n, x_1x_2, \ldots, x_n^d]^T$ is the vector of monomials of degree up to $d$.



Optimizing over set of sos polynomials is an SDP!

# Positivstellensätze

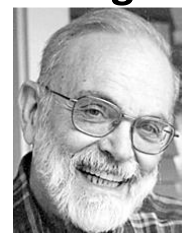
$p(x) > 0, \forall x \in \mathbb{R}^n$

**Artin**

If $p(x) \geq 0, \forall x \in \mathbb{R}^n$,
then $\exists$ sos $q$ s.t. $p \cdot q$ sos.

1927

**Stengle**



1974

**20th century**

1991

1993

$p(x) > 0,$
$\forall x \in S = \{x \mid g_i(x) \geq 0\}$

**Schmudgen**

Requires compactness

If $p(x) > 0, \forall x \in S$,
then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x) g_i(x) + \sum_{ij} \sigma_{ij}(x) g_i(x) g_j(x) + \cdots$, where $\sigma_0, \sigma_i$, … sos

**Putinar**

Requires the **Archimedean property**

If $p(x) > 0, \forall x \in S$,
then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x) g_i(x)$,
where $\sigma_0, \sigma_i$ are sos

**Search for these sos polynomials (when degree is fixed) --->SDP.**

# Infeasibility proofs for polynomial (in)equalities

**Stengle's Positivstellensatz**

$$\{g_i(x) \geq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, t\}$$

infeasible

if and only if

there exist polynomials $\tau_i$ and sos polynomials $\sigma$ such that

$$-1 = \sum \tau_i h_i + \sigma_0 + \sum \sigma_i g_i + \sum \sigma_{ij} g_i g_j + \sum \sigma_{ijk} g_i g_j g_k + \cdots + \sigma_{1\dots m} \Pi g_1 \dots g_m.$$

**Search for these sos certificates of infeasibility (when deg. is fixed) ---> SDP.**


# Back to coercivity

# Coercivity

Definition: A function $f$ is *coercive* if for every sequence $\{x_k\}$ such that $||x_k|| \to \infty$, we have $f(x) \to \infty$.

- A coercive function attains its infimum
- Checking whether a polynomial is coercive is NP-hard

Past work:

- Jeyakumar, Lasserre (2014)
    - SDP hierarchy
- Bajbar, Stein (2015)
- Bajbar, Behrends (2017)

We provide a condition which is (i) both necessary and sufficient for a polynomial to be coercive and (ii) amenable to an SDP hierarchy

# An sos hierarchy for testing coercivity (1/2)

**Theorem (AAA, Zhang)**

*A polynomial $p$ of degree $d$ is coercive if and only if for some integer $r \geq 1$ the following SDP is feasible*

$$
\begin{aligned}
-1 =& \sigma_0(x,\gamma) + \sigma_1(x,\gamma)(\gamma - p(x)) + \sigma_2(x,\gamma)\left(\sum_{i=1}^n x_i^2 - \gamma^{2r} - 2^r\right) \\
&+ \sigma_3(x,\gamma)(\gamma - p(x))\left(\sum_{i=1}^n x_i^2 - \gamma^{2r} - 2^r\right), \\
&\sigma_0 \text{ is sos and of degree } \leq 4r, \\
&\sigma_1 \text{ is sos and of degree } \leq \max\{4r - d, 0\}, \\
&\sigma_2 \text{ is sos and of degree } \leq 2r, \\
&\sigma_3 \text{ is sos and of degree } \leq \max\{2r - d, 0\}.
\end{aligned}
$$

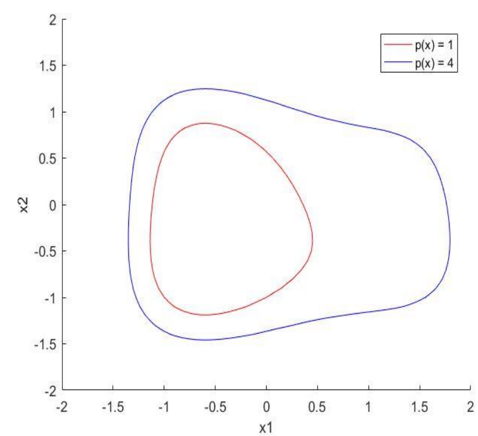
# An sos hierarchy for testing coercivity (2/2)

A function is coercive if and only if all its sublevel sets are compact.

**Theorem (AAA, Zhang)**

A polynomial $p$ is coercive if and only if there exist an even integer $c > 0$ and a scalar $k \geq 0$ such that for all $\gamma \in \mathbb{R}$, the $\gamma$-sublevel set of $p$ is contained within a ball of radius $\gamma^c + k$.

$$p(x) = x^4 + y^4 - 2x^3 + y^2 + 3x + y$$




Equivalently, $p$ is coercive if and only if there exist an even integer $c'$ and a scalar $k'$ for which the set $\{(x, \gamma) | p(x) \leq \gamma, \sum x_i^2 \geq \gamma^{c'} + k'\}$ is empty.

# Coercivity hierarchy revisited

**Theorem (AAA, Zhang)**

*A polynomial $p$ of degree $d$ is coercive if and only if for some integer $r \geq 1$ the following SDP is feasible*

$$-1 = \sigma_0(x,\gamma) + \sigma_1(x,\gamma)(\gamma - p(x)) + \sigma_2(x,\gamma)\left(\sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r\right) + \sigma_3(x,\gamma)(\gamma - p(x))\left(\sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r\right),$$

$\sigma_0$ is sos and of degree $\leq 4r$,

$\sigma_1$ is sos and of degree $\leq \max\{4r - d, 0\}$,

$\sigma_2$ is sos and of degree $\leq 2r$,

$\sigma_3$ is sos and of degree $\leq \max\{2r - d, 0\}$.

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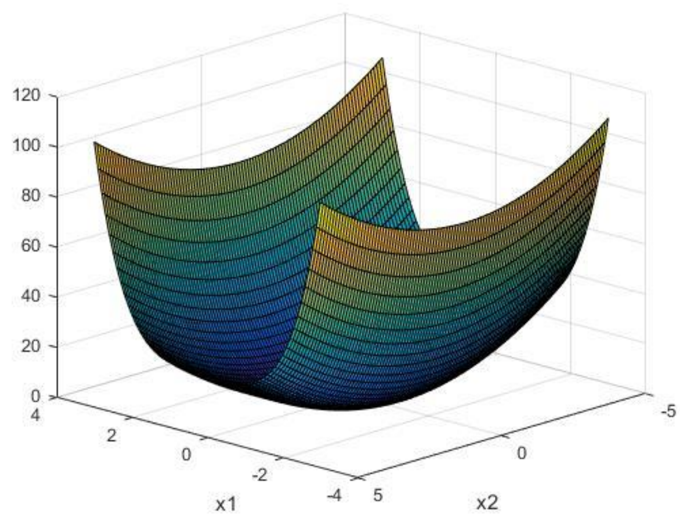
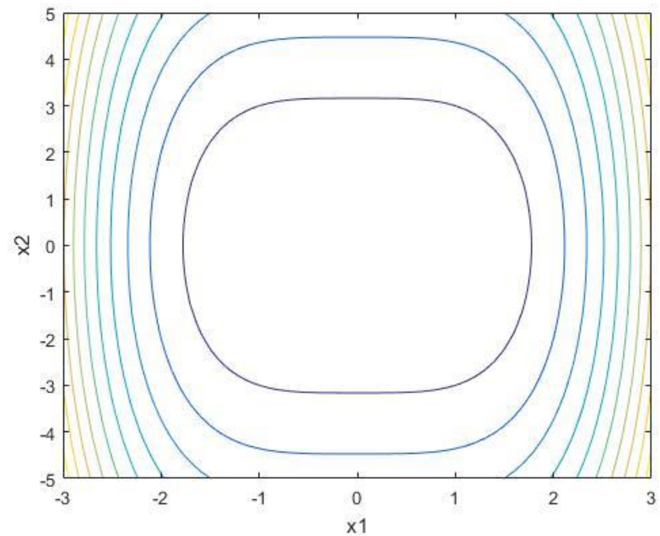
# Toy example

The polynomial $p(x) = x_1^4 + x_2^2$ is coercive as certified by the following algebraic identity:

$$-1 = \left(\frac{2}{3}\left(x_1^2 - \frac{1}{2}\right)^2 + \frac{2}{3}\left(\gamma - \frac{1}{2}\right)^2\right) + \frac{2}{3}(\gamma - x_1^4 - x_2^2) + \frac{2}{3}(x_1^2 + x_2^2 - \gamma^2 - 2)$$

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   (brief and independent)

# Recall what a Positivstellensatz establishes

$$p(x) > 0, \forall x \in \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

**Putinar**

Under the Archimedean property

If $p(x) > 0, \forall x \in S$,
then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x) g_i(x)$,
where $\sigma_0, \sigma_i$ are sos

**Search for these sos polynomials (when degree is fixed) --->SDP.**
**Similar situation for Psatze of Stengle and Schmudgen.**

# An optimization-free Positivstellensatz (1/2)

$$p(x) > 0, \forall x \in \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \dots, m\}$$

$2d$ =maximum degree of $p, g_i$

$\Updownarrow$ Under compactness assumptions, i.e., $\{x \mid g_i(x) \geq 0\} \subseteq B(0, R)$

$\exists\, r \in \mathbb{N}$ such that

$$\left(f(v^2 - w^2) - \frac{1}{r}\left(\sum_i \left(v_i^2 - w_i^2\right)^2\right)^d + \frac{1}{2r}\left(\sum_i \left(v_i^4 + w_i^4\right)\right)^d\right) \cdot \left(\sum_i v_i^2 + \sum_i w_i^2\right)^{r^2}$$

has **nonnegative coefficients**,

where $f$ is a form in $n + m + 3$ variables and of degree $4d$, which can be explicitly written from $p, g_i$ and $R$.


[AAA, Hall, *Math of OR* '18] (2018 INFORMS Young Researchers Prize)

# An optimization-free Positivstellensatz (2/2)

$$p(x) > 0 \text{ on } \{x \mid g_i(x) \geq 0\} \Leftrightarrow$$

$$\exists r \in \mathbb{N} \text{ s.t. } \left( f(v^2 - w^2) - \frac{1}{r}\left(\sum_i (v_i^2 - w_i^2)^2\right)^d + \frac{1}{2r}\left(\sum_i (v_i^4 + w_i^4)\right)^d \right) \cdot \left(\sum_i v_i^2 + \sum_i w_i^2\right)^{r^2}$$

**has $\geq 0$ coefficients**

- $p(x) > 0$ on $\{x \mid g_i(x) \geq 0\} \Leftrightarrow f$ is pd
- **Result by Polya (1928):**

$f$ **even** and pd $\Rightarrow \exists r \in \mathbb{N}$ such that $f(z) \cdot \left(\sum_i z_i^2\right)^r$ has nonnegative coefficients.

- Make $f(z)$ even by considering $f(v^2 - w^2)$. We lose positive definiteness of $f$ with this transformation.
- Add the positive definite term $\frac{1}{2r}\left(\sum_i (v_i^4 + w_i^4)\right)^d$ to $f(v^2 - w^2)$ to make it positive definite. **Apply Polya's result.**
- The term $-\frac{1}{r}\left(\sum_i (v_i^2 - w_i^2)^2\right)^d$ ensures that the converse holds as well.

Want to know more? aaa.princeton.edu

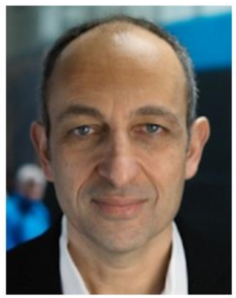
# You are cordially invited…

# Princeton Day of Optimization

September 28, 2018

http://orfe.princeton.edu/pdo/

D. Bertsimas

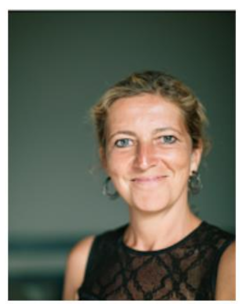


M. Dahleh

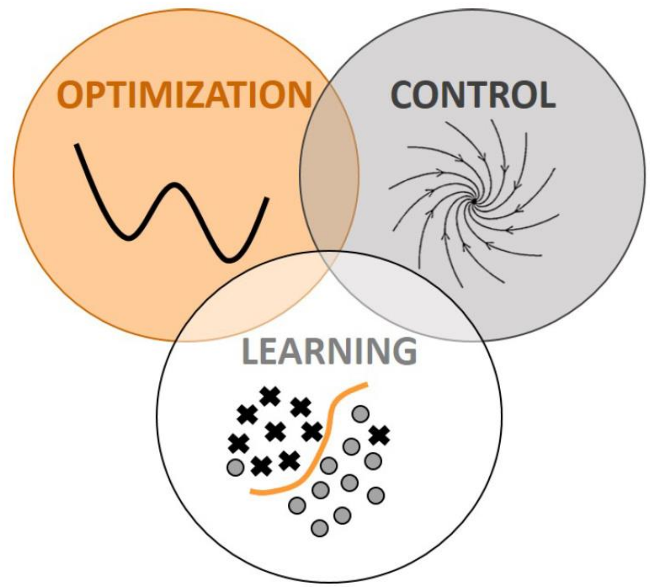
E. Hazan

P. Parrilo

B. Recht



K. Scheinberg

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Thank you.