Two Problems at the Interface of Optimization and Dynamical Systems

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Outline

"Optimization ----> Dynamical systems"

1) Stability analysis of polynomial ODEs

- Power/limitations of SOS Lyapunov functions
  - Joint work with Bachir El Khadir (Princeton)

"Dynamical systems ----> Optimization"

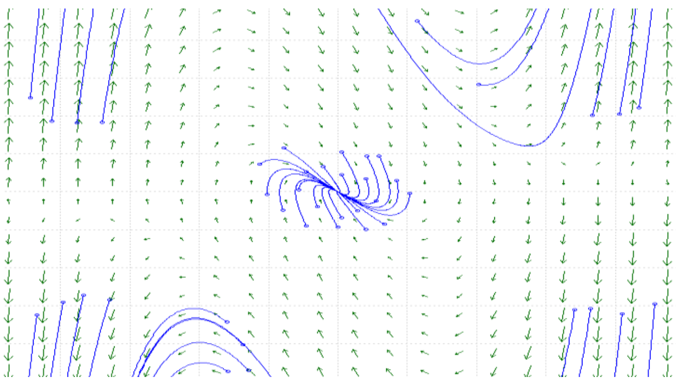
2) Robust-to-dynamics optimization

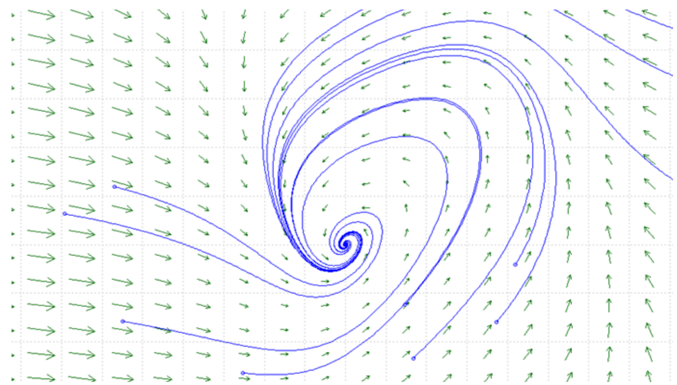
- A new class of robust optimization problems
  - Joint work with Oktay Gunluk (IBM Research)

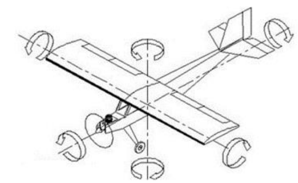
Asymptotic stability

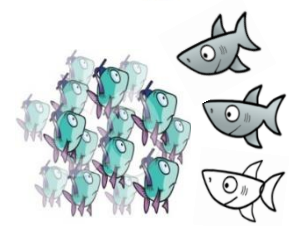
$$\dot{x} = f(x) \qquad f : \mathbb{R}^n \to \mathbb{R}^n$$ polynomial with rational coefficients $$f(0) = 0$$

Example $$\dot{x}_1 = -x_2 + \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 x_2$$
$$\dot{x}_2 = 3x_1 - x_1 x_2$$









Locally Asymp. Stable (LAS) if

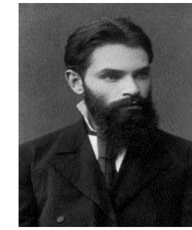
- $\forall \epsilon > 0, \exists \delta > 0,$ s.t.
  $x(0) \in B_\delta \Rightarrow x(t) \in B_\epsilon \;\; \forall t$
- $\exists \alpha > 0$ s.t.
  $x(0) \in B_\alpha \Rightarrow \lim_{t \to \infty} x(t) = 0.$

Globally Asymp. Stable (GAS) if

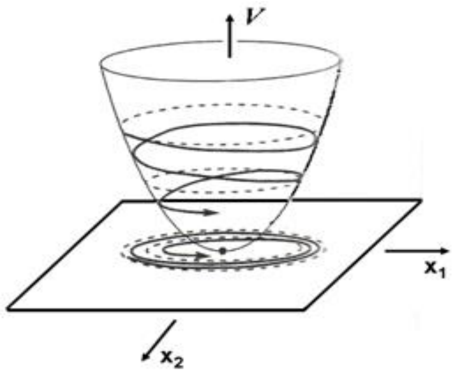
- $\forall \epsilon > 0, \exists \delta > 0,$ s.t.
  $x(0) \in B_\delta \Rightarrow x(t) \in B_\epsilon \;\; \forall t$
- $\forall x_0 \in \mathbb{R}^n, \; \lim_{t \to \infty} x(t) = 0.$

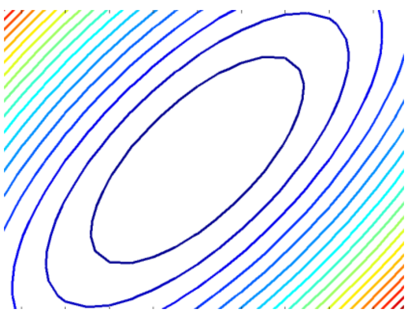
- Stability of equilibrium prices in economics
- Convergence analysis of algorithms, …

Lyapunov's theorem on asymptotic stability



$$\dot{x} = f(x)$$





Existence of a (Lyapunov) function

$$V(x) : \mathbb{R}^n \to \mathbb{R}$$

$$\dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle$$

such that

$$V(x) > 0$$

$$\dot{V}(x) < 0$$

in a neighborhood of the origin, then LAS.

(If inequalities hold everywhere, then GAS.)

Such a function is guaranteed to exist! But how to find one?

Very popular since 2000: Use SDP to find polynomial Lyapunov functions.

How to prove nonnegativity?

$$p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4$$

Nonnegative

$\Uparrow$

$$p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2.$$

SOS

- Extends to the constrained case!

Well-known fact:

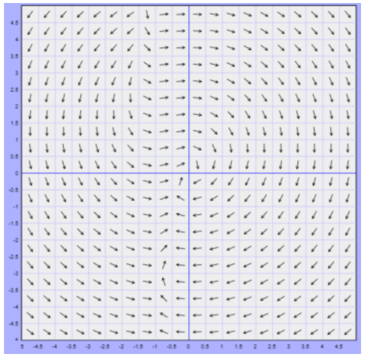
- Optimization over sum of squares (SOS) polynomials is an SDP!

Sum of squares Lyapunov functions (GAS)

$$\left.\begin{array}{rl} V(x) & \text{sos} \\ -\dot{V}(x) & \text{sos} \end{array}\right. \Rightarrow \begin{array}{rl} V(x) & > 0 \\ -\dot{V}(x) & > 0 \end{array} \Rightarrow \text{GAS}$$

(stolen from Pablo's homepage)

(M. Krstić) Find a Lyapunov function for *global asymptotic stability*:



$$\dot{x} = -x + (1+x)\,y$$
$$\dot{y} = -(1+x)\,x.$$

Using SOSTOOLS we easily find a quartic polynomial:

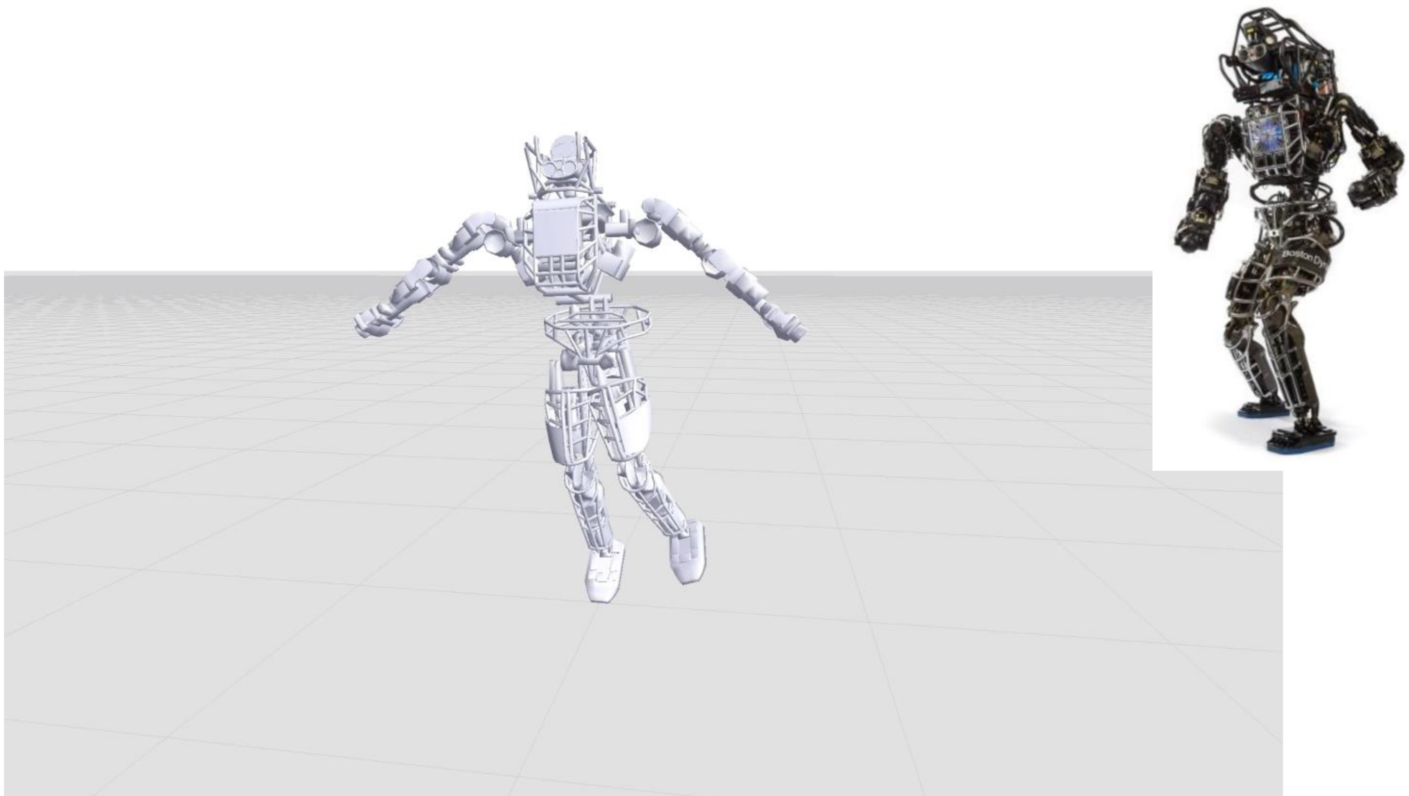
$$V(x,y) = 6x^2 - 2xy + 8y^2 - 2y^3 + 3x^4 + 6x^2y^2 + 3y^4.$$

Both $V(x,y)$ and $(-\dot{V}(x,y))$ are SOS:

$$V(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}, \quad -\dot{V}(x,y) = \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}^T \begin{bmatrix} 10 & 1 & -1 & 1 \\ 1 & 2 & 1 & -2 \\ -1 & 1 & 12 & 0 \\ 1 & -2 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^2 \\ xy \end{bmatrix}$$

The matrices are positive *definite*; this *proves* asymptotic stability.

Sum of squares Lyapunov functions (LAS)



[Majumdar, AAA, Tedrake]

Complexity of deciding asymptotic stability?

$$\dot{x} = Ax$$

- d=1 (linear systems): decidable, and polynomial time
  - **Iff *A* is Hurwitz** (i.e., eigenvalues of *A* have negative real part)
  - **Quadratic Lyapunov functions** always exist:
    - $V(x) = x^T P x, \dot{V}(x) = x^T (A^T P + PA) x$
      $(P \succ 0, A^T P + PA \prec 0).$
  - A polynomial time algorithm is the following:
    - Solve $A^T P + PA = -I$
    - Check if P is positive definite

What if deg(*f*)>1? …

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Complexity of deciding asymptotic stability?

What if deg(f)>1? …

- **Conjecture of Arnol'd (1976): undecidable** (still open)

Existence of **polynomial Lyapunov functions**, together with a **computable upper bound** on the degree would imply decidability (e.g., by quantifier elimination).

Thm: Deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard.

[AAA]

(In particular, this rules out tests based on polynomially-sized convex programs.)

Thm: Deciding asymptotic stability of cubic *homogeneous* vector fields is strongly NP-hard.

Homogeneous means:

$$\dot{x} = f(x)$$
$$f(\lambda x) = \lambda^d f(x)$$

- All monomials in f have the same degree
- Local Asymptotic Stability = Global Asymptotic Stability

Proof

Thm: Deciding asymptotic stability of cubic homogeneous vector fields is strongly NP-hard.

Reduction from: **ONE-IN-THREE 3SAT**

$$1 \quad 0 \quad 0 \qquad 0 \quad 1 \quad 0 \qquad 0 \quad 0 \quad 1$$

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

$$x_1 = 1, x_2 = 1, x_3 = 0$$

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

Goal: Design a cubic differential equation which is a.s. iff ONE-IN-THREE 3SAT has no solution

Proof (cont'd)

ONE-IN-THREE 3SAT

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee \bar{x}_5) \wedge (x_1 \vee x_3 \vee x_4)$$

↓

Positivity of quartic forms

$$\begin{aligned} p(x) \;=\; & \textstyle\sum_{i=1}^{5} x_i^2 (1-x_i)^2 \\ & +(x_1 + (1-x_2) + x_4 - 1)^2 + ((1-x_2) + (1-x_3) + x_5 - 1)^2 \\ & +((1-x_1) + x_3 + (1-x_5) - 1)^2 + (x_1 + x_3 + x_4 - 1)^2 \end{aligned}$$

$$p_h(x,y) = y^4 p\left(\tfrac{x}{y}\right)$$

↓

Asymptotic stability of cubic homogeneous vector fields

$$z := (x, y)$$

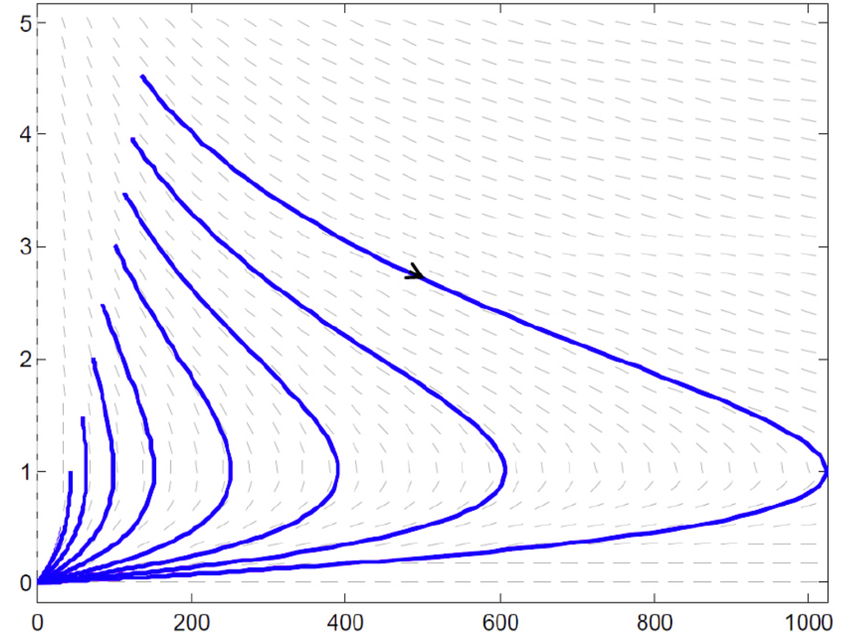
$$\dot{z} = -\nabla p_h(z)$$

Stability ?==>? Polynomial Lyapunov function (1/4)

$$\begin{aligned}\dot{x} &= -x + xy \\ \dot{y} &= -y\end{aligned}$$

Claim 1: System is GAS.

Claim 2: No polynomial Lyapunov function (of any degree) exists!



Proof:

$$V(x,y) = \ln(1+x^2) + y^2$$

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= -\frac{x^2+2y^2+x^2y^2+(x-xy)^2}{1+x^2}$$

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[AAA, Krstic, Parrilo, CDC'11]

Stability ?==>? Polynomial Lyapunov function (2/4)

$$\begin{aligned}\dot{x} &= -x + xy \\ \dot{y} &= -y\end{aligned}$$

Claim 2: No polynomial Lyapunov function (of any degree) exists!

Proof:

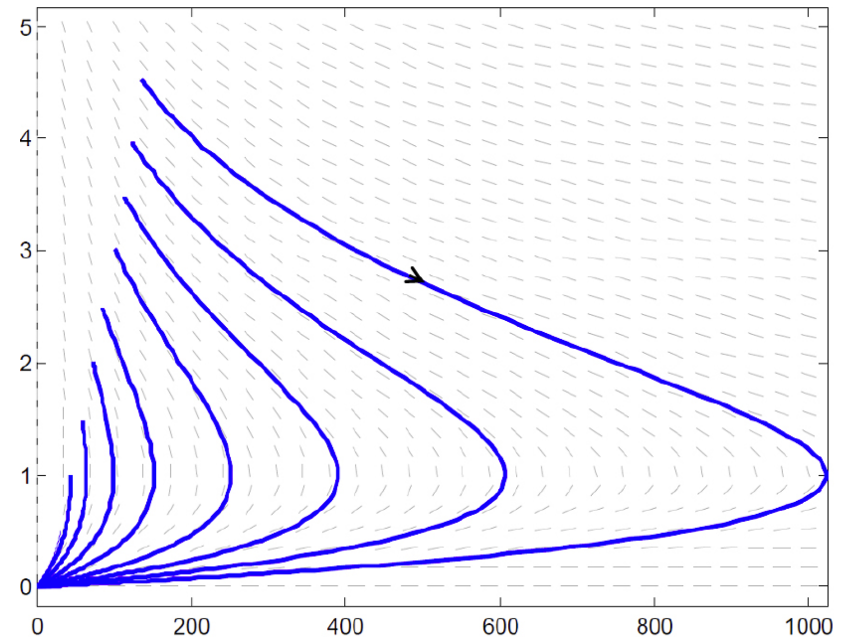
$$\begin{aligned}x(t) &= x(0)e^{[y(0)-y(0)e^{-t}-t]} \\ y(t) &= y(0)e^{-t}\end{aligned}$$

$$t^* = \ln(k)$$

$$(k, \alpha k) \longrightarrow (e^{\alpha(k-1)}, \alpha)$$

$$V(e^{\alpha(k-1)}, \alpha) < V(k, \alpha k)$$

Impossible. ∎



- No rational Lyapunov function either.
- But a quadratic Lyapunov function locally.

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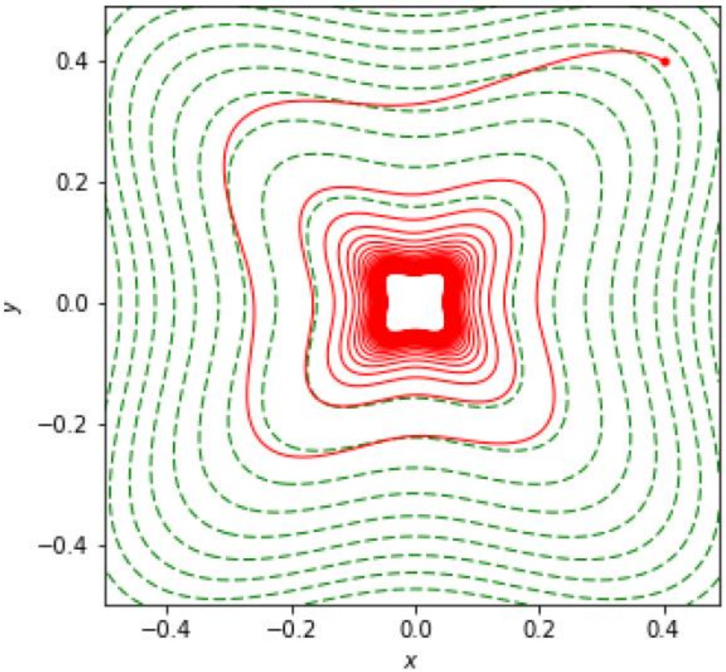
Stability ?==>? Polynomial Lyapunov function (3/4)

$$f(x,y) = \begin{pmatrix} -2y(-x^4 + 2x^2y^2 + y^4) \\ 2x(x^4 + 2x^2y^2 - y^4) \end{pmatrix} - (x^2 + y^2) \begin{pmatrix} 2x(x^4 + 2x^2y^2 - y^4) \\ 2y(-x^4 + 2x^2y^2 + y^4) \end{pmatrix}$$

Claim 1: System is GAS.

Claim 2: No polynomial Lyapunov function (of any degree) **even locally**!



Proof:

$$W(x,y) = \frac{x^4 + y^4}{x^2 + y^2}$$

[AAA, El Khadir, *Systems & Control Letters'18*]

Stability ?==>? Polynomial Lyapunov function (4/4)

$$f(x,y) = \underbrace{\begin{pmatrix} -2y(-x^4+2x^2y^2+y^4) \\ 2x(x^4+2x^2y^2-y^4) \end{pmatrix}}_{f_0(x,y)} \underbrace{- (x^2+y^2)\begin{pmatrix} 2x(x^4+2x^2y^2-y^4) \\ 2y(-x^4+2x^2y^2+y^4) \end{pmatrix}}_{f_1(x,y)}$$

Claim 2: No polynomial Lyapunov function (of any degree) **even locally**!

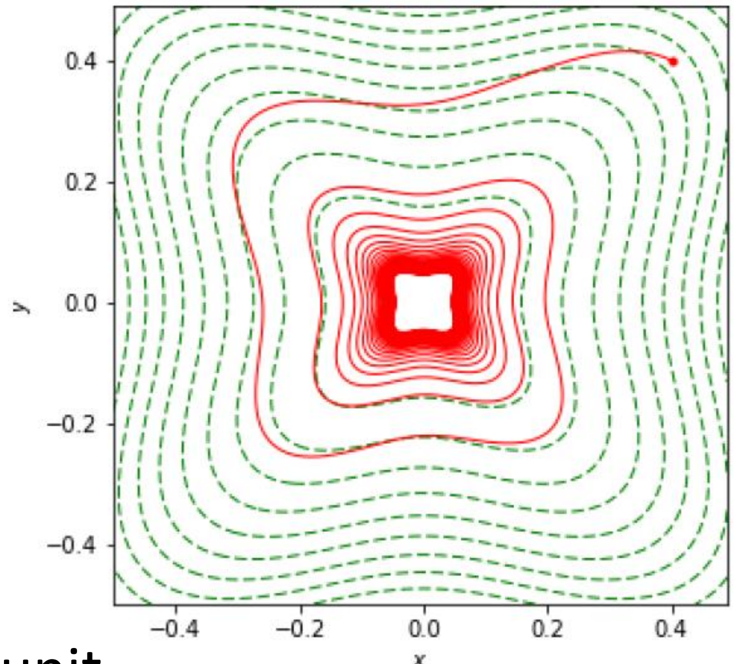
Proof idea:

Suppose we had one: $p = \sum_{k=0}^{\infty} p_k$

➔ $\langle \nabla p_{k_0}(x,y), f_0(x,y) \rangle \leq 0$

➔ $\langle \nabla p_{k_0}(x,y), f_0(x,y) \rangle = 0.$

➔ A polynomial must be constant on the unit level set of $W(x,y) = (x^4+y^4)/(x^2+y^2)$



Algebraic proofs of stability for homogeneous vector fields

Homogeneous means:

$$\dot{x} = f(x)$$

$$f(\lambda x) = \lambda^d f(x)$$

- All monomials in f have the same degree
- Local Asymptotic Stability = Global Asymptotic Stability

A positive result

Thm. A homogeneous polynomial vector field is asymptotically stable iff it admits a **rational Lyapunov function** of the type

$$V(x) = \frac{p(x)}{(\sum_{i=1}^{n} x_i^2)^r}$$

where p is a homogeneous polynomial.

Moreover, both V and $-\dot{V}$ have **"strict SOS certificates"** and hence V can be found by SDP.

$$f(cx) = c^d f(x)$$

Linear case, $d = 1$

i.e. $f(x) = Ax$

$r = 0, p(x) = x^T P x$

- Useful also for local asym. stability of non-homogeneous systems.

[AAA, El Khadir, *TAC'18*]

Proof outline

- Rosier: $\exists$ homog. (of deg. k) C^1 Lyap. fn. $R(x)$
- Approximate $R(x)$ and $\nabla R(x)$ on S^{n-1}.
  - Bernstein polys (of order m): $B_m(x) \approx R(x)$, $\nabla B_m(x) \approx R_m(x)$
- Homogenize to a rational fn:

$$C(x) = \frac{B_m(x) + B_m(-x)}{2}, \quad V(x) = \|x\|^k C\left(\frac{x}{\|x\|}\right)$$

- $V(x)$ will be of the form $V(x) = \frac{p(x)}{\|x\|^r}$ → homog. poly
  - Show $V(x) \approx B_m(x)$, $\nabla V(x) \approx B_m(x)$
- $W(x) = V^s(x)$ will have SOS certificates for large enough s (b/c of certain Psatz)

Nonexistence of degree bounds

- So homogeneous systems always admit rational Lyapunov functions
- Unlike the linear case though:

Thm. The degree of the numerator of a rational Lyapunov function **cannot be bounded as a function of the dimension and the degree** of the input (homogeneous) polynomial vector field.

- Nevertheless rational Lyapunov functions may be "arbitrarily better" than polynomial ones…

[AAA, El Khadir, *TAC'18*]

Potential merits of rational Lyapunov functions

Thm. For any integer M, there exists a homogeneous polynomial vector field f of degree 5 in 2 variables such that:

- f admits a rational Lyapunov function with numerator degree 4 and denominator degree 2, but
- f does *not* admit a polynomial Lyapunov function of degree less than M.

The SDP searching for our rational Lyapunov functions is no more expensive than the SDP searching for a polynomial one!

$$V(x) = \frac{p(x)}{(\sum_{i=1}^{n} x_i^2)^r}$$

[AAA, El Khadir, *TAC'18*]

Outline

"Optimization ----> Control"

1) Stability analysis of polynomial ODEs

- Power/limitations of SOS Lyapunov functions
  - Joint work with Bachir El Khadir (Princeton)

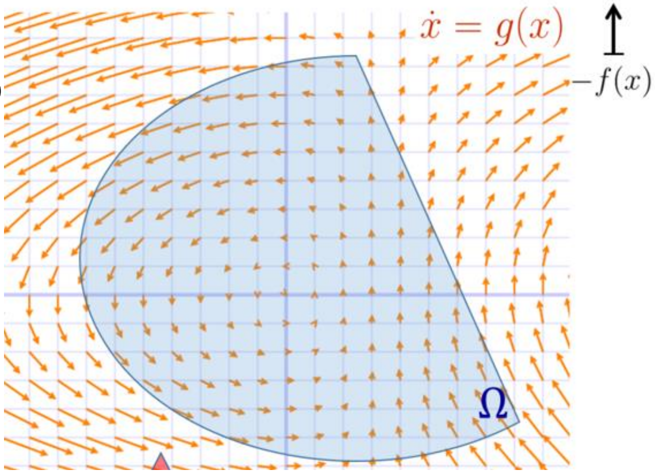
"Control ----> Optimization"

2) Robust-to-dynamics optimization (RDO)

- A new class of robust optimization problems
  - Joint work with Oktay Gunluk (IBM Research)

RDO (informally)

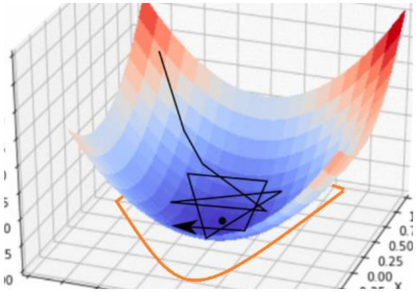
- You solve a constrained optimization problem at time zero
- An external dynamical system may move your optimal point in the future and make it infeasible
- You want your initial decision to be "safe enough" to not let this happen



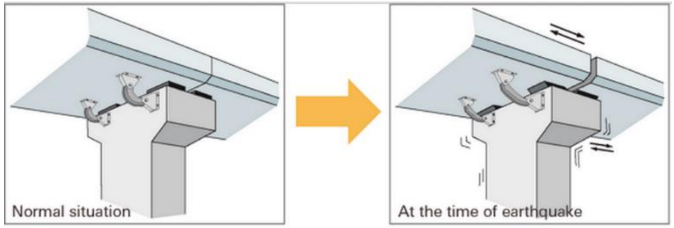
Population control

Learning a dynamical system

Drug design



Projection-free descent



Earthquake-resistant structures

Robust to Dynamics Optimization (RDO)

An RDO is described by two pieces of input:

1) An optimization problem: $\min_{x} \{ f(x) : x \in \Omega \}$

2) A dynamical system: $x_{k+1} = g(x_k)$ (discrete time case)

RDO is then the following problem:

$$\min_{x_0} \{ f(x_0) : x_k \in \Omega, k = 0, 1, 2, \ldots \}$$

This talk:

Optimization Problem	Dynamics
Linear Program	Linear
Quadratic Program	Nonlinear
Integer Program	Uncertain
Semidefinite Program	Time-varying
Polynomial Program, …	Hybrid, …

R-LD-LP

Robust to linear dynamics linear programming (R-LD-LP)

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots ; x_{k+1} = Gx_k\}$$

Input data: A, b, c, G

Alternative form:

$$\min_{x}\{c^T x : Ax \leq b, AGx \leq b, AG^2 x \leq b, AG^3 x \leq b, \ldots\}$$

(An infinite LP)

Feasible set of R-LD-LP:

$$S := \bigcap_{k=0}^{\infty} \{x \mid AG^k x \leq b\}$$

An example...

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots; x_{k+1} = Gx_k\}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0.6 & -0.4 \\ 0.8 & 0.5 \end{bmatrix}$$

$$\boldsymbol{S} = \cap_{k=0}^{\infty} \{\boldsymbol{A}\boldsymbol{G}^k \boldsymbol{x} \leq \boldsymbol{b}\} = \cap_{k=0}^{2} \{\boldsymbol{A}\boldsymbol{G}^k \boldsymbol{x} \leq \boldsymbol{b}\}$$

Obvious way to get lower bounds

$$\min_x \{c^T x : Ax \leq b, AGx \leq b, AG^2 x \leq b, AG^3 x \leq b, ..| \}$$

Truncate!

(outer approximations to the feasible set)

Natural questions:

- Is the feasible set of R-LD-LP always a polyhedron?
- When it is, how large are the number of facets?
- Does the feasible set have a tractable description?

- How to get **upper bounds**?!
  - (We'll see later: from semidefinite programming)

The feasible set of an R-LD-LP

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x \in \mathbb{R}^n \mid AG^k x \leq b\}$$

Theorem.

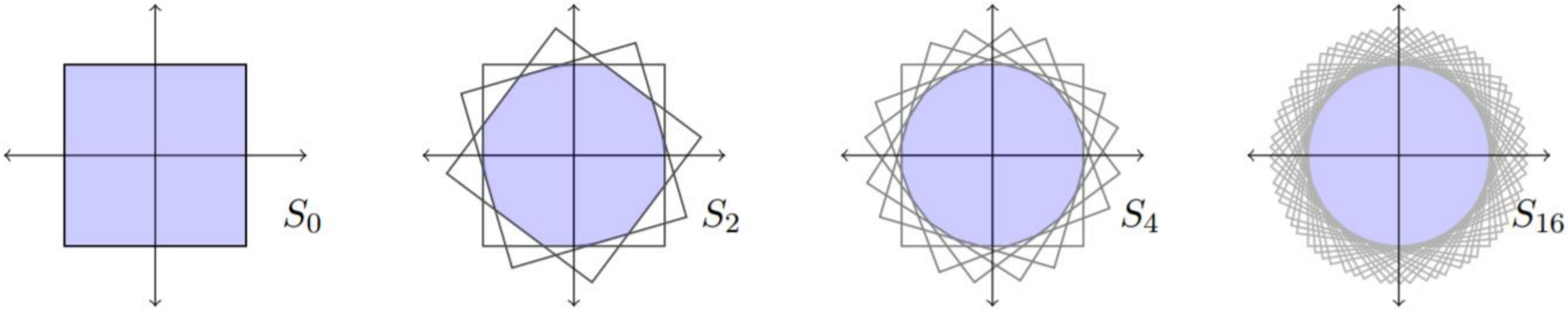
(1) This set is always closed, convex, and invariant.

(2) It is not always polyhedral.

(3) Given A, b, G, and $z \in \mathbb{Q}^n$, it is NP-hard to check whether $z \in S$.

Proof of (2).

$$\{Ax \leq b\} \quad G = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \ \theta \text{ irrational}$$



Finite convergence of outer approximations

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x \in \mathbb{R}^n |\ AG^k x \leq b\}$$

$$S_r := \bigcap_{k=0}^{r} \{x \in \mathbb{R}^n |\ AG^k x \leq b\}$$

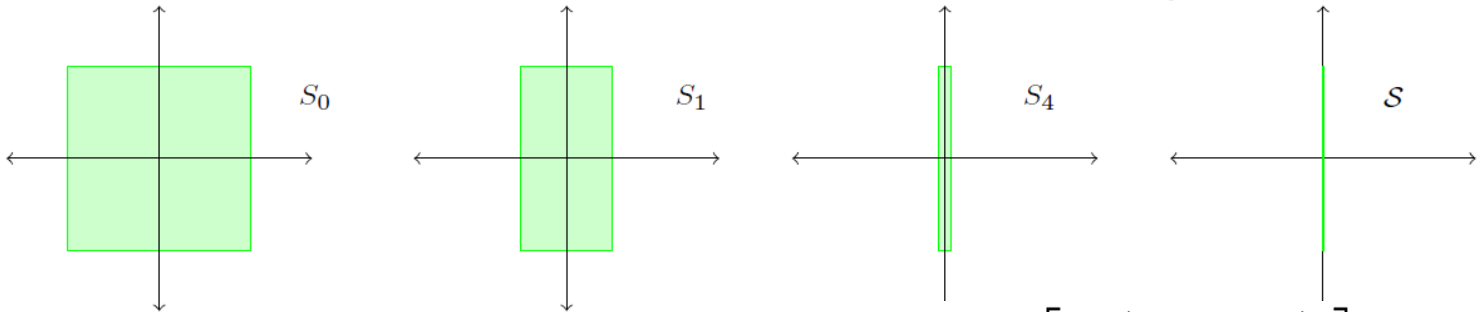
$$\mathcal{S} \subseteq \ldots S_{r+1} \subseteq S_r \subseteq \ldots \subseteq S_2 \subseteq S_1 \subseteq S_0 = P.$$

Lemma. If $S_r = S_{r+1}$, then $S_r = \mathcal{S}$.
(Poly-time checkable condition for fixed r.)

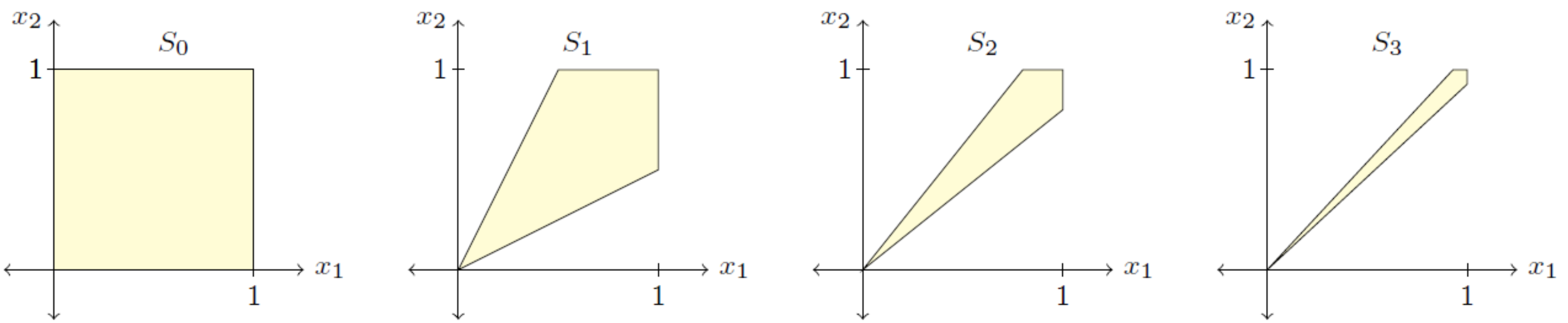
Proposition. There are **three barriers to finite convergence**:
(1) Having $\rho(G) \geq 1$.
(2) Having the origin on the boundary of P.
(3) Having an unbounded polyhedron P.

Barriers to finite convergence

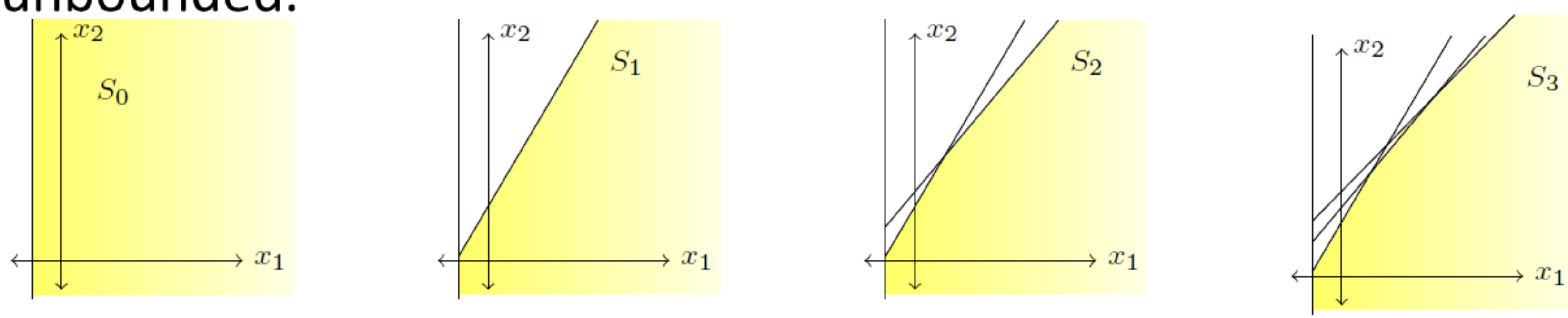
(1) $\rho(G) \geq 1$. $\quad G = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$



(2) The origin on the boundary of P. $\quad G = \begin{bmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{bmatrix} \quad \rho(G) = \frac{1}{2}$



(3) P unbounded.



Computing time to convergence

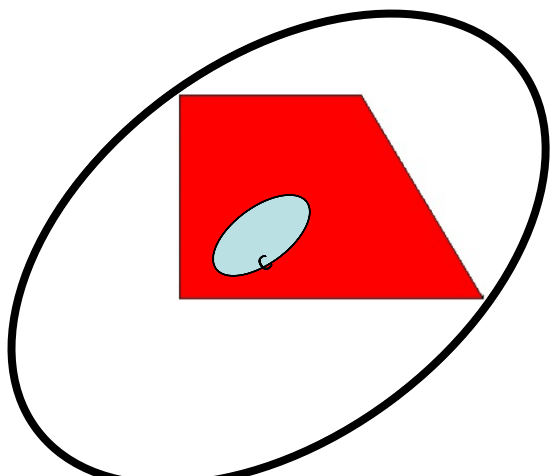
Theorem: If $\rho(G) < 1$, $P = \{Ax \leq b\}$ is bounded and contains the origin in its interior, then

(1) $S = S_r$ for an integer r that can be computed in time $poly\big(\sigma(A, b, G)\big)$.

(2) For any fixed $\rho^* < 1$, all instances of R-LD-LP with $\rho(G) \leq \rho^*$ can be solved in time $poly\big(\sigma(A, b, c, G)\big)$.

Proof idea.

Invariant ellipsoid:
$\{x^T P x \leq 1\}$



Upper bound on the number of iterations

- Find an invariant ellipsoid defined by a positive definite matrix P
- Find a shrinkage factor $\gamma \in (0,1)$; i.e., a scalar satisfying $G^T P G \preceq \gamma P$
- Find a scalar $\alpha_2 > 0$ such that

$$\{Ax \leq b\} \subseteq \{x^T P x \leq \alpha_2\}$$

- Find a scalar $\alpha_1 > 0$ such that

$$\{x^T P x \leq \alpha_1\} \subseteq \{Ax \leq b\}$$

- Let

$$r = \lceil \frac{\log \frac{\alpha_1}{\alpha_2}}{\log \gamma} \rceil$$

Finding an invariant ellipsoid

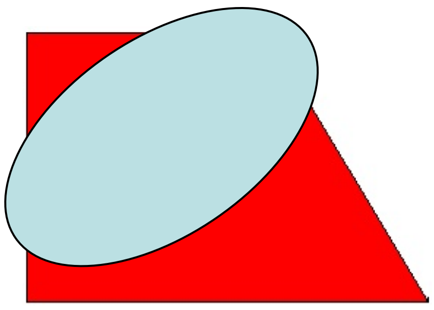
- **Computation of P.**

To find an invariant ellipsoid for G, we solve the linear system

$$G^T PG - P = -I,$$

where I is the $n \times n$ identity matrix. This is called the Lyapunov equation.

The matrix P will automatically turn out to be positive definite.



Finding the shrinkage factor

- **Computation of γ.**

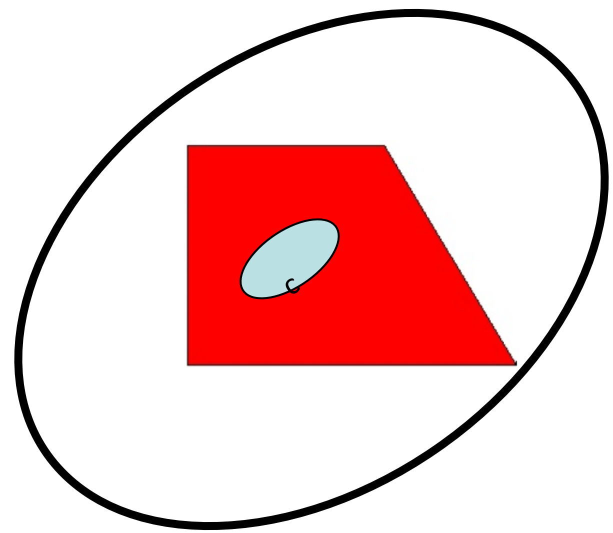
$$\gamma = 1 - \frac{1}{\max_i\{P_{ii} + \sum_{j\neq i} |P_{i,j}|\}}.$$

Proof idea.

$$\begin{aligned} x^T G^T P G x &= x^T P x - x^T x \\ &\leq x^T P x (1-\eta) \end{aligned}$$

where η is any number such that

$$\eta x^T P x \leq x^T x$$



Shrinkage is at least $1 - \frac{1}{\lambda_{max}(P)}$

$$\lambda_{max}(P) \leq \max_i \{P_{ii} + \sum_{j\neq i} |P_{i,j}|\}.$$

(Bound from Gershgorin's circle theorem)

Finding the outer ellipsoid

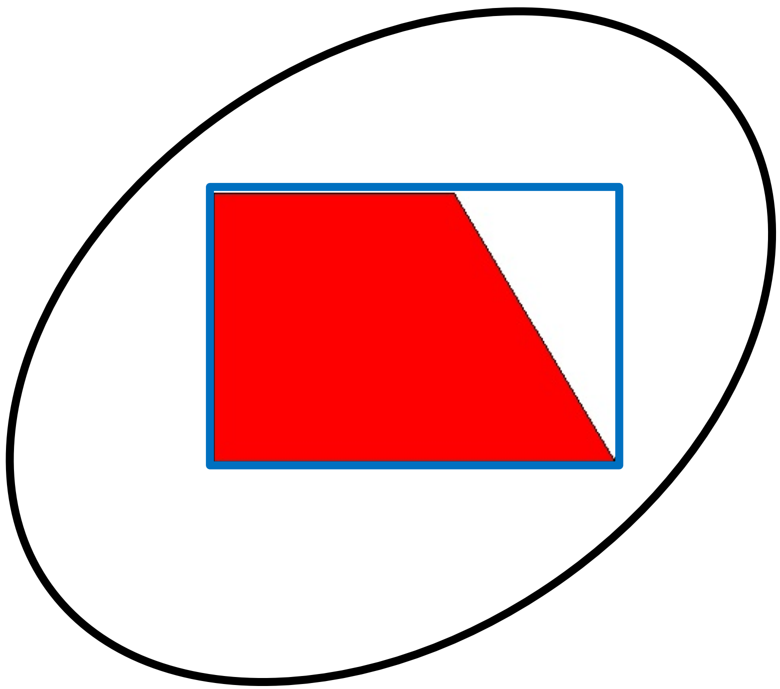
- **Computation of α_2.** By solving, e.g., n LPs, we can place our polytope $\{Ax \le b\}$ in a box; i.e., compute $2n$ scalars l_i, u_i such that

$$\{Ax \le b\} \subseteq \{l_i \le x_i \le u_i\}.$$

We then bound $x^T P x = \sum_{i,j} P_{i,j} x_i x_j$ term by term to get α_2:

$$\alpha_2 = \sum_{i,j} \max\{P_{i,j} u_i u_j, P_{i,j} l_i l_j, P_{i,j} u_i l_j, P_{i,j} l_i u_j\}.$$

This ensures that $\{l_i \le x_i \le u_i\} \subseteq \{x^T P x \le \alpha_2\}$. Hence, $\{Ax \le b\} \subseteq \{x^T P x \le \alpha_2\}$.



Finding the inner ellipsoid

- **Computation of α_1.** For $i = 1, \ldots, m$, we compute a scalar η_i by solving the convex program

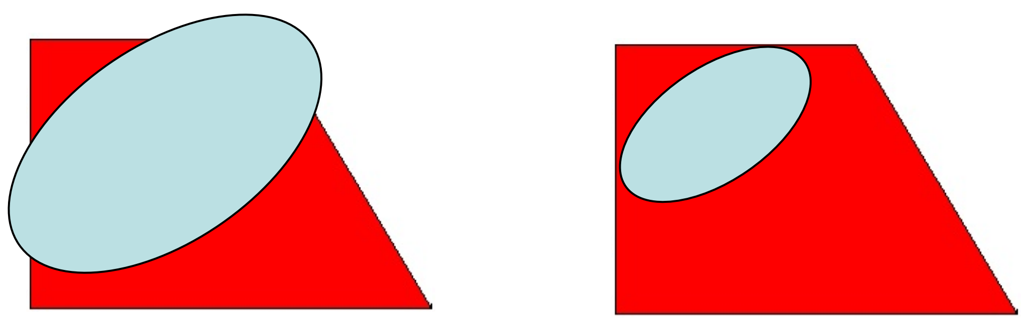
$$\eta_i := \max_{x \in {}^n} \{a_i^T x : x^T P x \leq 1\}$$

where a_i is the i-th row of the constraint matrix A. This problem has a closed form solution:

$$\eta_i = -\sqrt{a_i^T P^{-1} a_i}.$$

Note that P^{-1} exists since $P \succ 0$. We then let

$$\alpha_1 = \min_i \{\frac{b_i^2}{\eta_i^2}\}.$$



Recap

R-LD-LP:

$$\min_x \{c^T x : Ax \leq b, AGx \leq b, AG^2 x \leq b, AG^3 x \leq b, \dots\}$$

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x \in \mathbb{R}^n \mid AG^k x \leq b\}$$

Outer approximations:

(gives lower bounds on the optimal value)

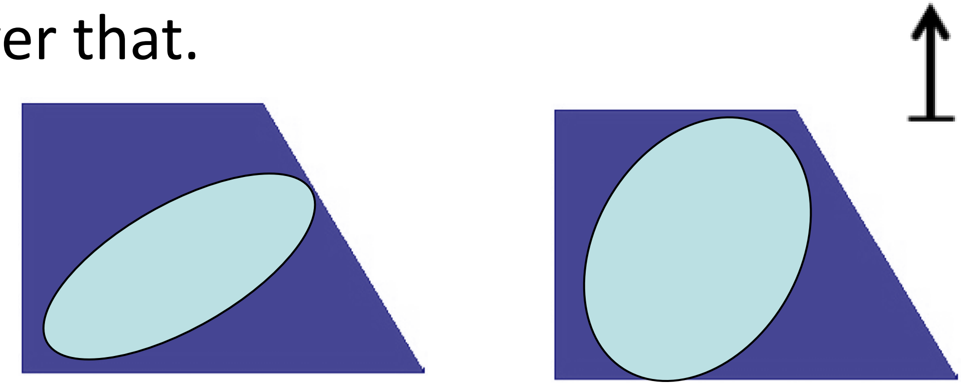
$$S_r := \bigcap_{k=0}^{r} \{x \in \mathbb{R}^n \mid AG^k x \leq b\}$$

$$\mathcal{S} \subseteq \dots S_{r+1} \subseteq S_r \subseteq \dots \subseteq S_2 \subseteq S_1 \subseteq S_0 = P.$$

What about upper bounds? Need inner approximations!

Upper bounds on R-LD-LP via SDP

- Goal: Find the best invariant ellipsoid inside the original polytope and optimize over that.



$$\min_{x,P} \quad c^T x$$

$$P \succ 0$$

$$G^T P G \preceq P$$

$$x^T P x \leq 1$$

$$[\forall z, \ \ z^T P z \leq 1 \Rightarrow Az \leq b]$$

Non-convex formulation

(even after the application of the S-lemma)

Upper bounds on R-LD-LP via SDP

- If we parameterize in terms of P^{-1} instead, then it becomes convex!

An improving sequence of SDPs

- Goal: Find the best point that lands in an invariant set.

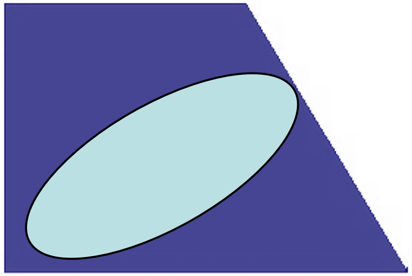
$$\min_{x,P} \quad c^T x$$

$$P \succ 0$$

$$G^T P G \preceq P$$

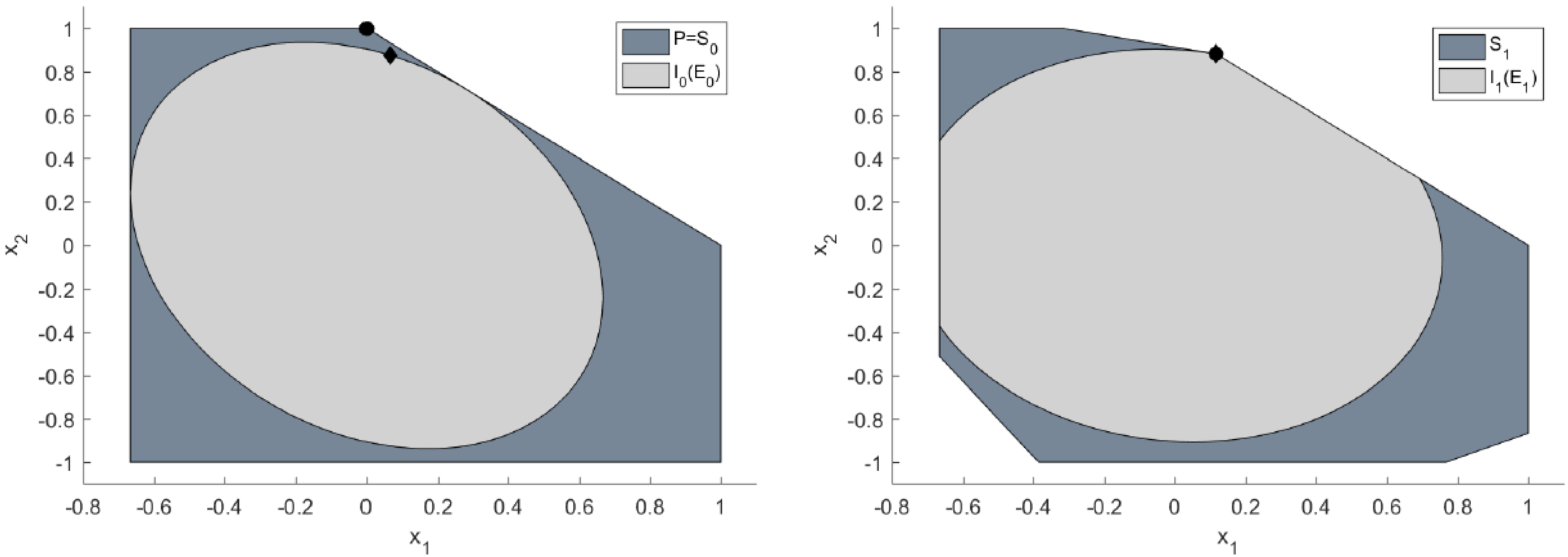
$$x^T P x \leq 1$$

$$\left[\forall z, \quad z^T P z \leq 1 \Rightarrow Az \leq b\right]$$



An example

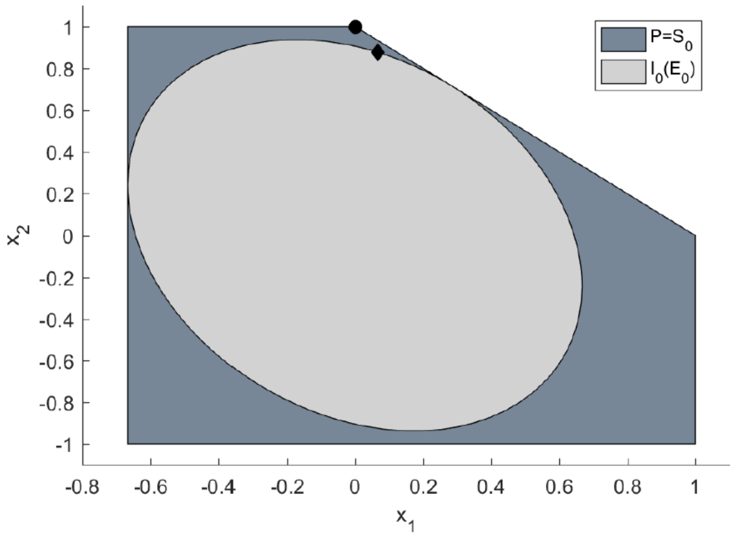
$$A = \begin{pmatrix} 1 & 0 \\ -1.5 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad c = -\begin{pmatrix} 0.5 & 1 \end{pmatrix}, \quad G = \frac{4}{5}\begin{pmatrix} \cos(\theta) & \sin(\theta), \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \text{ where } \theta = \frac{\pi}{6}.$$



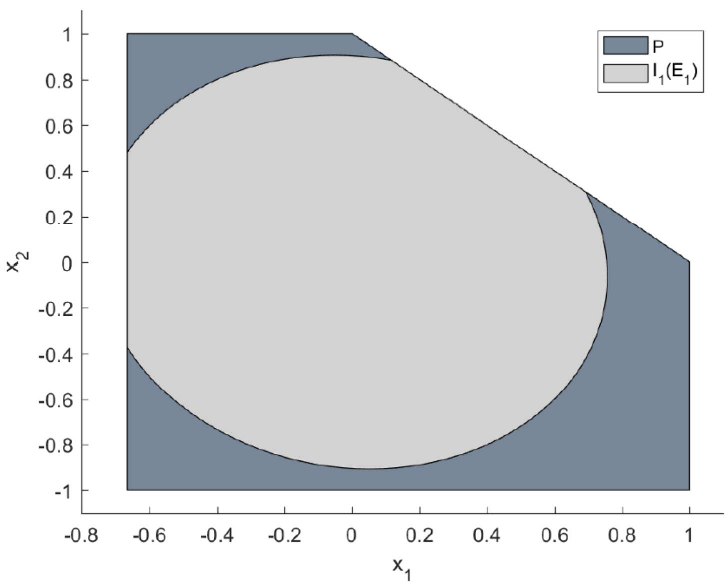
Thm. The SDP upper bound monotonically improves and gives the exact optimal value of R-LD-LP in r^* steps, where r^* is polynomially computable.

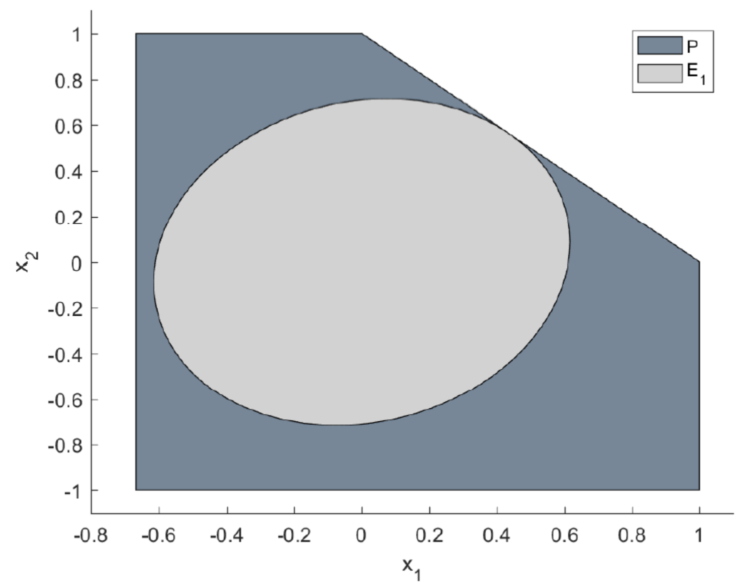
Another interpretation

r=0



r=1



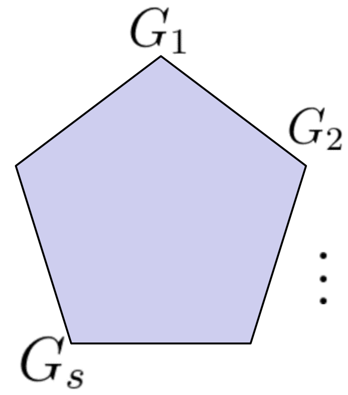


LP
+
Uncertain & time-varying linear systems

R-ULD-LP

Robust to uncertain linear dynamics linear programming (R-ULD-LP)

$$x_{k+1} \in conv\{G_1, \ldots, G_s\} x_k$$



Models **uncertainty** and **variations with time** in the dynamics

$$\min_x \{c^T x : AGx \leq b, \forall G \in \mathbb{G}^*\}$$ (An infinite LP)

$\mathbb{G}^*$: set of all finite products of $G_1, \ldots, G_s$

Input data: $A, b, c, G_1, \ldots, G_s$

Finite convergence of outer approximations

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x \in \mathbb{R}^n | \ AGx \leq b, \forall G \in \mathcal{G}^*\}$$

$$S_r := \bigcap_{k=0}^{r} \{x \in \mathbb{R}^n | \ AGx \leq b, \forall G \in \mathcal{G}^k\}$$

$$\mathcal{S} \subseteq \ldots S_{r+1} \subseteq S_r \subseteq \ldots \subseteq S_2 \subseteq S_1 \subseteq S_0 = P.$$

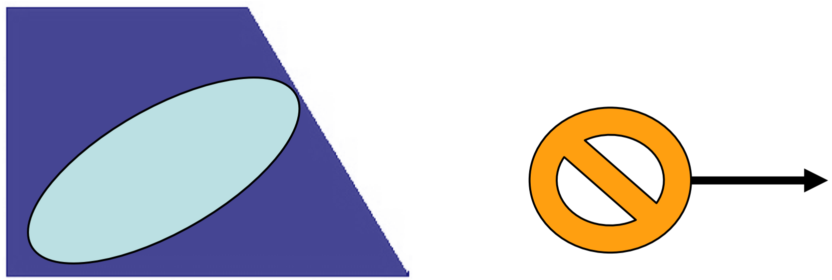
Joint spectral radius (JSR):

$$\rho(G_1, \dots, G_s) = \lim_{k \to \infty} \max_{\sigma \in \{1,\dots,s\}^m} ||G_{\sigma_1} \cdots G_{\sigma_k}||^{\frac{1}{k}}$$

Theorem. If $\rho(G_1, \dots, G_s) < 1$, and $P = \{Ax \leq b\}$ is bounded and contains the origin in its interior, then $S = S_r$, for some r.

(However, number of facets of S is typically very large.)

What about inner approximations?



$$\min_{x,Q} \; c^T x$$

$$Q \succ 0$$

$$G Q G^T \preceq Q$$

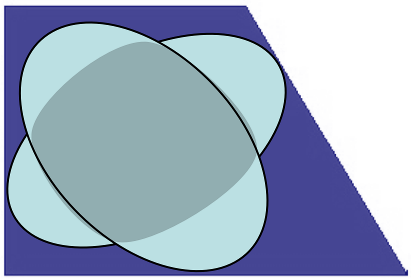
$$\begin{bmatrix} Q & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

$$a_i^T Q a_i \leq 1$$

Invariant ellipsoid may not exist even when JSR<1

Idea: search for intersection of ellipsoids instead!

Guaranteed to exist!

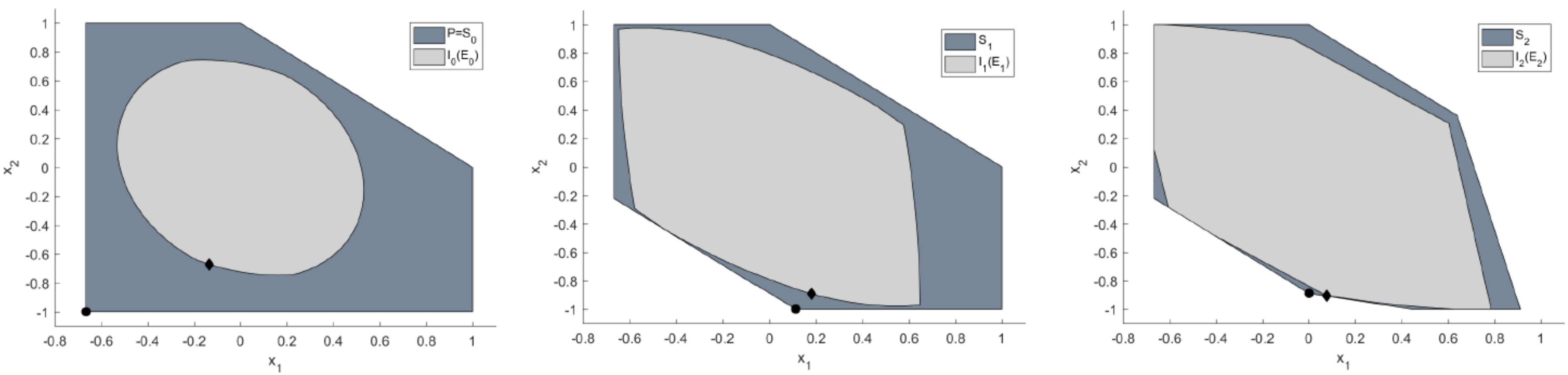


- The convexification tricks go through!
- Finite convergence of upper bounds is guaranteed.

$$
\begin{aligned}
\underset{x\in\mathbb{R}^n, Q_{1,2}\in S^{n\times n}}{\text{minimize}} \quad & c^T x \\
\text{s.t.} \quad & Q_1 \succ 0, Q_2 \succ 0 \\
& G_1 Q_1 G_1^T \preceq Q_1 \\
& G_2 Q_1 G_2^T \preceq Q_2 \\
& G_1 Q_2 G_1^T \preceq Q_1 \\
& G_2 Q_2 G_2^T \preceq Q_2 \\
& \begin{bmatrix} Q_1 & \tilde{G}x \\ (\tilde{G}x)^T & 1 \end{bmatrix} \succeq 0, \ \forall \tilde{G} \in \mathcal{G}^r \\
& \begin{bmatrix} Q_2 & \tilde{G}x \\ (\tilde{G}x)^T & 1 \end{bmatrix} \succeq 0, \ \forall \tilde{G} \in \mathcal{G}^r \\
& a_i^T Q_1 a_i \leq 1 \\
& a_i^T Q_2 a_i \leq 1 \\
& A\tilde{G}x \leq 1, \ \forall \tilde{G} \in \mathcal{G}^k, k = 0, \ldots, r-1
\end{aligned}
$$

A numerical example of an R-ULD-LP

$$A = \begin{pmatrix} 1 & 0 \\ -1.5 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, c = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, G_1 = \begin{pmatrix} -1/4 & -1/4 \\ -1 & 0 \end{pmatrix}, \text{ and } G_2 = \begin{pmatrix} 3/4 & 3/4 \\ -1/2 & 1/4 \end{pmatrix}.$$



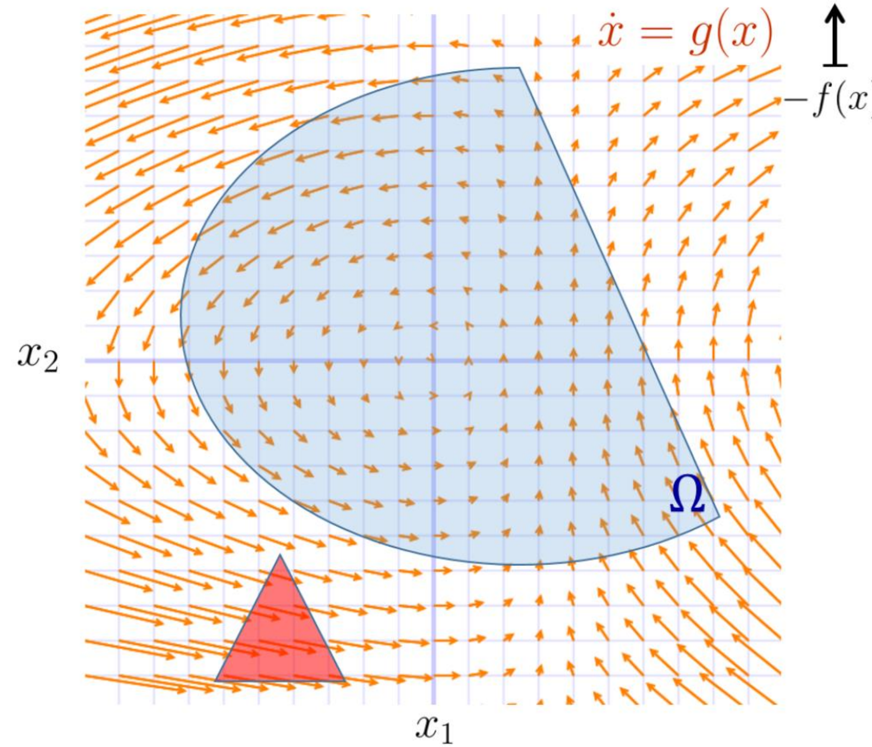
(a) $r = 0$ (b) $r = 1$ (c) $r = 2$

	$r = 0$	$r = 1$	r=2
Lower bounds	-1.3333	-0.9444	-0.8889
Upper bounds	-0.7395	-0.8029	-0.8669

A broader agenda

Optimization problems with dynamical systems (DS) constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \Omega \cap \Omega_{DS}. \end{array}$$



Optimization Problem "f, Ω"	Type of Dynamical System "g"	DS Constraint "Ω_{DS}"
Linear program*	Linear*	Invariance*
Convex quadratic program*	Linear and uncertain/stochastic	Inclusion in region of attraction
Semidefinite program	Linear and time-varying*	Collision avoidance
Robust linear program	Nonlinear (polynomial)	Reachability
Polynomial program	Nonlinear and time-varying	Orbital stability
Integer program	Discrete/continuous/hybrid of both	Stochastic stability
⋮	⋮	⋮

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