Computation of the Joint Spectral Radius with Optimization Techniques

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Joint work with:

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- **Pablo A. Parrilo** (MIT)
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Workshop on Real Algebraic Geometry with a View Towards Systems Control and Free Positivity
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The Joint Spectral Radius

Given a finite set of $n \times n$ matrices

$$\mathcal{A} := \{A_1, \ldots, A_m\}$$

Joint spectral radius (JSR):

$$\rho(\mathcal{A}) = \lim_{k \to \infty} \max_{\sigma \in \{1,\ldots,m\}^k} \|A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

If only one matrix:

$$\mathcal{A} = \{A\}$$

Spectral Radius

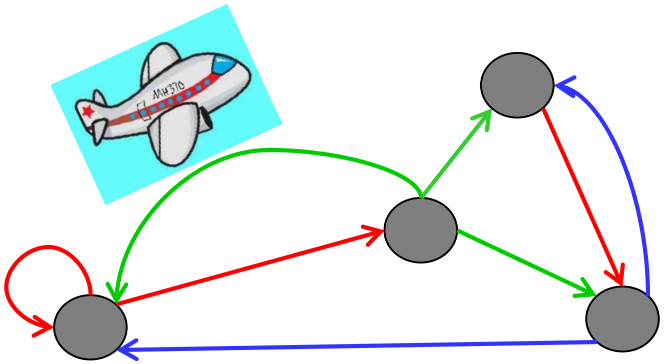
$$\rho(A) = \lim_{k \to \infty} ||A^k||^{\frac{1}{k}}$$

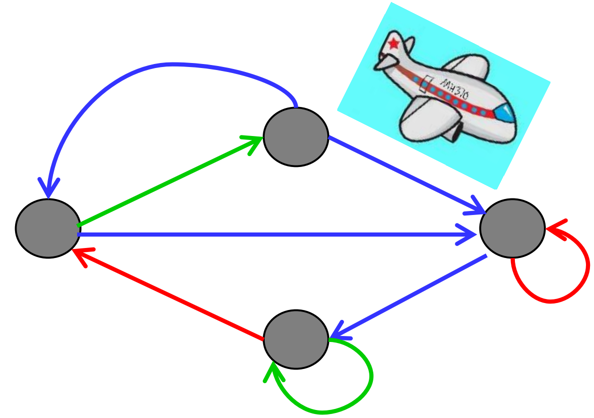
G. C. Rota and W. G. Strang

A note on the joint spectral radius

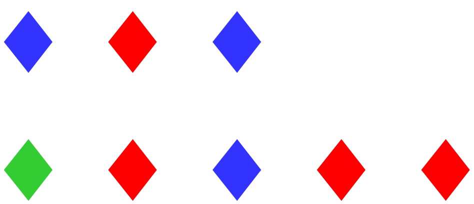
Indag. Math., 22:379–381, 1960.

Trackability of Graphs





Noisy observations:



How does the number of possible paths grow with length of observation?

N(t): max. number of possible paths over all observations of length *t*

Graph is called ***trackable*** if ***N(t)*** is bounded by a polynomial in *t*

$$\mathcal{A} = \{A_1, A_2, A_3\}$$

$$\rho(\mathcal{A}) = \lim_{t\to\infty} N(t)^{1/t}$$

Graph ***trackable*** iff $\rho(\mathcal{A}) \leq 1$

JSR and Switched/Uncertain Linear Systems

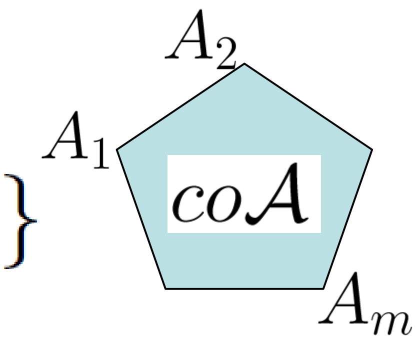
Linear dynamics: $x_{k+1} = Ax_k$

Spectral radius: $\rho(A) = \lim_{k\to\infty} ||A^k||^{\frac{1}{k}}$

"Stable" iff $\rho(A) < 1$

Switched linear dynamics: $x_{k+1} = A_i x_k$

$$\mathcal{A} := \{A_1, \ldots, A_m\}$$

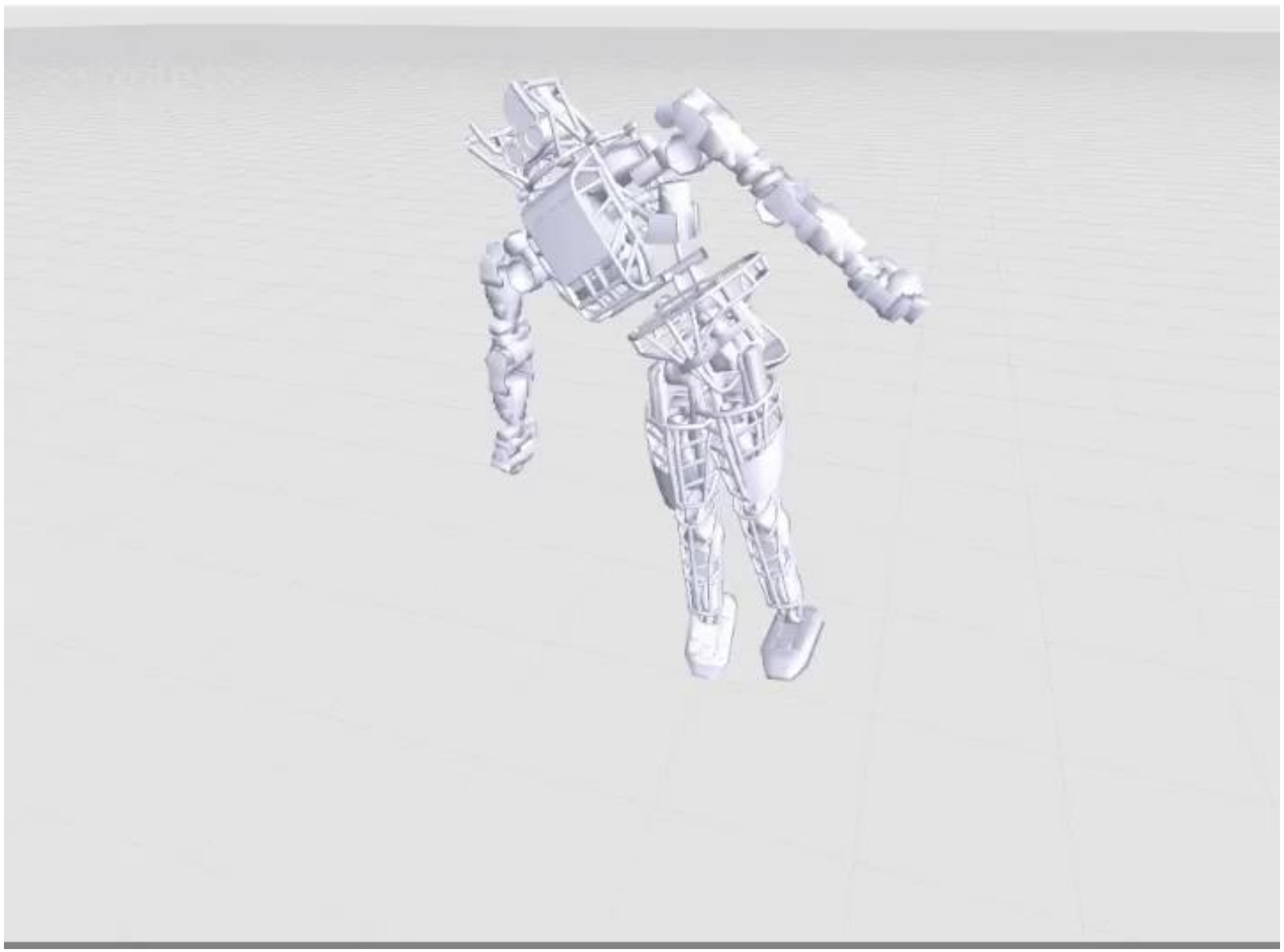


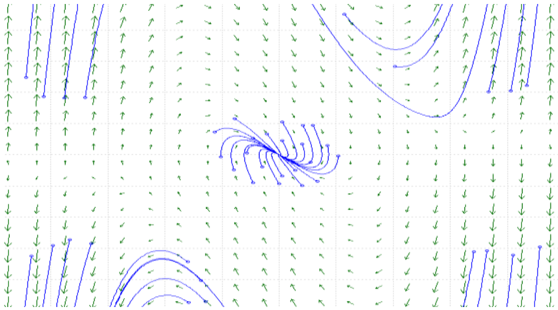
Joint spectral radius (JSR):

$$\rho(\mathcal{A}) = \lim_{k\to\infty} \max_{\sigma\in\{1,\ldots,m\}^k} ||A_{\sigma_k} \ldots A_{\sigma_2} A_{\sigma_1}||^{1/k}$$

"Uniformly stable" iff $\rho(\mathcal{A}) < 1$

Stability around an equilibrium point





Controller design for this humanoid presented in:

[Majumdar, AAA, Tedrake, CDC'14 – submitted]

Done by **SDSOS Optimization**
[AAA, Majumdar,'13]

Computation of JSR

If only one matrix: $\mathcal{A} = \{A\}$

Testing "$\rho(A) < 1$?" decidable in polynomial time

For more than one matrix:

Testing "$\rho(\mathcal{A}) \leq 1$?" undecidable **[Blondel, Tsitsiklis]**

(even for 2 matrices of size 47x47 !!)

- **Open problem:** decidability of testing "$\rho(\mathcal{A}) < 1$?"
- Would become decidable if *rational finiteness conjecture* is true
- **Lower bounds on JSR:** $\rho(A_{\sigma_k} \dots A_{\sigma_1})^{1/k} \leq \rho(\mathcal{A})$
  - Finiteness conjecture: equality achieved at finite *k*
- **Upper bounds on JSR:** from Lyapunov theory

This Talk

1. **A meta-SDP-algorithm for computing upper bounds**

   Lyapunov theory + basic automata theory

2. **Exact JSR of rank-one matrices**

   via dynamic programming

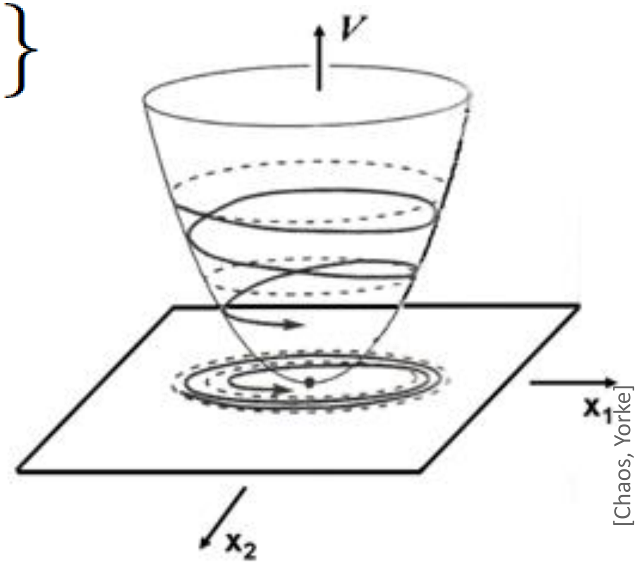
3. **Uncertain nonlinear systems**

   SOS-convex Lyapunov functions

Common Lyapunov function

$$x_{k+1} = A_i x_k$$

$$\mathcal{A} := \{A_1, ..., A_m\}$$



If we can find a function $V(x) : \mathbb{R}^n \to \mathbb{R}$

such that $V(x) > 0,$

$$V(A_i x) < V(x),\ \forall i = 1, \ldots, m$$

then, $\rho(\mathcal{A}) < 1$

Such a function always exists! But may be extremely difficult to find!!

Computationally-friendly common Lyapunov functions

$$x_{k+1} = A_i x_k \qquad \mathcal{A} := \{A_1, \ldots, A_m\}$$

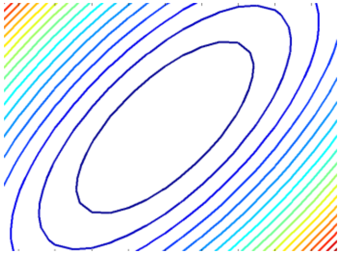
If we can find a function $V(x) : \mathbb{R}^n \to \mathbb{R}$

such that

$$V(x) > 0,$$
$$V(A_i x) < V(x), \; \forall i = 1, \ldots, m$$

then, $\rho(\mathcal{A}) < 1$

- Common quadratic Lyapunov function: $V(x) = x^T P x$

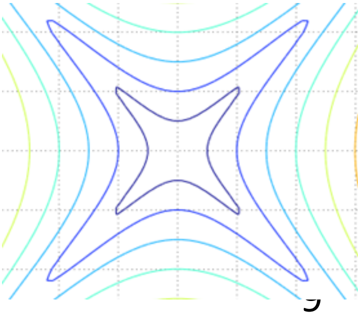


$$A_i^T P A_i \preceq P \; \forall i = 1, \ldots, m$$

[Ando, Shih]

[Blondel, Nesterov, Theys]

- Common SOS Lyapunov function [Parrilo, Jadbabaie]



Our approach: use **multiple Lyapunov functions**

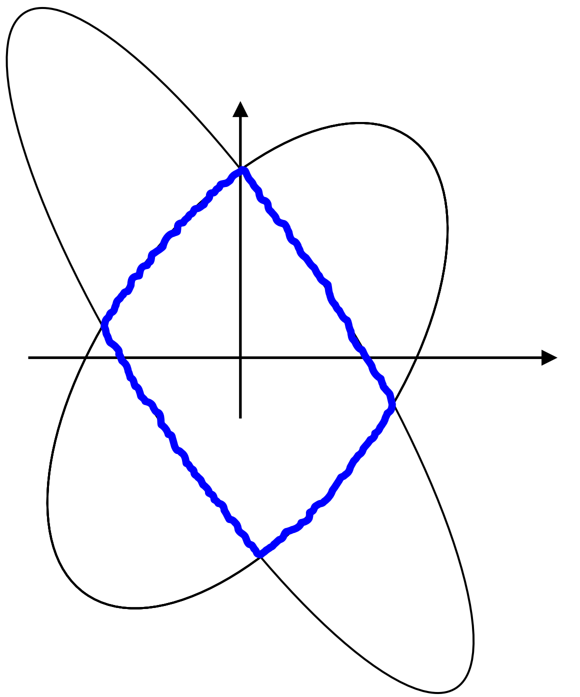
Multiple Lyapunov functions

- Can we do better with more than one Lyapunov function?
- How?
- Consider the SDP:

$$
\begin{array}{rcl}
A_1^T P_1 A_1 & \preceq & P_1, \\
A_2^T P_1 A_2 & \preceq & P_2, \\
A_1^T P_2 A_1 & \preceq & P_1, \\
A_2^T P_2 A_2 & \preceq & P_2, \\
P_i & \succ & 0.
\end{array}
$$

$\Rightarrow \rho(\mathcal{A}) \leq 1$

max-of-quadratics



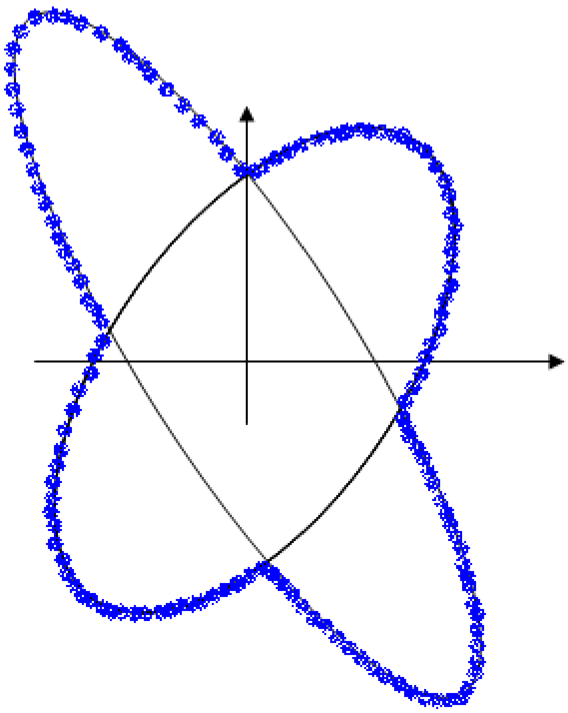
Multiple Lyapunov functions

- Consider another SDP:

$$
\begin{aligned}
A_1^T P_1 A_1 &\preceq P_1, \\
A_2^T P_2 A_2 &\preceq P_1, \\
A_1^T P_1 A_1 &\preceq P_2, \\
A_2^T P_2 A_2 &\preceq P_2, \\
P_i &\succ 0.
\end{aligned}
$$

$\Rightarrow \rho(\mathcal{A}) \leq 1$

min-of-quadratics



Even stranger SDPs…

- Feasibility of the following SDP also implies stability:

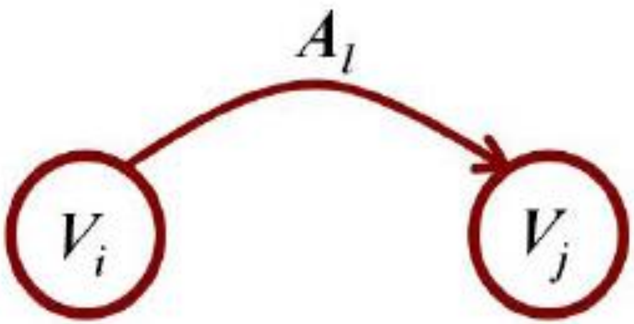
$$\begin{array}{rcl} A_1^T P A_1 & \preceq & P, \\ (A_2 A_1)^T P (A_2 A_1) & \preceq & P, \\ (A_2^2)^T P (A_2^2) & \preceq & P, \\ P & \succ & 0. \end{array}$$

$$\Rightarrow \rho(\mathcal{A}) \leq 1$$

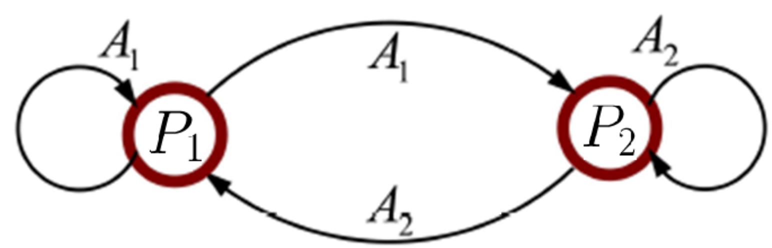
- Where do these conditions come from?
- Can we give a unifying framework?

Representation of Lyapunov inequalities via labeled graphs

$$V_j(A_l x) \le V_i(x) \quad \forall x \in \mathbb{R}^n$$



$$\begin{aligned} A_1^T P_1 A_1 &\preceq P_1 \\ A_1^T P_2 A_1 &\preceq P_1 \\ A_2^T P_1 A_2 &\preceq P_2 \\ A_2^T P_2 A_2 &\preceq P_2 \\ P_{1,2} &\succ 0 \end{aligned}$$

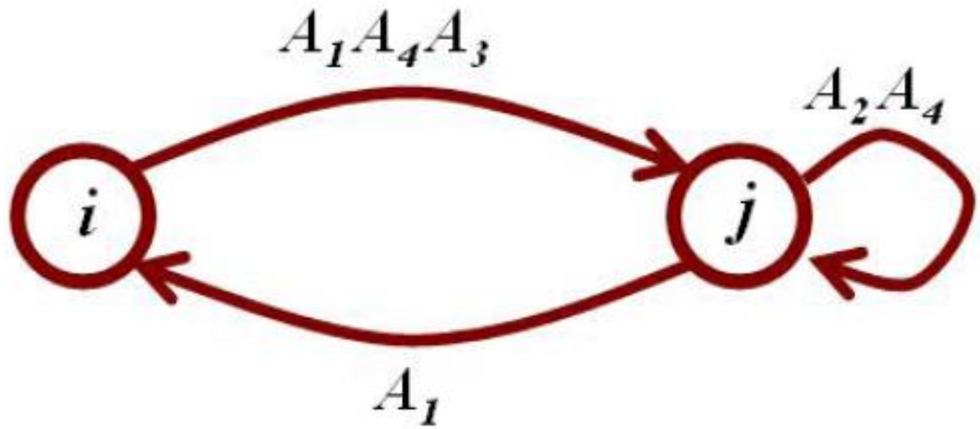


[AAA, Jungers, Parrilo, Roozbehani
SIAM J. on Control and Opt.,'13]

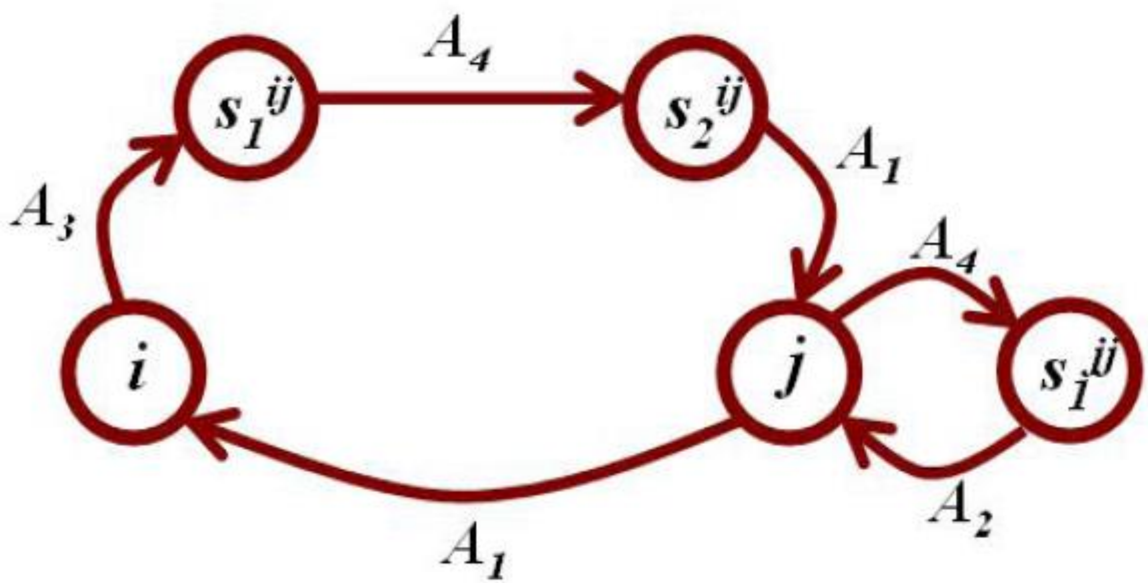
- What property of the graph implies stability?

Graph expansion

Graph $G(N,E)$



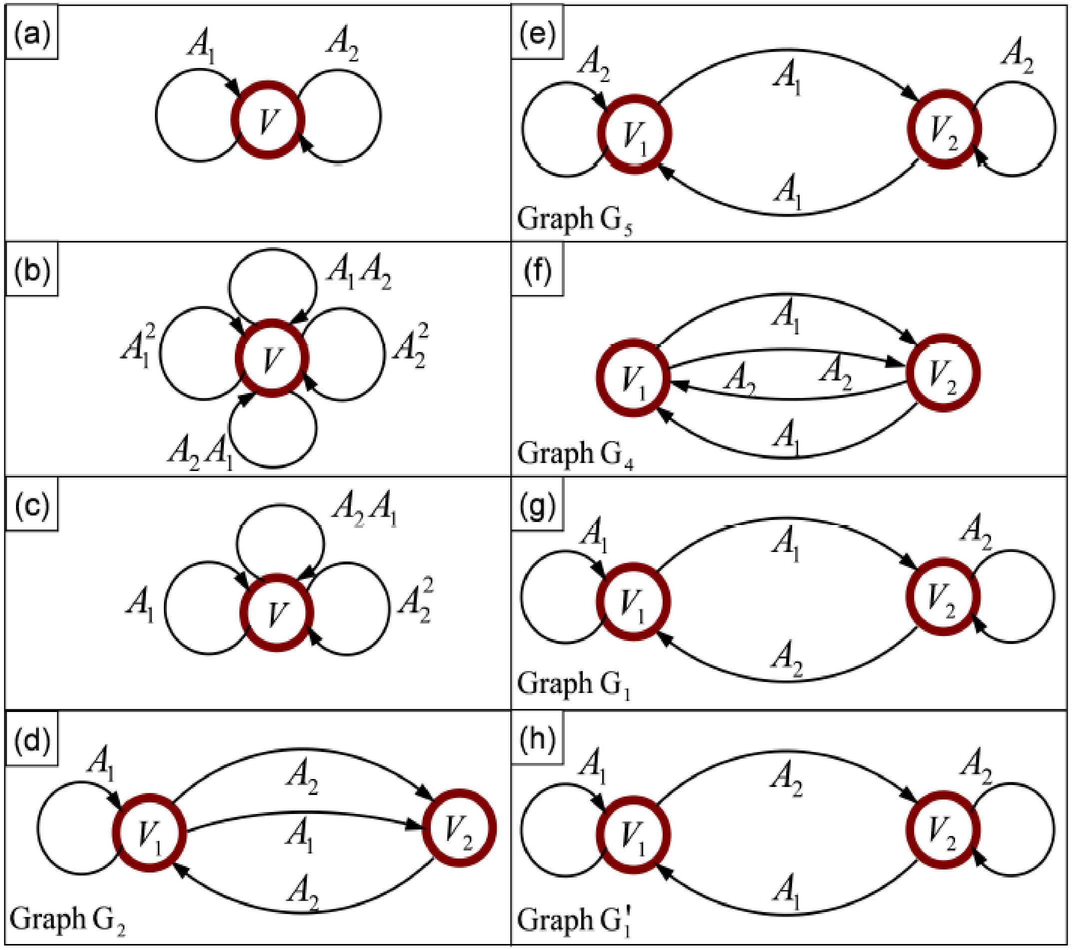
Expanded Graph $G^e(N^e,E^e)$



Path-complete graphs

Defn. A labeled directed graph $G(N,E)$ is path-complete if for every word of finite length there is an associated directed path in its expanded graph $G^e(N^e,E^e)$.

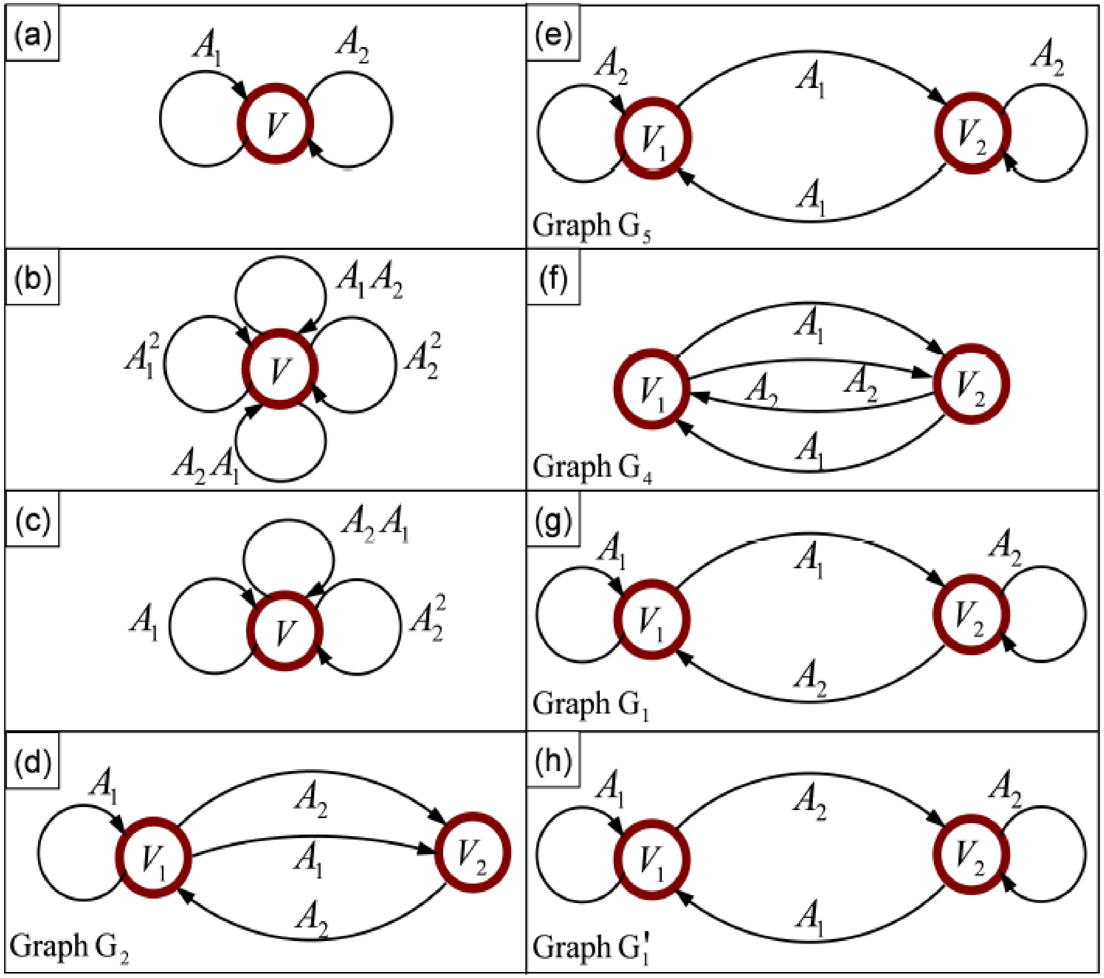
- Path-completeness can be checked with standard algorithms in **automata theory**



Path-complete graphs and stability

THM. If Lyapunov functions satisfying Lyapunov inequalities associated with **any path-complete graph** are found, then the switched system is uniformly stable (i.e., JSR<1).

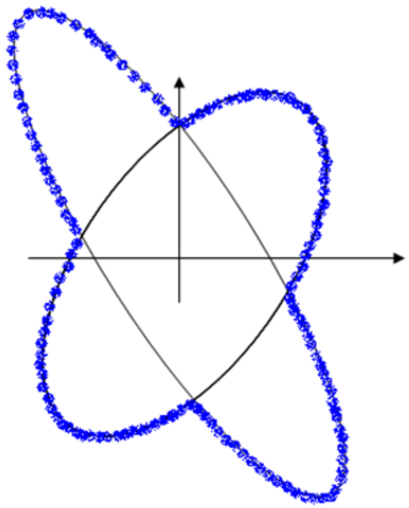
- Gives immediate proofs for existing methods
- Introduces numerous new methods



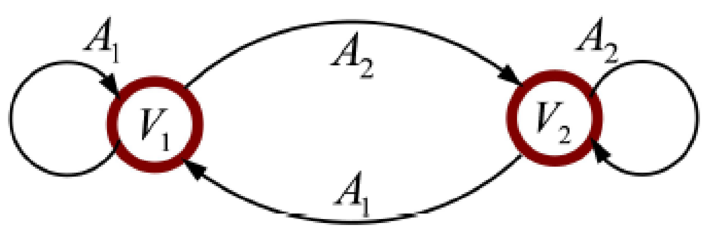
Quick proofs

For example:

min-of-quadratics

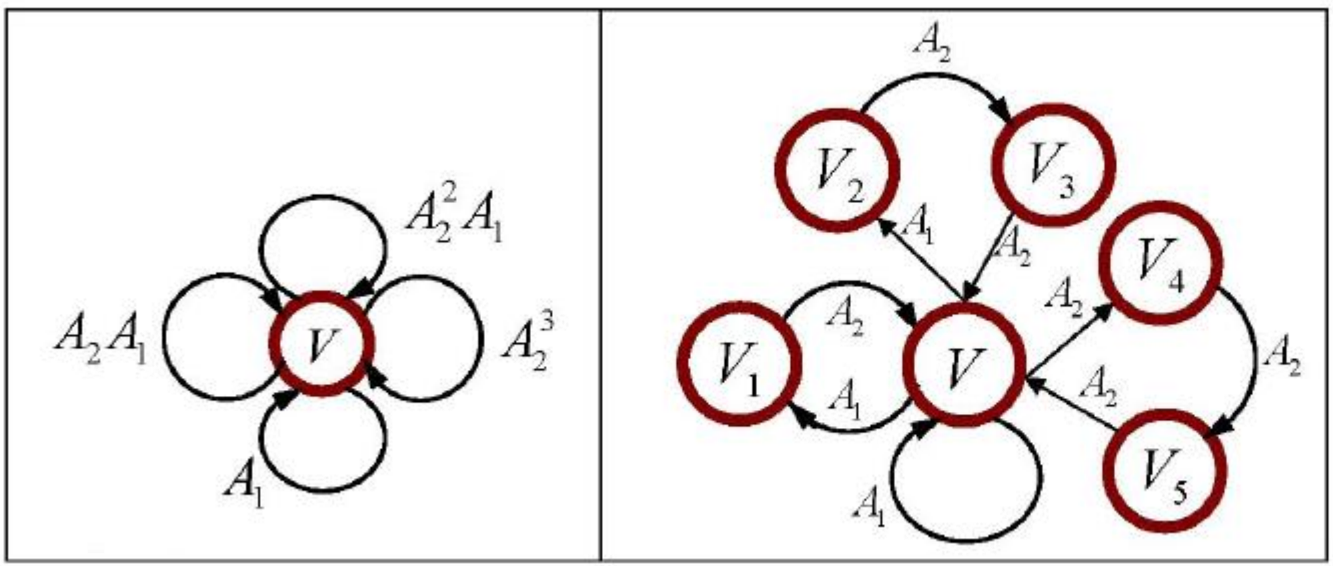


$$
\begin{aligned}
A_1^T P_1 A_1 &\preceq P_1 \\
A_2^T P_2 A_2 &\preceq P_1 \\
A_1^T P_1 A_1 &\preceq P_2 \\
A_2^T P_2 A_2 &\preceq P_2
\end{aligned}
$$



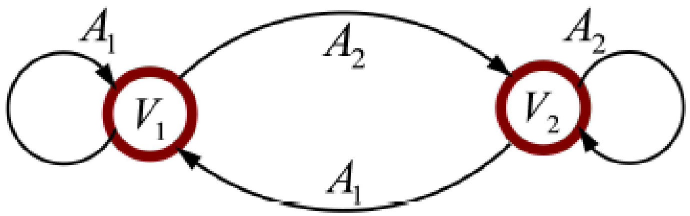
Let's revisit our strange SDP

$$\begin{aligned} A_1^T P A_1 &\preceq P \\ (A_2 A_1)^T P (A_2 A_1) &\preceq P \\ (A_2^2 A_1)^T P (A_2^2 A_1) &\preceq P \\ (A_2^3)^T P (A_2^3) &\preceq P \end{aligned}$$

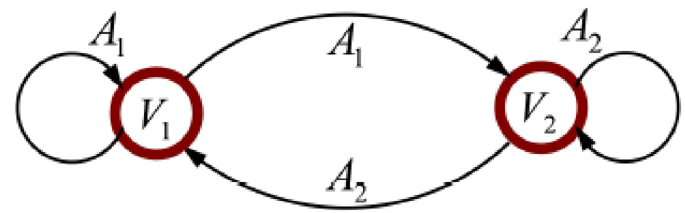


Approximation guarantees

min-of-quadratics



max-of-quadratics



$$\frac{1}{\sqrt[4]{n}}\hat{\rho}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}(\mathcal{A})$$

- tighter than known SOS bounds
- proof relies on the John's ellipsoid thm

THM. Given any desired accuracy

$$\frac{1}{\sqrt[2l]{n}}\hat{\rho}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \hat{\rho}(\mathcal{A})$$

we can explicitly construct a graph G (**with m^{l-1} nodes**) such that the corresponding **SDP** achieves the accuracy.

No bound on size of SDP

THM. [AAA, Jungers, IFAC'14]
Given any positive integer *d*,
there are families of switched systems that are uniformly stable (i.e., have JSR<1), but yet this fact cannot be proven with

- a polynomial Lyapunov function of degree ≤ *d*
- a max-of-quadratics Lyapunov function with ≤ *d* pieces
- a min-of-quadratics Lyapunov function with ≤ *d* pieces
- a polytopic Lyapunov function with ≤ *d* facets.

Kozyakin'90:

$$A_1 = \frac{(1-t^4)}{(1-3\pi t^3/2)}\begin{bmatrix}\sqrt{1-t^2} & -t \\ 0 & 0\end{bmatrix}, \quad A_2 = (1-t^4)\begin{bmatrix}\sqrt{1-t^2} & -t \\ t & \sqrt{1-t^2}\end{bmatrix}$$

$t = \sin\frac{2\pi}{2k+1}$ JSR<1 $\qquad$ $t = \sin\frac{2\pi}{2k}$ JSR>1

JSR of Rank One Matrices and the Maximum Cycle Mean Problem

[AAA, Parrilo, *IEEE Conf. on Decision and Control,*'12]

Basic facts about rank one matrices

A rank one iff $A = xy^T$

spectral radius: $|y^T x|$

Products of rank one matrices have rank at most one:

$$A_i A_j = x_i y_i^T x_j y_j^T$$
$$= (y_i^T x_j) x_i y_j^T$$

Cycles and cycle gains

Easy definitions:

- Cycle
- Simple cycle
- Cycle gain

$$\rho_c = \prod_{i=1}^{k} w(e_i)$$

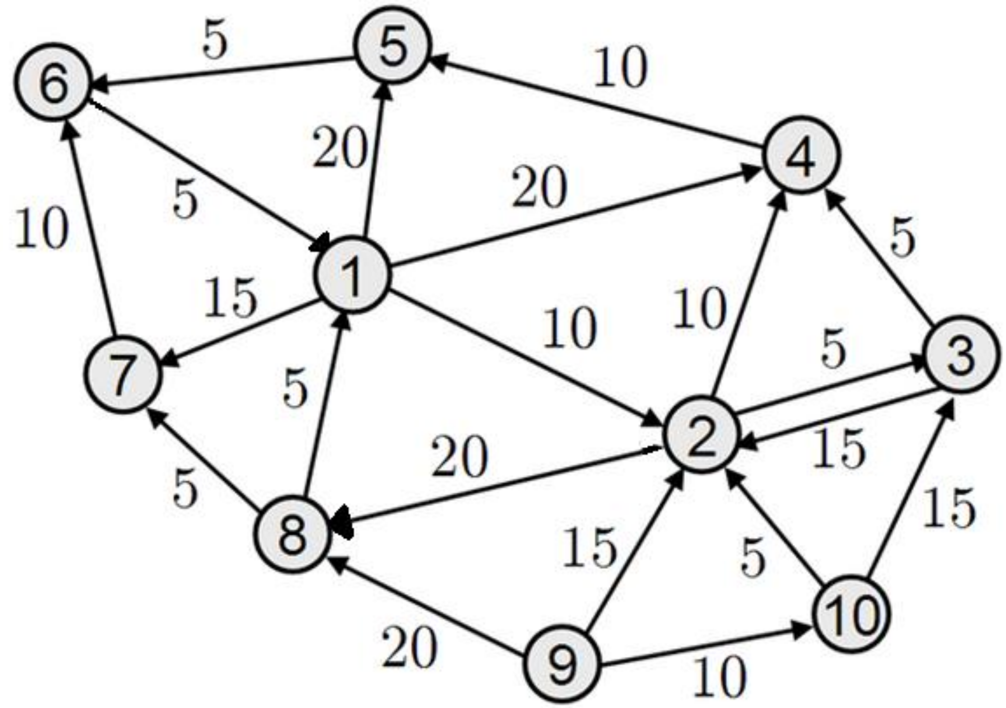
$$g(c) = \rho_c^{1/k}$$

- Maximum cycle gain

$$\max_c g(c)$$

- Gain-maximizing cycle

$$c_{max}$$



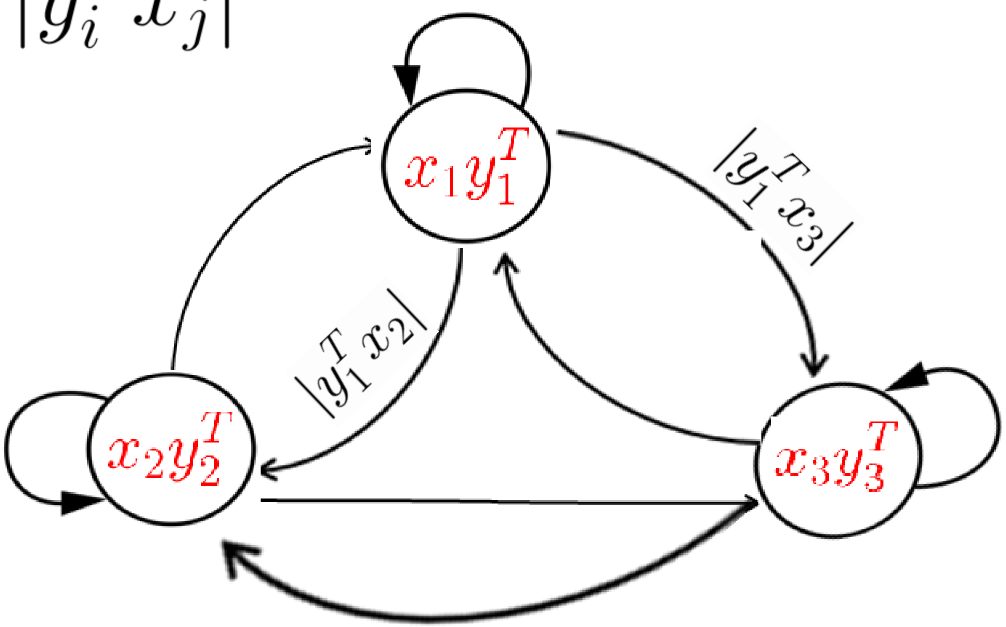
From matrix products to cycles in graphs

$$\mathcal{A} = \{A_1, \ldots, A_m\} \quad A_i = x_i y_i^T$$

$G_{\mathcal{A}}$ Complete directed graph on *m* nodes:

Nodes: matrices A_i

Edge weights: $w(e_{ij}) = |y_i^T x_j|$



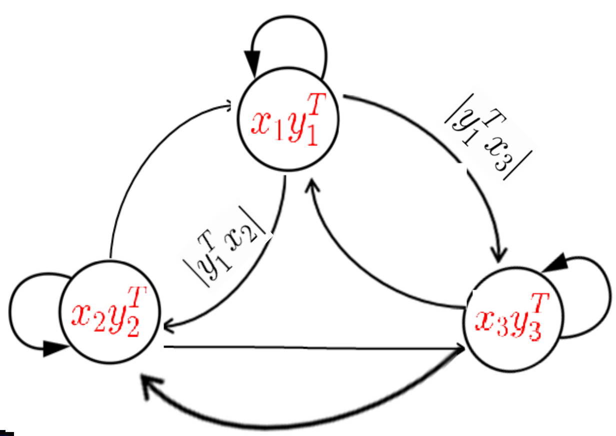
Maximum cycle gain gives the JSR

Thm: Let c_{max} be a gain-maximizing cycle, with l_{max} and ρ_{cmax} denoting its length and the product of the weights on its edges, respectively.
Then, the joint spectral radius is given by:

$$\rho(\mathcal{A}) = \rho_{c_{max}}^{1/l_{max}}$$

Proof sketch:

$$\rho(A_{\sigma_k} \cdots A_{\sigma_1}) = \rho_c$$



$$\begin{aligned} \rho(A_{\sigma_1} \cdots A_{\sigma_k})^{1/k} &= \left(\prod_{i=1}^{s} \rho_{c_i}^{m_i}\right)^{1/k} \\ &= \prod_{i=1}^{s} (\rho_{c_i}^{1/l_i})^{m_i l_i / k} \\ &\leq \rho_{c_{max}}^{1/l_{max}}, \end{aligned}$$

$$\rho(\mathcal{A}) = \limsup_{k \to \infty} \max_{A \in \mathcal{A}^k} \rho^{\frac{1}{k}}(A)$$

■

Finiteness property and the optimal product

Corollary:

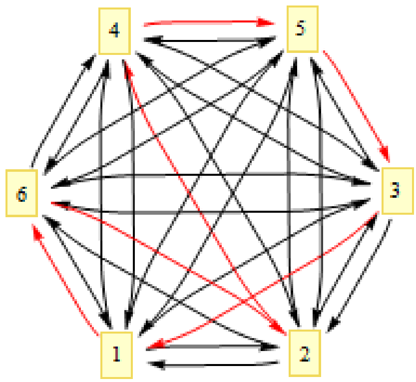
•The JSR is achieved by the spectral radius of a finite matrix product, of length at most m.
(In particular, the finiteness property holds. – independently shown by Gurvits et al.)

•There always exists an optimal product where no matrix appears more than once.

Proof: A simple cycle does not visit a node twice. ■

Bound of m is tight.

Not fun to enumerate all simple cycles…



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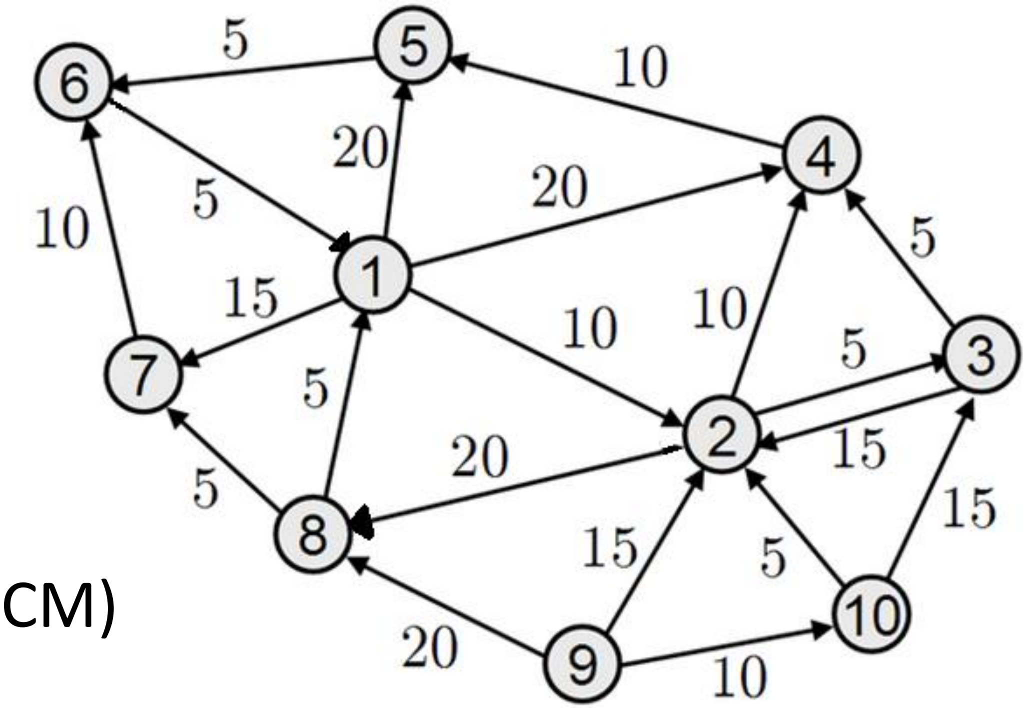
Maximum Cycle Mean Problem (MCMP)

- Cycle mean

$$m(c) = \sum_{i=1}^{k} \frac{w(e_i)}{k}$$

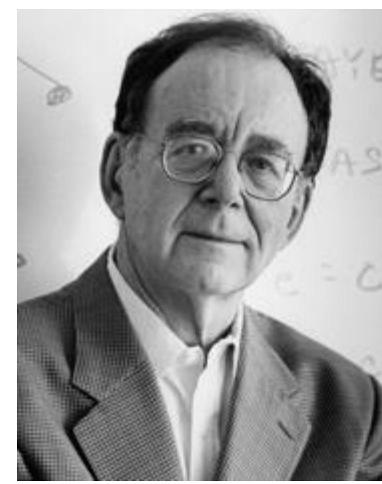
- Maximum cycle mean (MCM)

$$\lambda^* = \max_c m(c)$$



Karp's algorithm for MCMP

- Proposed by Karp in 1978, based on *dynamic programming*
- Let *s* be an arbitrary vertex
- For every vertex *v* and integer *k*, define $F_k(v)$ as the minimum weight of an edge progression of length *k* from *s* to *v*
- $F_k(v)$ can be computed via a simple DP recursion
- From this, the MCM can be computed as:



$$\min_{v} \max_{0 \le k \le n-1} \left[\frac{F_n(v) - F_k(v)}{n-k} \right]$$

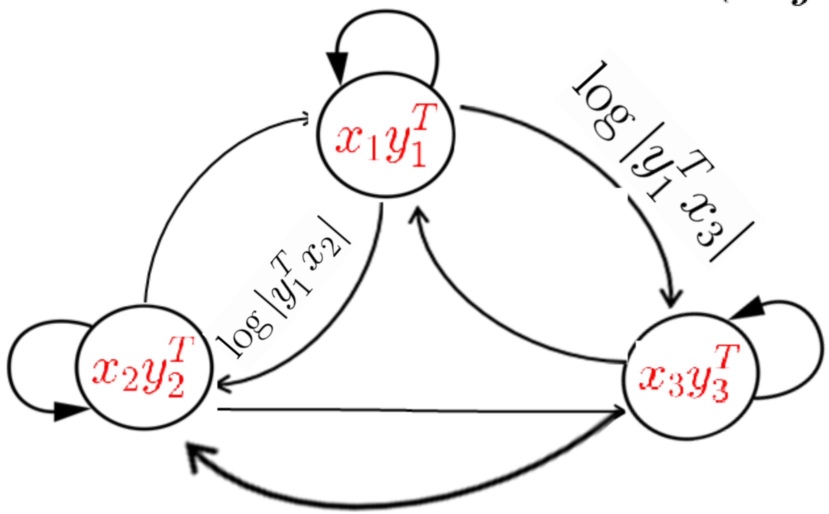
- Run time O(|N||E|)

Take logs and apply Karp

$$\mathcal{A} = \{A_1, \ldots, A_m\} \quad A_i = x_i y_i^T$$

$\tilde{\mathrm{G}}_{\mathcal{A}}$ Complete directed graph on *m* nodes:

Nodes: matrices A_i Edge weights: $w(e_{ij}) = \log |y_i^T x_j|$



$$\rho(\mathcal{A}) = e^{\lambda^* / k^*}$$

Run time: $O(m^3+m^2n)$

Common quadratic Lyapunov function can fail

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

- $\rho(\mathcal{A}) = 1$ (can be proven e.g. using our algorithm)
- An LMI searching for a common quadratic Lyapunov function can only prove

$$\rho(\mathcal{A}) \leq \sqrt{2}$$

Nonlinear Switched Systems
&
SOS-Convex Lyapunov Functions

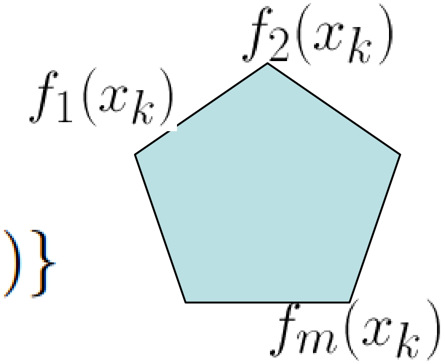
[AAA, Jungers, *IEEE Conf. on Decision and Control,*'13]

Nonlinear switched systems

$$x_{k+1} = \tilde{f}(x_k)$$

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \ldots, f_m(x_k)\}$$

$$f_1, \ldots, f_m : \mathbb{R}^n \rightarrow \mathbb{R}^n$$



Lemma:
Unlike the linear case, a common Lyapunov function for the corners **does not imply** stability of the convex hull.

Ex.

$$\begin{aligned} f_1(x) &= (x_1 x_2, 0)^T \\ f_2(x) &= (0, x_1 x_2)^T \end{aligned}$$

Common Lyapunov function:

$$V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2)$$

$$V(f_i(x)) = x_1^2 x_2^2 < V(x) = x_1^2 x_2^2 + (x_1^2 + x_2^2)$$

But unstable:

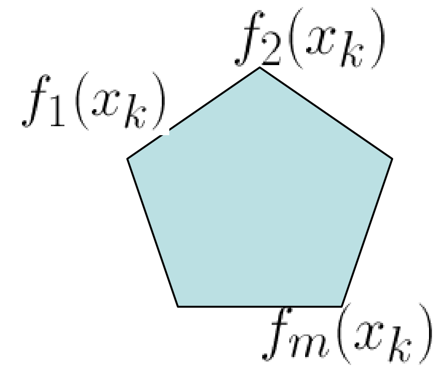
$$f(x) = \left(\frac{x_1 x_2}{2}, \frac{x_1 x_2}{2}\right) \in conv\{f_1(x_k), f_2(x_k)\}$$

But a *convex* Lyapunov function implies stability

$$x_{k+1} = \tilde{f}(x_k)$$

$$\tilde{f}(x_k) \in conv\{f_1(x_k), \ldots, f_m(x_k)\}$$

$$f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}^n$$



Suppose we can find a **convex** common Lyapunov function:

$$V(x) > 0, \quad V(f_i(x)) < V(x) \text{ for } i = 1, \ldots, m$$

Then, then we have stability of the convex hull.

Proof: $V(\tilde{f}(x)) = V(\sum_{i=1}^{m} \alpha_i f_i(x)) \leq \sum_{i=1}^{m} \alpha_i V(f_i(x)) < V(x)$

SOS-Convexity

sos-convex polynomial: $y^T H(x) y$ sos (Helton &Nie)

sos-convex Lyapunov function:

$$V(x) \quad \text{sos-covex}$$

$$V(x) - V(f_i(x)) \quad \text{sos}$$

- Search for an sos-convex Lyapunov function is an SDP!
- But except for some specific degrees and dimensions, there are convex polynomials that are not sos-convex:

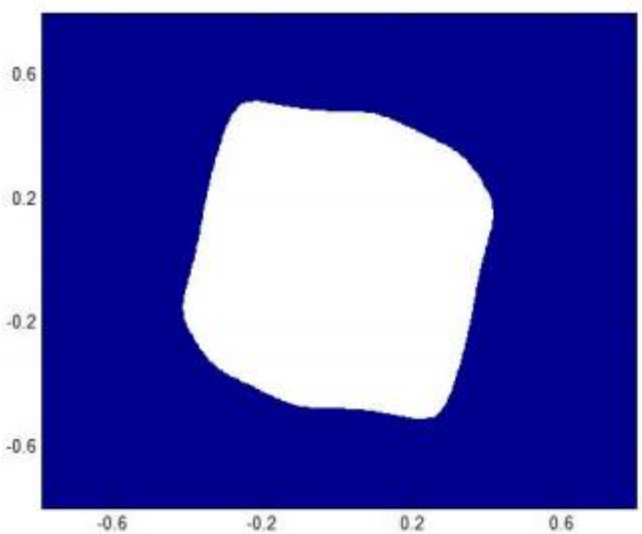
$$\begin{aligned} p(x) = {} & 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^4 - 43x_1^4x_2^2x_3^2 \\ & - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6 + 16x_2^8 \\ & + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8 \end{aligned}$$

[AAA, Parrilo – *Math Prog.*, ’11]

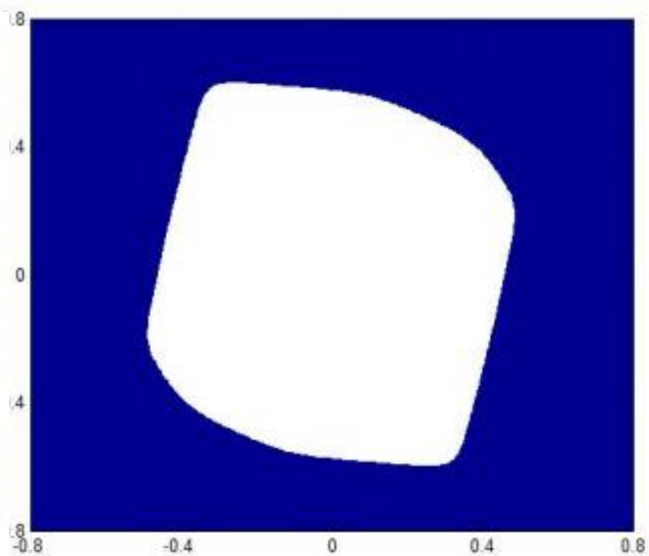
ROA Computation via SDP

$$f_1(x) = \begin{pmatrix} 0.687x_1 + 0.558x_2 - .0001x_1x_2 \\ -0.292x_1 + 0.773x_2 \end{pmatrix}$$

$$f_2(x) = \begin{pmatrix} 0.369x_1 + 0.532x_2 - .0001x_1^2 \\ -1.27x_1 + 0.12x_2 - .0001x_1x_2 \end{pmatrix}$$



non-convex, deg=12



sos-convex, deg=14

- **Left:**
  Cannot make any statements about ROA
- **Right:**
  Level set is part of ROA under arbitrary switching

A converse Lyapunov theorem

$$x_{k+1} \in conv\{A_i x_k\}, \quad i = 1, \ldots, m$$

Thm: SOS-convex Lyapunov functions are *universal* (i.e., *necessary and sufficient*) for stability.

Proof idea:

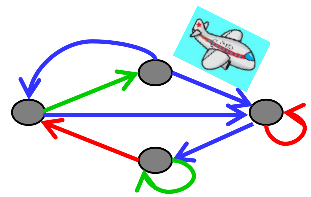
- Approximate original Lyapunov function with convex polynomials
- In a second step, go from convex to sos-convex
  - Uses a Positivstellensatz result of Claus Scheiderer:

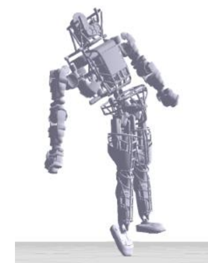
Given any two positive definite forms g and h, there exists an integer k such that $g.h^k$ is sos.

Joint Spectral Radius

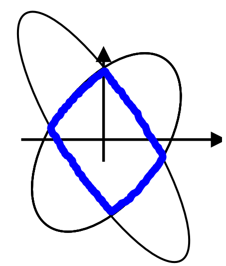
$$\rho(\mathcal{A}) = \lim_{k\to\infty} \max_{\sigma\in\{1,\dots,m\}^k} \|A_{\sigma_k} \dots A_{\sigma_2} A_{\sigma_1}\|^{1/k}$$

Has lots of applications…





Powerful approximation algorithms based on Lyapunov theory + optimization…



$$\begin{aligned} A_1^T P_1 A_1 &\preceq P_1 \\ A_2^T P_1 A_2 &\preceq P_2 \\ A_1^T P_2 A_1 &\preceq P_1 \\ A_2^T P_2 A_2 &\preceq P_2 \end{aligned}$$

A lot less understood for nonlinear switched systems…

Want to know more? http://aaa.princeton.edu/