Exploiting Structure in SDPs with Chordal Sparsity

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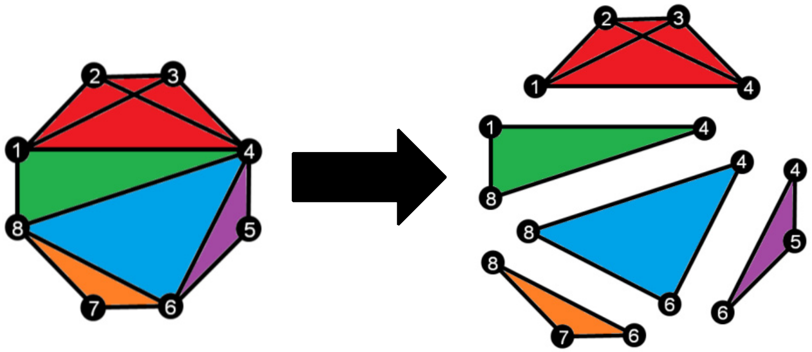
Joint work with Yang Zheng, Giovanni Fantuzzi, Paul Goulart and Andrew Wynn



CDC 2016 Pre-conference Workshop

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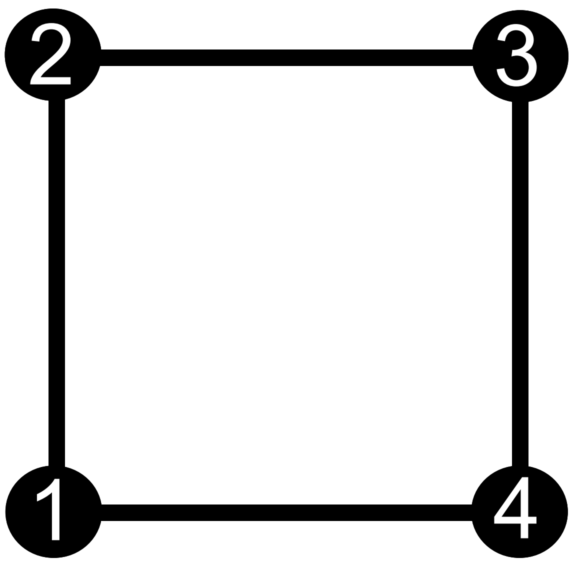
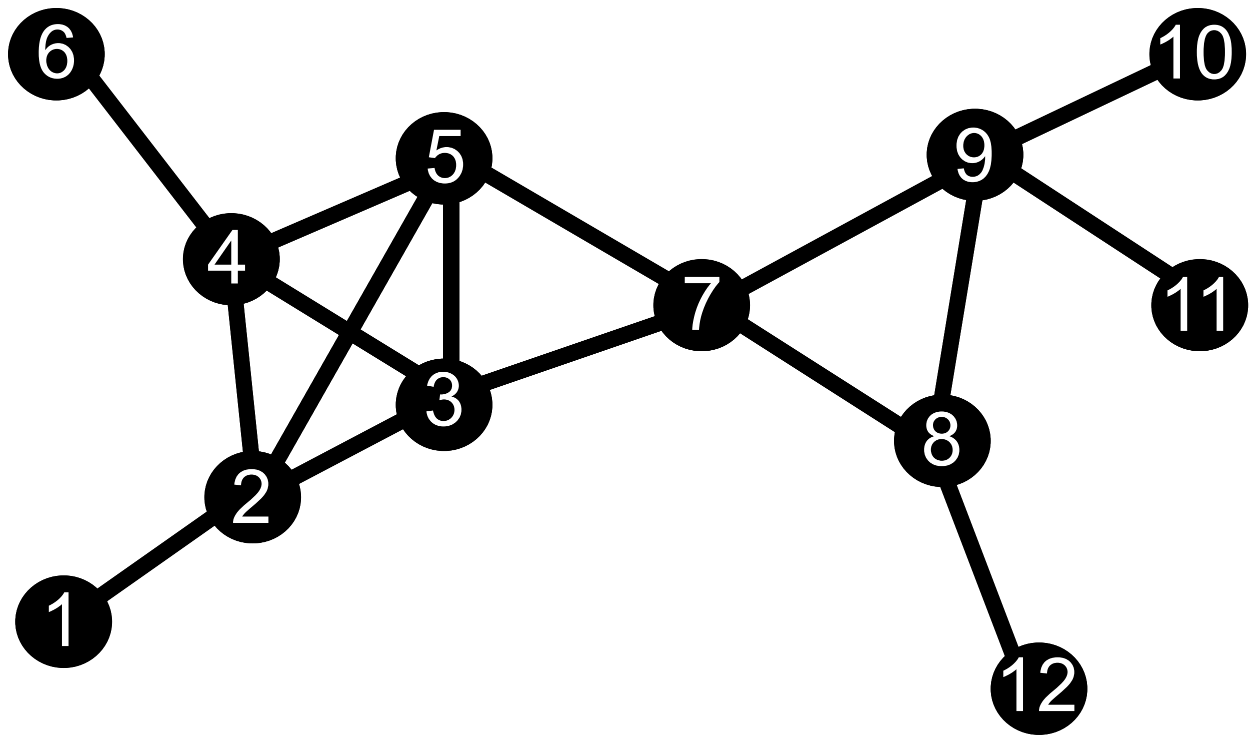
OUTLINE

Chordal Graphs

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is chordal if every cycle of length $\geq$ 4 has a chord.





Can recognise chordal graphs in $O(|\mathcal{V}| + |\mathcal{E}|)$ time

Chordal Graphs

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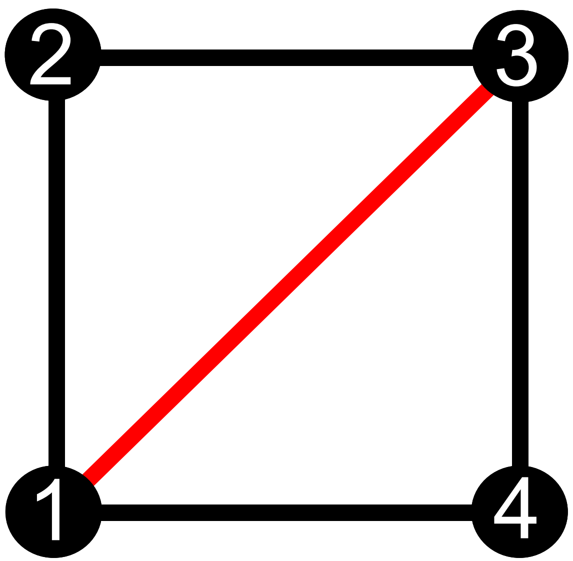
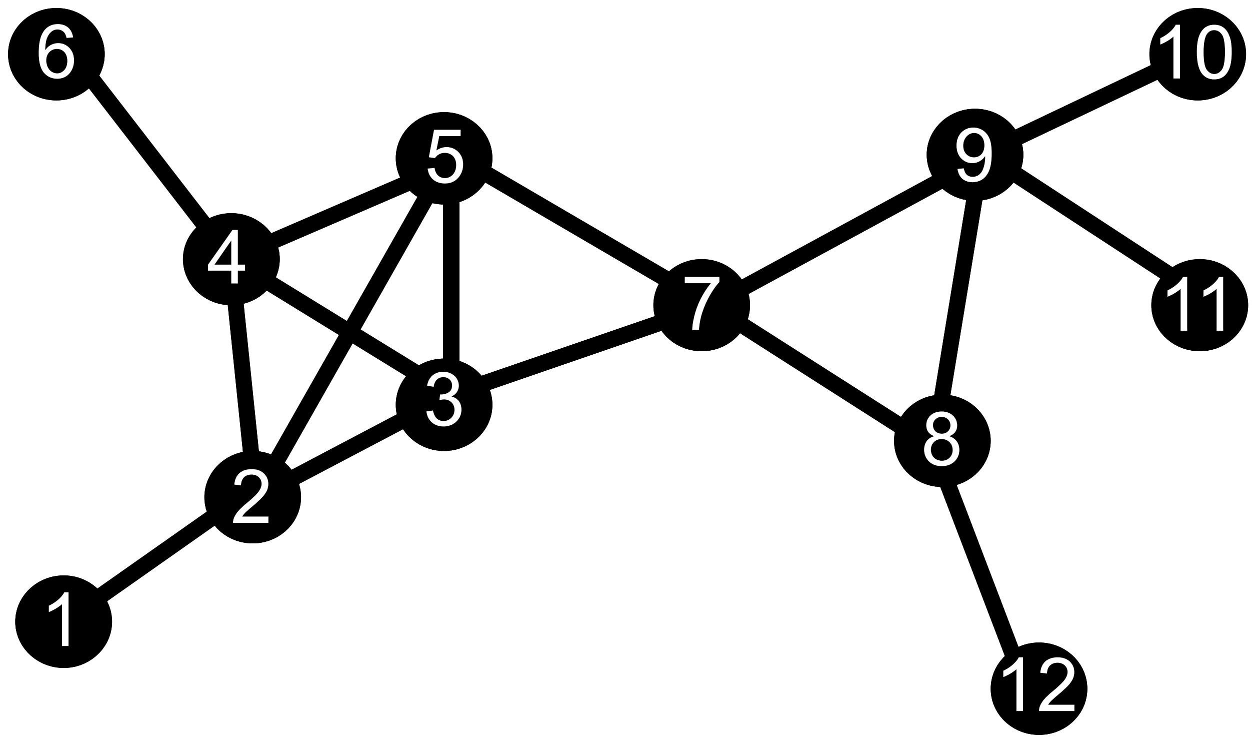




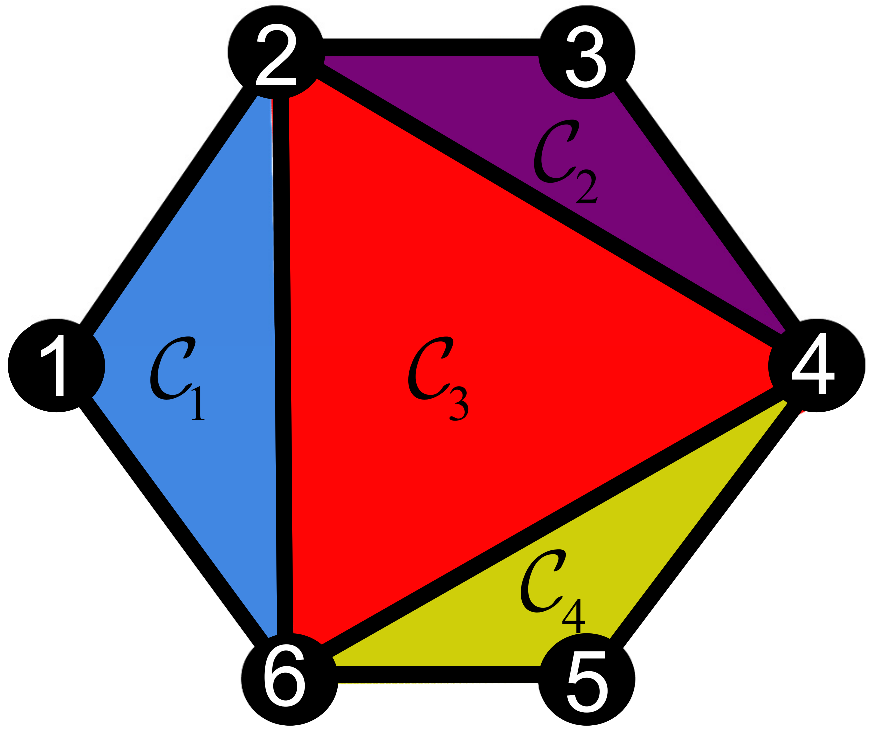
Can recognise chordal graphs in $O(|\mathcal{V}| + |\mathcal{E}|)$ time

Maximal Cliques

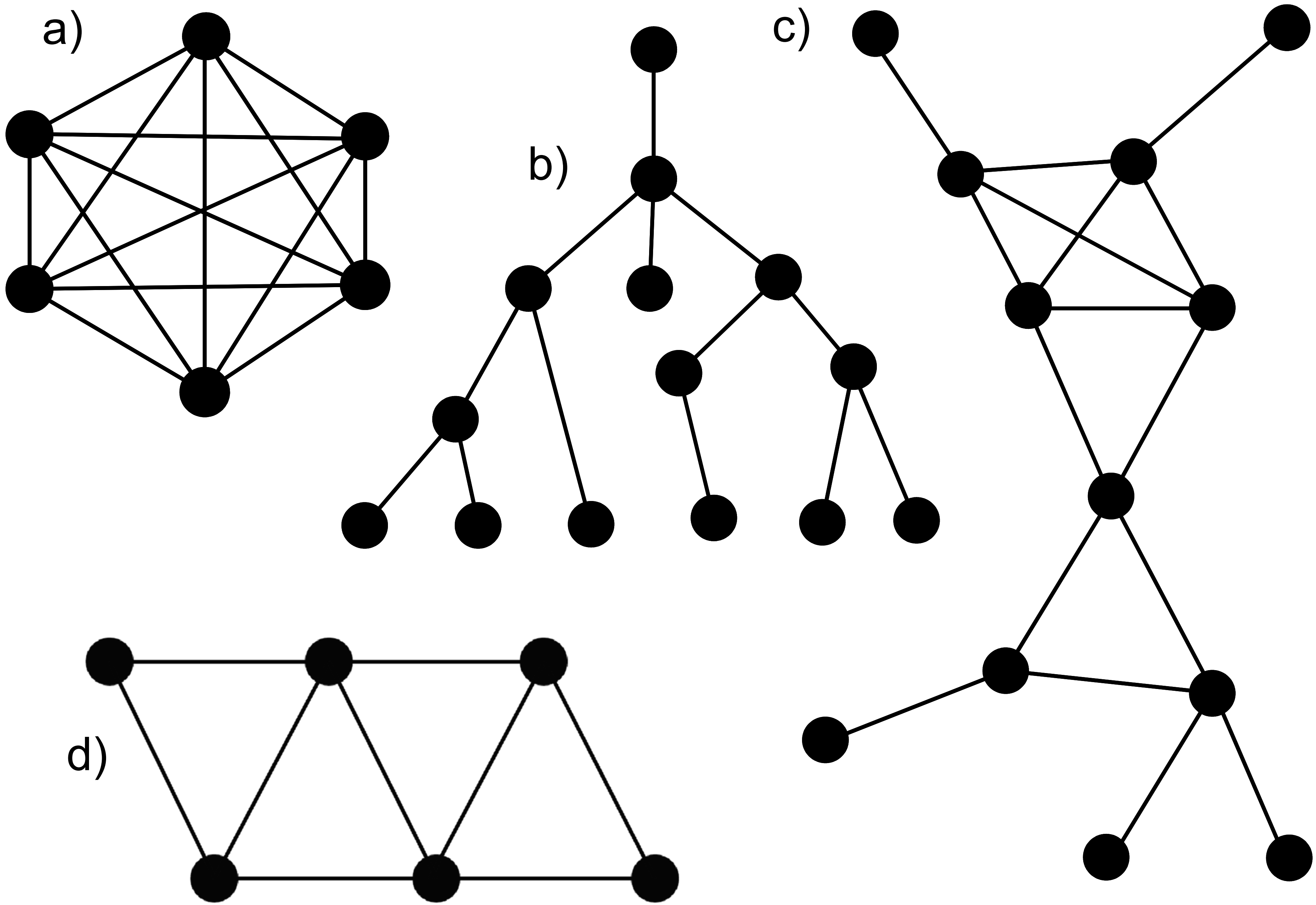
A maximal clique is a clique that is not a subset of another clique.

e.g. $\mathcal{C}_1 = \{1, 2, 6\}$.

Can find the maximal cliques of a chordal graph in $O(|\mathcal{V}| + |\mathcal{E}|)$ time.



Examples of Chordal Graphs

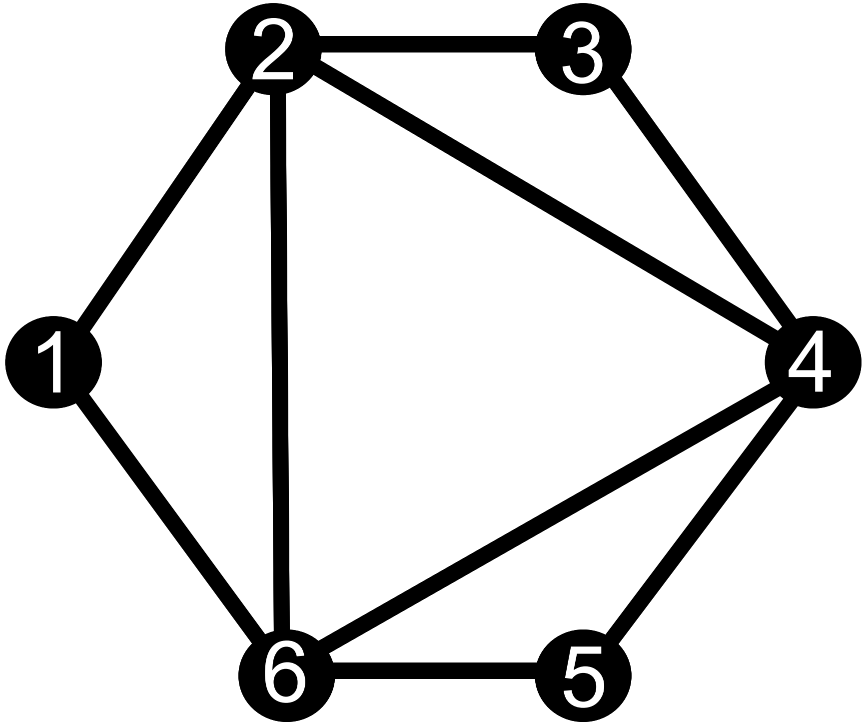


Sparse Matrices

$$Z$$

$$\begin{bmatrix} Z_{11} & Z_{12} & 0 & 0 & 0 & Z_{16} \\ Z_{12} & Z_{22} & Z_{23} & Z_{24} & 0 & Z_{26} \\ 0 & Z_{23} & Z_{33} & Z_{34} & 0 & 0 \\ 0 & Z_{24} & Z_{34} & Z_{44} & Z_{45} & Z_{46} \\ 0 & 0 & 0 & Z_{45} & Z_{55} & Z_{56} \\ Z_{16} & Z_{26} & 0 & Z_{46} & Z_{56} & Z_{66} \end{bmatrix}$$

$$\mathcal{G}(Z) = (\mathcal{V}, \mathcal{E})$$



$$Z \in \mathbb{S}^n(\mathcal{E},0) = \left\{ Z \in \mathbb{S}^n \mid Z_{ij} = 0, \forall (i,j) \notin \mathcal{E} \right\}$$

$$\mathbb{S}^n_+(\mathcal{E},0) = \{ Z \in \mathbb{S}^n(\mathcal{E},0) \mid Z \succeq 0 \}$$

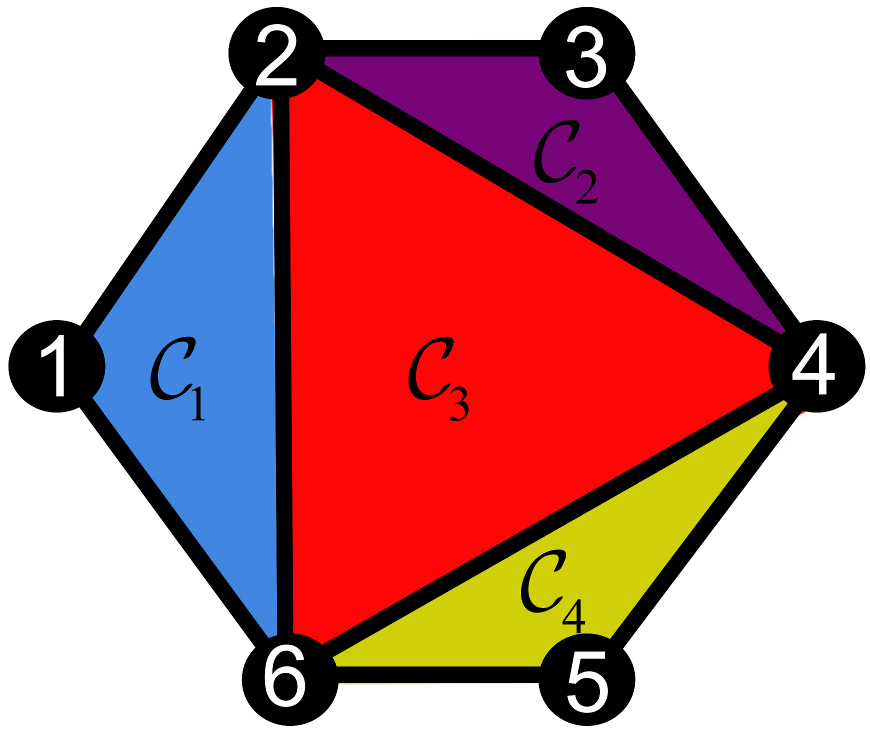
Agler's Theorem

Theorem: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a chordal graph with set of maximal cliques $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Suppose that $Z \in \mathbb{S}^n(\mathcal{E}, 0)$. Then $Z \in \mathbb{S}^n_+(\mathcal{E}, 0)$ if and only if there exists a set of matrices $\{Z^1, Z^2, \ldots, Z^p\}$ such that $Z = \sum_{k=1}^{p} Z^k$, $Z^k \succeq 0$, $Z^k_{ij} = 0$ if $(i, j) \notin \mathcal{E}_k \times \mathcal{E}_k$.

Example: Applying Agler's Theorem

$$Z$$

$$\begin{bmatrix} z_{11} & z_{12} & 0 & 0 & 0 & z_{16} \\ z_{12} & z_{22} & z_{23} & z_{24} & 0 & z_{26} \\ 0 & z_{23} & z_{33} & z_{34} & 0 & 0 \\ 0 & z_{24} & z_{34} & z_{44} & z_{45} & z_{46} \\ 0 & 0 & 0 & z_{45} & z_{55} & z_{56} \\ z_{16} & z_{26} & 0 & z_{46} & z_{56} & z_{66} \end{bmatrix}$$



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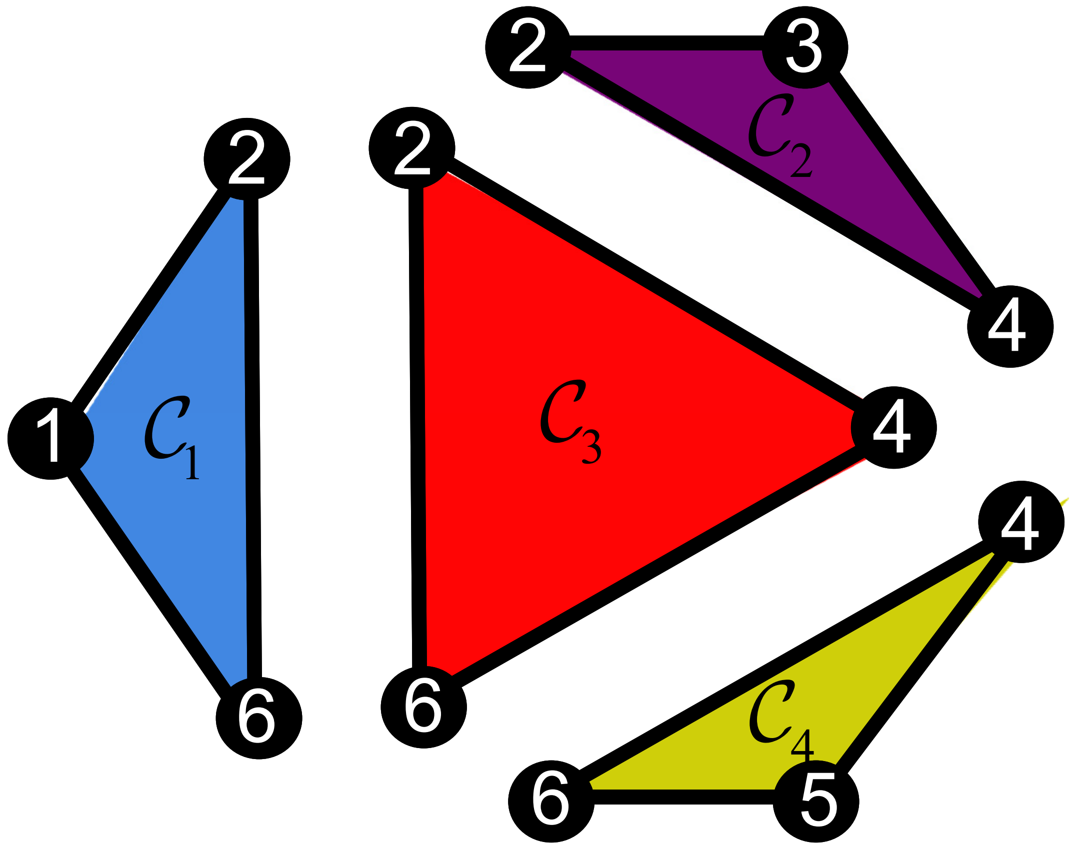
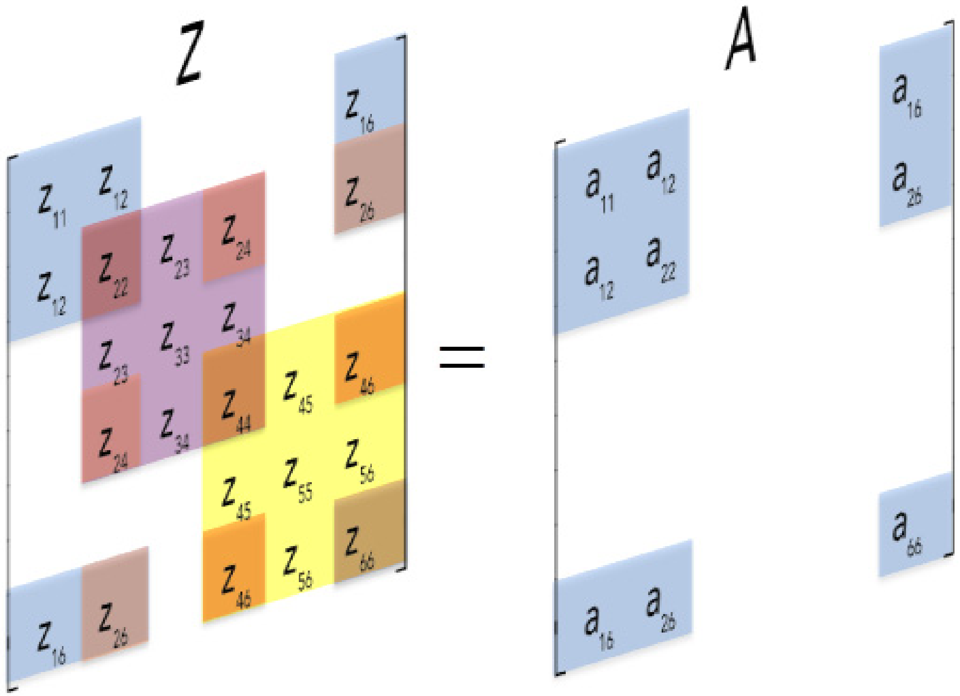


Example: Applying Agler's Theorem

$$Z \succeq 0 \iff \exists\, A,\ B,\ C,\ D \text{ such that}$$

$$A + B + C + D = Z, \quad A,\ B,\ C,\ D \succeq 0$$



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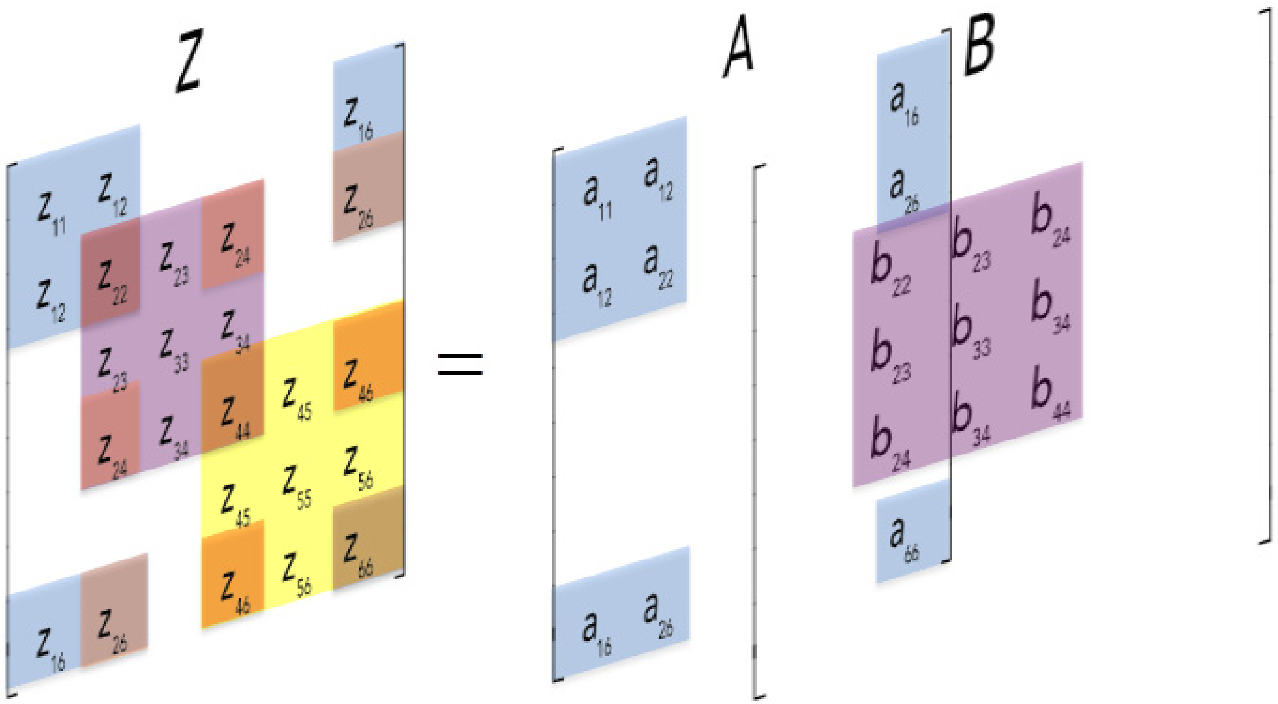

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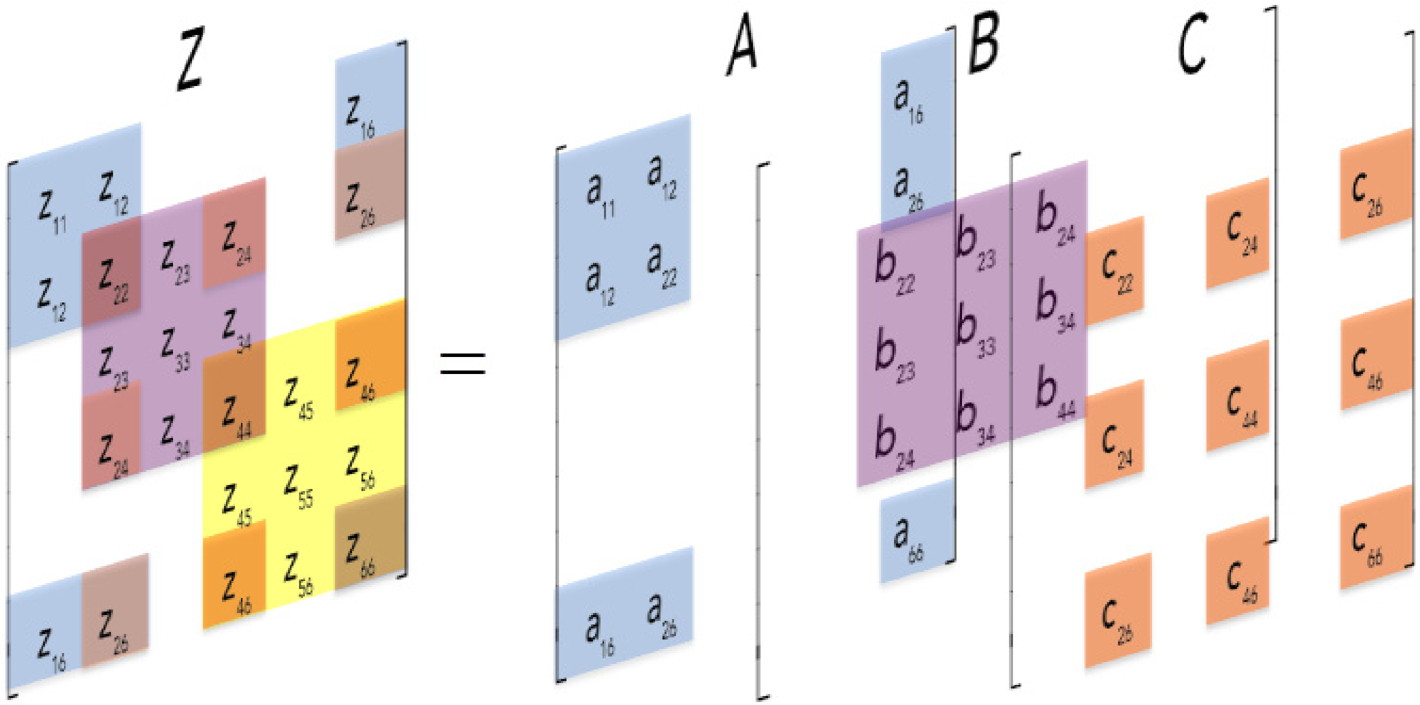

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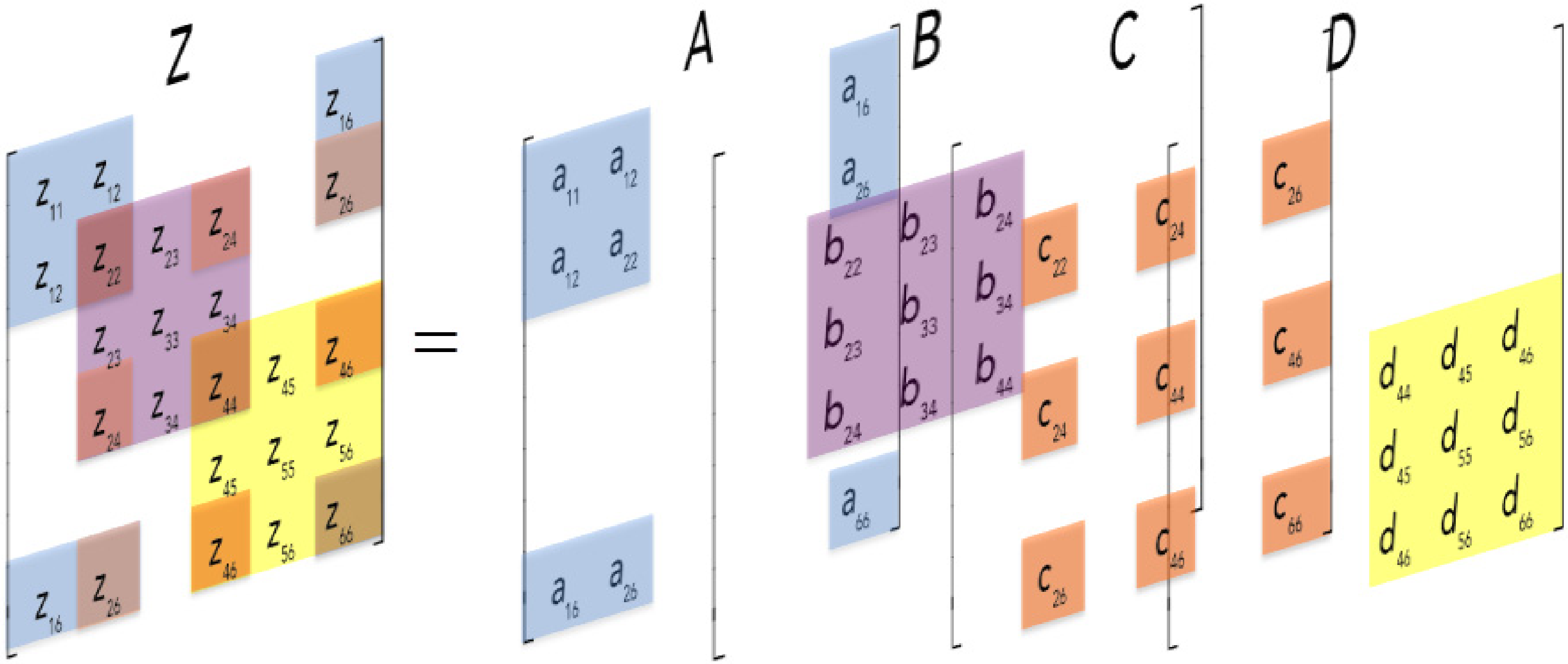
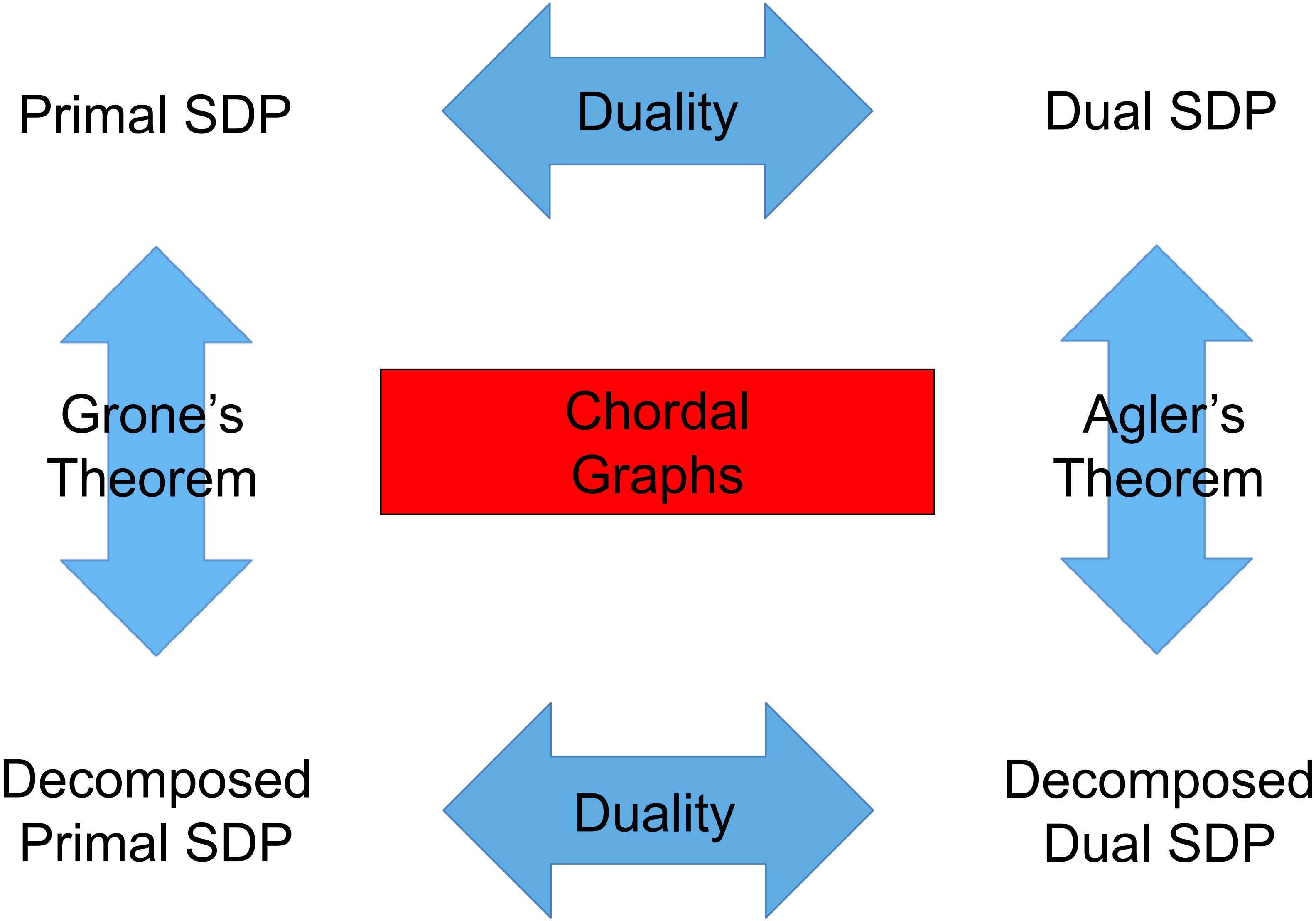


SDPs with Chordal Sparsity

$$\begin{aligned} \min_{X} \quad & \langle C, X \rangle \\ \text{subject to} \quad & \mathcal{A}(X) = b \\ & X \in \mathbb{S}_{+}^{n} \end{aligned} \quad \xleftrightarrow{\text{Dual}} \quad \begin{aligned} \max_{y,Z} \quad & \langle b, y \rangle \\ \text{subject to} \quad & \mathcal{A}^{*}(y) + Z = C \\ & Z \in \mathbb{S}_{+}^{n} \end{aligned}$$

- Applications: control theory, fluid mechanics, machine learning, polynomial optimization, combinatorics, operations research, finance, etc.
- Second-order solvers : SeDuMi, SDPA, SDPT3
- Large-scale cases: exploit the inherent structure of the instances (De Klerk, 2010):
  - ➢ Low Rank and Algebraic Symmetry.
  - ➢ **Chordal Sparsity:**
    - ✓ Second-order methods: Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Andersen *et al.*, 2010;
    - ✓ **First-order methods:** Madani *et al.,* 2015; Sun *et al.,* 2014.

Decomposing Sparse SDPs



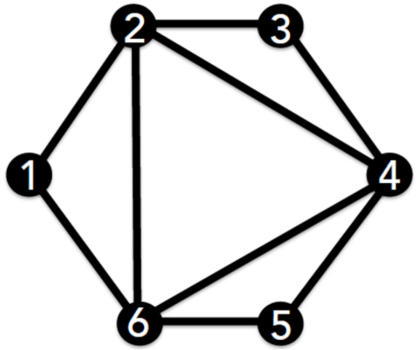
SDPs with Chordal Sparsity

$$\mathbb{S}^n(\mathcal{E},0) = \left\{ X \in \mathbb{S}^n \mid X_{ij} = 0, \forall (i,j) \notin \mathcal{E} \right\}$$

$$\mathbb{S}^n_+(\mathcal{E},0) = \{ X \in \mathbb{S}^n(\mathcal{E},0) \mid X \succeq 0 \}$$

$$\begin{array}{c|c} & \begin{matrix} 1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 & \;\;6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 & x_{16} \\ x_{12} & x_{22} & x_{23} & x_{24} & 0 & x_{26} \\ 0 & x_{23} & x_{33} & x_{34} & 0 & 0 \\ 0 & x_{24} & x_{34} & x_{44} & x_{45} & x_{46} \\ 0 & 0 & 0 & x_{45} & x_{55} & x_{56} \\ x_{16} & x_{26} & 0 & x_{46} & x_{56} & x_{66} \end{bmatrix} \end{array}$$

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



$$\begin{array}{c|c} & \begin{matrix} 1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 & \;\;6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} x_{11} & x_{12} & ? & ? & ? & x_{16} \\ x_{12} & x_{22} & x_{23} & x_{24} & ? & x_{26} \\ ? & x_{23} & x_{33} & x_{34} & ? & ? \\ ? & x_{24} & x_{34} & x_{44} & x_{45} & x_{46} \\ ? & ? & ? & x_{45} & x_{55} & x_{56} \\ x_{16} & x_{26} & ? & x_{46} & x_{56} & x_{66} \end{bmatrix} \end{array}$$

$\mathbb{S}^n(\mathcal{E},?) = n \times n$ partial symmetric matrices with entries defined on $\mathcal{E}$

$$\mathbb{S}^n_+(\mathcal{E},?) = \{ X \in \mathbb{S}^n(\mathcal{E},?) \mid \exists M \geq 0, M_{ij} = X_{ij}, \forall (i,j) \in \mathcal{E} \}$$

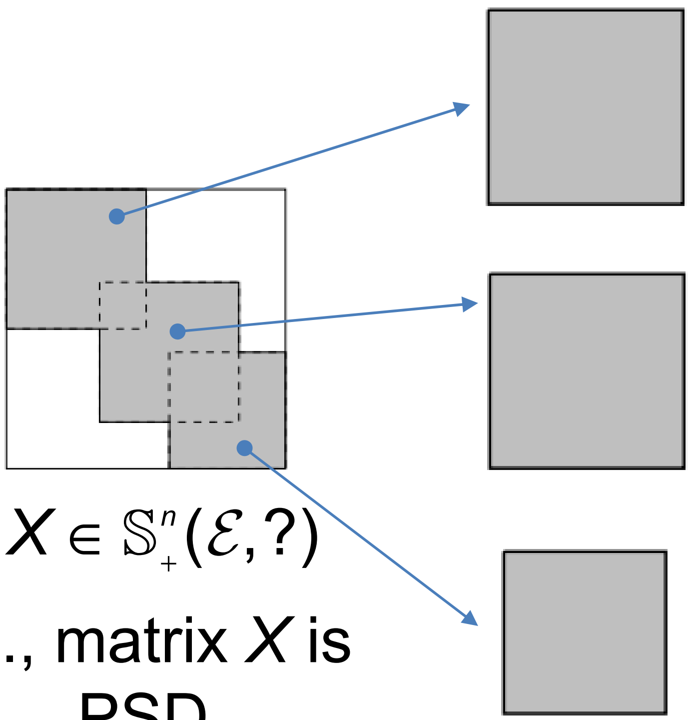
$\boxed{\mathbb{S}^n_+(\mathcal{E},?) \text{ and } \mathbb{S}^n_+(\mathcal{E},0) \text{ are dual cones of each other}}$

Grone's theorem

Consider a choral graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a set of maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_p$.

Grone's Theorem:

$$X \in \mathbb{S}^n_+(\mathcal{E}, ?) \text{ if and only if } X(\mathcal{C}_k) \succeq 0, \; k = 1, \ldots, p.$$



$X(\mathcal{C}_1) \succeq 0$

$X(\mathcal{C}_2) \succeq 0$

$X(\mathcal{C}_3) \succeq 0$

$X \in \mathbb{S}^n_+(\mathcal{E}, ?)$

i.e., matrix X is PSD completable

Cone Decomposition of Primal and Dual SDPs

Primal

$$\min_X \quad \langle C, X\rangle$$
$$\text{subject to} \quad \mathcal{A}(X) = b$$
$$\boxed{X \in \mathbb{S}_+^n}$$

$$\Downarrow$$

$$X \in \mathbb{S}_+^n(\mathcal{E}, ?)$$

$$\Downarrow$$

$$\min_X \quad \langle C, X\rangle$$
$$\text{s.t.} \quad \mathcal{A}(X) = b$$
$$\boxed{E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^T \in \mathbb{S}_+^{|\mathcal{C}_k|}, \ k = 1, \dots, p}$$

Dual

$$\max_{y,Z} \quad \langle b, y\rangle$$
$$\text{subject to} \quad \mathcal{A}^*(y) + Z = C$$
$$\boxed{Z \in \mathbb{S}_+^n}$$

$$\Downarrow$$

$$Z \in \mathbb{S}_+^n(\mathcal{E}, 0)$$

$$\Downarrow$$

$$\max_{y,Z} \quad \langle b, y\rangle$$
$$\text{s.t} \quad \mathcal{A}^*(y) + \sum_{k=1}^{p} E_{\mathcal{C}_k}^T Z_k E_{\mathcal{C}_k} = C$$
$$\boxed{Z \in \mathbb{S}_+^{|\mathcal{C}_k|}, \quad k = 1, \dots, p}$$

OUTLINE

1. Chordal Graphs and Positive Semidefinite Matrices
2. ADMM for Primal and Dual Sparse SDPs
3. CDCS: Cone Decomposition Conic Solver
4. Conclusion

Alternating Direction Method of Multipliers

An operator splitting method for a problem of the form

$$\begin{aligned} \min \quad & f(x)+g(z) \\ \text{s.t.} \quad & x=z \end{aligned}$$

f, g may be nonsmooth.

Lagrangian:

$$\mathcal{L}=f(x)+g(z)+\frac{\rho}{2}\left\| x-z+\frac{1}{\rho}\lambda \right\|_2^2$$

ADMM:

$$x^{n+1}=\operatorname*{argmin}_x \left(f(x)+\frac{\rho}{2}\left\| x-z^n+\frac{1}{\rho}\lambda^n \right\|_2^2 \right)$$

$$z^{n+1}=\operatorname*{argmin}_z \left(g(z)+\frac{\rho}{2}\left\| x^{n+1}-z+\frac{1}{\rho}\lambda^n \right\|_2^2 \right)$$

$$\lambda^{n+1}=\lambda^n+\rho(x^{n+1}-z^{n+1})$$

Reformulation and decomposition of the PSD constraint

$$\begin{aligned}\min_{X}\quad & \langle C, X\rangle \\ \text{s.t.}\quad & \mathcal{A}(X) = b \\ & X_k = E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^T,\ k = 1,\dots,p \\ & X_k \in \mathbb{S}_+^{|\mathcal{C}_k|},\ k = 1,\dots,p\end{aligned} \quad \Rightarrow \quad \begin{aligned}\min_{x,x_1,\dots,x_p}\quad & c^T x \\ \text{s.t.}\quad & Ax = b \\ & x_k = H_k x,\ k = 1,\dots,p \\ & x_k \in \mathcal{S}_k,\ k = 1,\dots,p\end{aligned}$$

- Using indicator functions

$$\begin{aligned}\min_{x,x_1,\dots x_p}\quad & c^T x + \delta_0(Ax - b) + \sum_{k=1}^{p} \delta_{\mathcal{S}_k}(x_k) \\ \text{s.t.}\quad & x_k = H_k x,\ k = 1,\dots,p\end{aligned}$$

- Augmented Lagrangian

$$\mathcal{L} = c^T x + \delta_0(Ax - b) + \sum_{k=1}^{p} \left[\delta_{\mathcal{S}_k}(x_k) + \frac{\rho}{2} \left\| x_k - H_k x + \frac{1}{\rho}\lambda_k \right\|^2 \right]$$

- Regroup the variables

$$\mathcal{X} \triangleq \{x\};\ \ \mathcal{Y} \triangleq \{x_1,\dots x_p\};\ \ \mathcal{Z} \triangleq \{\lambda_1,\dots \lambda_p\}$$

ADMM for Primal SDPs

$$\mathcal{L} = c^T x + \delta_0(Ax - b) + \sum_{k=1}^{p}\left[\delta_{\mathcal{S}_k}(x_k) + \frac{\rho}{2}\left\| x_k - H_k x + \frac{1}{\rho}\lambda_k \right\|^2\right]$$

$$\mathcal{X} \triangleq \{x\};\quad \mathcal{Y} \triangleq \{x_1, \ldots x_p\};\quad \mathcal{Z} \triangleq \{\lambda_1, \ldots \lambda_p\}$$

1) Minimization over $\mathcal{X}$

$$\min_x \quad c^T x + \frac{\rho}{2}\sum_{k=1}^{p}\left\| x_k^{(n)} - H_k x + \frac{1}{\rho}\lambda_k^{(n)} \right\|^2$$

$$\text{s.t.} \quad Ax = b$$

QP with linear constraint

✓ the KKT system matrix only depends on the problem data.

2) Minimization over $\mathcal{Y}$

$$\min_{x_k} \quad \left\| x_k - H_k x^{(n+1)} + \frac{1}{\rho}\lambda_k^{(n)} \right\|^2$$

$$\text{s.t.} \quad x_k \in \mathcal{S}_k$$

Projections in parallel

3) Update multipliers

$$\lambda_k^{(n+1)} = \lambda_k^{(n)} + \rho\left(x_k^{(n+1)} - H_k x^{(n+1)}\right)$$

ADMM for Primal and Dual SDPs

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{Primal SDP} \arrow[r, "\text{Duality}"] \arrow[d, "\text{Grone's Theorem}"'] & \text{Dual SDP} \arrow[d, "\text{Agler's Theorem}"] \\
\text{Decomposed Primal SDP} \arrow[r, "\text{Duality}"] \arrow[d, "\text{ADMM}"'] & \text{Decomposed Dual SDP} \arrow[d, "\text{ADMM}"] \\
\text{Algorithm 1} \arrow[r, "\text{Scaling}"] & \text{Algorithm 2}
\end{tikzcd}$$

The duality between the primal and dual SDP is inherited by the decomposed problems by virtue of the duality between Grone's and Agler's theorems.

ADMM for the Homogeneous self-dual embedding

$$\begin{aligned} \min_{x, x_k} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x_k = H_k x \\ & x_k \in \mathcal{S}_k, \ k = 1, \dots, p \end{aligned}$$

$$\begin{aligned} \min_{y, z_k} \quad & -b^T y \\ \text{s.t.} \quad & A^T y + \sum_{k=1}^{p} H_k^T v_k = c \\ & z_k - v_k = 0, \ \ k = 1, \dots, p \\ & z_k \in \mathcal{S}_k, \ \ k = 1, \dots, p \end{aligned}$$

- Notational simplicity

$$s \triangleq \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \ z \triangleq \begin{bmatrix} z_1 \\ \vdots \\ z_p \end{bmatrix}, \ v \triangleq \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}, \ H \triangleq \begin{bmatrix} H_1 \\ \vdots \\ H_p \end{bmatrix}$$

- KKT conditions
  - Primal feasible $\quad Ax^* - r^* = b, \ \ r^* = 0,$

    $s^* + w^* = Hx^*, \ w^* = 0, \ s^* \in \mathcal{S}$
  - Dual feasible $\quad A^T y^* + H^T v^* + h^* = c, \ \ h^* = 0,$

    $z^* - v^* = 0, \ z^* \in \mathcal{S}$
  - Zero-duality gap $\quad c^T x^* - b^T y^* = 0$

ADMM for the Homogeneous self-dual embedding

$$\begin{bmatrix} h \\ z \\ r \\ w \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & 0 & -A^T & -H^T & c \\ 0 & 0 & 0 & I & 0 \\ A & 0 & 0 & 0 & -b \\ H & -I & 0 & 0 & 0 \\ -c^T & 0 & b^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ s \\ y \\ v \\ \tau \end{bmatrix}$$

τ, κ are two non-negative and complementary variables

- Notational simplicity

$$v \triangleq \begin{bmatrix} h \\ z \\ r \\ w \\ \kappa \end{bmatrix}, \ u \triangleq \begin{bmatrix} x \\ s \\ y \\ v \\ \tau \end{bmatrix}, \ Q \triangleq \begin{bmatrix} 0 & 0 & -A^T & -H^T & c \\ 0 & 0 & 0 & I & 0 \\ A & 0 & 0 & 0 & -b \\ H & -I & 0 & 0 & 0 \\ -c^T & 0 & b^T & 0 & 0 \end{bmatrix}, \ \mathcal{K} = \mathbb{R}^{n^2} \times \mathcal{S} \times \mathbb{R}^m \times \mathbb{R}^{n_d} \times \mathbb{R}_+$$

- Feasibility problem

$$\begin{aligned} \text{find} \quad & (u, v) \\ \text{s.t.} \quad & v = Qu \\ & (u, v) \in \mathcal{K} \times \mathcal{K}^* \end{aligned}$$

ADMM for the Homogeneous self-dual embedding

$$\begin{aligned} &\text{find} \quad (u,v) \\ &\text{s.t.} \quad v = Qu, \quad (u,v) \in \mathcal{K} \times \mathcal{K}^* \end{aligned}$$

- ADMM steps (similar to solver SCS [1])

$$\hat{u}^{k+1} = (I+Q)^{-1}(u^k + v^k) \longrightarrow \text{Projection to a subspace}$$

$$u^{k+1} = \mathrm{P}_{\mathcal{K}}(\hat{u}^{k+1} - v^k) \longrightarrow \text{Projection to cones}$$

$$v^{k+1} = v^k - \hat{u}^{k+1} + u^{k+1}$$

$\boxed{Q \text{ is highly structured and sparse}}$

$$Q \triangleq \begin{bmatrix} 0 & 0 & -A^T & -H^T & c \\ 0 & 0 & 0 & I & 0 \\ A & 0 & 0 & 0 & -b \\ H & -I & 0 & 0 & 0 \\ -c^T & 0 & b^T & 0 & 0 \end{bmatrix}$$

✓ Block elimination can be applied here to speed up the projection;

✓ Then, the per-iteration cost is the same as applying a splitting method to the primal or dual alone.

[1] O'Donoghue, B., Chu, E., Parikh, N. and Boyd, S. (2016). Conic optimization via operator splitting and homogeneous self-dual embedding. Journal of Optimization Theory and Applications, 169(3), 1042– 1068

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CDCS: Cone Decomposition Conic Solver

- An open source MATLAB solver for partially decomposable conic programs;
- CDCS supports constraints on the following cones:
  - ✓ Free variables
  - ✓ non-negative orthant
  - ✓ second-order cone
  - ✓ the positive semidefinite cone.
- Input-output format is aligned with SeDuMi;
- Works with latest YALMIP release.

Syntax:

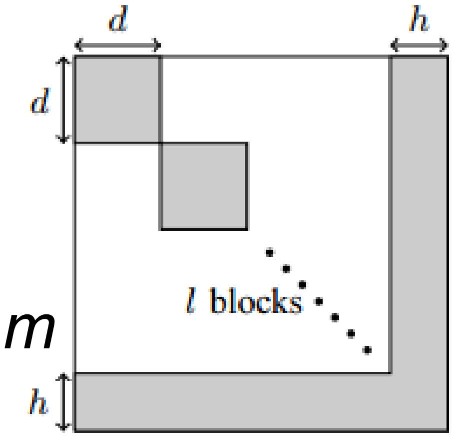
```
[x,y,z,info] = cdcs(At,b,c,K,opts);
```

Download from https://github.com/OxfordControl/CDCS

CDCS: Cone Decomposition Conic Solver

Random SDPs with block-arrow pattern

- Block size: *d*
- Number of Blocks: *l*
- Arrow head: *h*
- Number of constraints: *m*

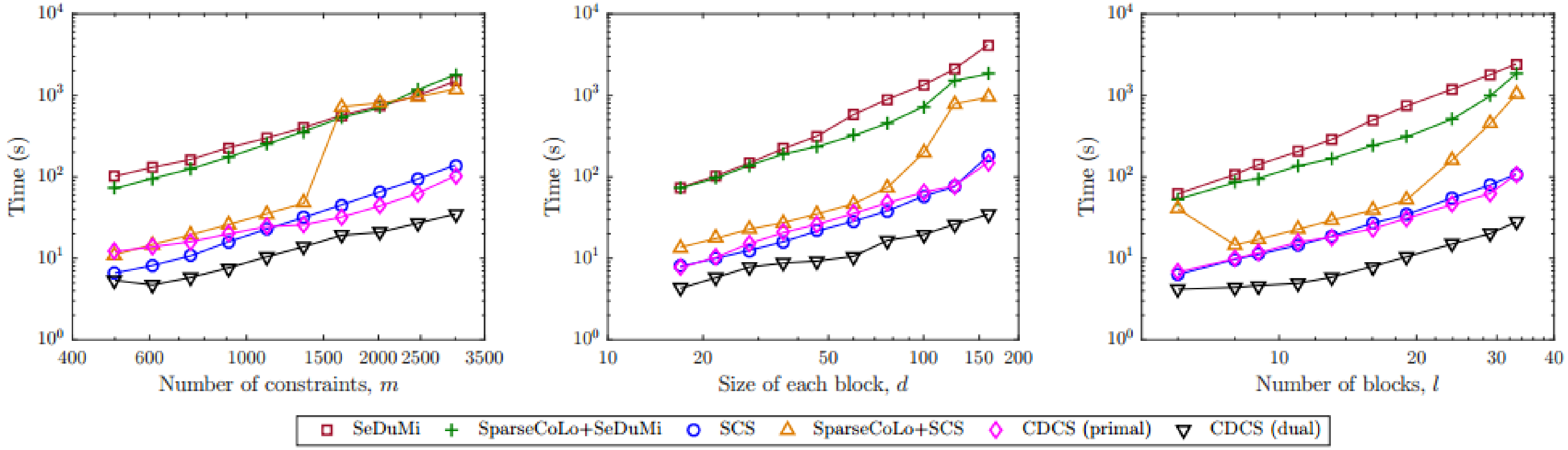


Numerical Comparison

- SeDuMi
- sparseCoLO+SeDuMi
- SCS
- sparseCoLO+SCS

CDCS and SCS $\epsilon_{\text{tol}} = 10^{-3}$

Numerical Result



CPU time for SDPs with block-arrow patterns. Left to right: varying number of constraints; varying number of blocks; varying block size.

Benchmark problems in SDPLIB

Three sets of benchmark problems in SDPLIB (Borchers, 1999):

1) Four small and medium-sized SDPs (theta1, theta2, qap5 and qap9);
2) Four large-scale sparse SDPs (maxG11, maxG32, qpG11 and qpG51);
3) Two infeasible SDPs (infp1 and infd1).

Table 1. Details of the SDPLIB problems considered in this work.

	Small and medium-size ($n \leq 100$)				Large-scale and sparse ($n \geq 800$)				Infeasible	
	theta1	theta2	qap5	qap9	maxG11	maxG32	qpG11	qpG51	infp1	infd1
Original cone size, n	50	100	26	82	800	2000	1600	2000	30	30
Affine constraints, m	104	498	136	748	800	2000	800	1000	10	10
Number of cliques, p	1	1	1	1	598	1499	1405	1675	1	1
Maximum clique size	50	100	26	82	24	60	24	304	30	30
Minimum clique size	50	100	26	82	5	5	1	1	30	30

CDCS: Cone Decomposition Conic Solver

Result: small and medium-sized instances

Table 2. Results for some small and medium-sized SDPs in SDPLIB.

		SeDuMi	SparseCoLO+ SeDuMi	SCS	CDCS (primal)	CDCS (dual)	Self-dual
theta1	Total time (s)	0.262	0.279	0.145	0.751	0.707	0.534
	Pre- time (s)	0	0.005	0.011	0.013	0.010	0.012
	Iterations	14	14	240	317	320	230
	Objective	2.300×10^1	2.300×10^1	2.300×10^1	2.299×10^1	2.299×10^1	2.303×10^1
theta2	Total time (s)	1.45	1.55	0.92	1.45	1.30	0.60
	Pre- time (s)	0	0.014	0.018	0.046	0.036	0.031
	Iterations	15	15	500	287	277	110
	Objective	3.288×10^1	3.288×10^1	3.288×10^1	3.288×10^1	3.288×10^1	3.287×10^1
qap5	Total time (s)	0.365	0.386	0.412	0.879	0.748	1.465
	Pre- time (s)	0	0.006	0.026	0.011	0.009	0.009
	Iterations	12	12	320	334	332	783
	Objective	-4.360×10^2	-4.360×10^2	-4.359×10^2	-4.360×10^2	-4.364×10^2	-4.362×10^2
qap9	Total time (s)	6.291	6.751	3.261	7.520	7.397	1.173
	Pre- time (s)	0	0.012	0.010	0.064	0.036	0.032
	Iterations	25	25	2000	2000	2000	261
	Objective	-1.410×10^3	-1.410×10^3	-1.409×10^3	-1.407×10^3	-1.409×10^3	-1.410×10^3

CDCS: Cone Decomposition Conic Solver

Result: large-sparse instances

Table 3. Results for some large-scale sparse SDPs in SDPLIB.

		SeDuMi	SparseCoLO+ SeDuMi	SCS	CDCS (primal)	CDCS (dual)	Self-dual
maxG11	Total time (s)	92.0	9.83	160.5	126.6	114.1	23.9
	Pre- time (s)	0	2.39	0.07	3.33	4.28	2.45
	Iterations	13	15	1860	1317	1306	279
	Objective	6.292×10^2	6.292×10^2	6.292×10^2	6.292×10^2	6.292×10^2	6.295×10^2
maxG32	Total time (s)	1.385×10^3	577.4	2.487×10^3	520.0	273.8	87.4
	Pre- time (s)	0	7.63	0.589	53.9	55.6	30.5
	Iterations	14	15	2000	1796	943	272
	Objective	1.568×10^3	1.568×10^3	1.568×10^3	1.568×10^3	1.568×10^3	1.568×10^3
qpG11	Total time (s)	675.3	27.3	1.115×10^3	273.6	92.5	32.1
	Pre- time (s)	0	11.2	0.57	6.26	6.26	3.85
	Iterations	14	15	2000	1355	656	304
	Objective	2.449×10^3	2.449×10^3	2.449×10^3	2.449×10^3	2.449×10^3	2.450×10^3
qpG51	Total time (s)	1.984×10^3	–	2.290×10^3	1.627×10^3	1.635×10^3	538.1
	Pre- time (s)	0	–	0.90	10.82	12.77	7.89
	Iterations	22	–	2000	2000	2000	716
	Objective	1.182×10^3	–	1.288×10^3	1.183×10^3	1.186×10^3	1.181×10^3

Result: Infeasible instances

Table 4. Results for two infeasible SDPs in SDPLIB.

		SeDuMi	SparseCoLO+ SeDuMi	SCS	CDCS (primal)	CDCS (dual)	Self-dual
infp1	Total time (s)	0.063	0.083	0.062	*	*	0.18
	Pre- time (s)	0	0.010	0.016	*	*	0.010
	Iterations	2	2	20	*	*	104
	Status	Infeasible	Infeasible	Infeasible	*	*	Infeasible
infd1	Total time (s)	0.125	0.140	0.050	*	*	0.144
	Pre- time (s)	0	0.009	0.013	*	*	0.009
	Iterations	4	4	40	*	*	90
	Status	Infeasible	Infeasible	Infeasible	*	*	Infeasible

CDCS: Cone Decomposition Conic Solver

- **Result: CPU time per iteration**

Table 5. CPU time per iteration (s) for some SDPs in SDPLIB

	SCS	CDCS (primal)	CDCS (dual)	Self-dual
theta1	6×10^{-4}	2.3×10^{-3}	2.2×10^{-3}	2.3×10^{-3}
theta2	1.8×10^{-3}	5.1×10^{-3}	4.7×10^{-3}	5.5×10^{-3}
qap5	1.2×10^{-3}	2.6×10^{-3}	2.2×10^{-3}	1.9×10^{-3}
qap9	1.5×10^{-3}	3.6×10^{-3}	3.7×10^{-3}	4.2×10^{-3}
maxG11	0.086	0.094	0.084	0.077
maxG32	1.243	0.260	0.231	0.209
qpG11	0.557	0.198	0.132	0.093
qpG51	1.144	0.808	0.811	0.741

✓ Our codes are currently written in MATLAB

✓ SCS is implemented in C.

Conclusion

$$\begin{tikzcd}
\text{Primal SDP} \arrow[r, "\text{Duality}"] \arrow[d, "\text{Grone's Theorem}"'] & \text{Dual SDP} \arrow[d, "\text{Agler's Theorem}"] \\
\text{Decomposed Primal SDP} \arrow[r, "\text{Duality}"] \arrow[d, "\text{ADMM}"'] & \text{Decomposed Dual SDP} \arrow[d, "\text{ADMM}"] \\
\text{Algorithm 1} \arrow[r, "\text{Scaling}"] & \text{Algorithm 2}
\end{tikzcd}$$

- Introduced a conversion framework for sparse SDPs
- Developed efficient ADMM algorithms
  - ✓ Primal and dual standard form;
  - ✓ The homogeneous self-dual embedding;

suitable for first-order methods

Ongoing work

- Develop ADMM algorithms for sparse SDPs arising in SOS.
- Applications in networked systems and power systems

Thank you for your attention!

- CDCS: Download from https://github.com/OxfordControl/CDCS

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