The Lasserre hierarchy for polynomial optimization: A tutorial

Etienne de Klerk

Tilburg University and Delft University of Technology, The Netherlands

INFORMS OS Workshop 2016, Princeton

Outline

PART I: Software

- SOSTools for sum of square computations;
- Gloptipoly and YALMIP for global polynomial optimization.

PART II: Research directions & open problems

- Convex polynomial optimization;
- Convergence rates and error bounds;
- Open questions.

Part I

Software

Gram matrix method

Lemma:

The vector space of n-variate polynomials of (total) degree at most d has dimension $\binom{n+d}{d}$.

Gram matrix method

Lemma:

The vector space of n-variate polynomials of (total) degree at most d has dimension $\binom{n+d}{d}$.

For example, a basis for 2-variate polynomials with total degree at most 3 is

$$[x]_3 := [1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3]^T.$$

Gram matrix method

Lemma:

The vector space of n-variate polynomials of (total) degree at most d has dimension $\binom{n+d}{d}$.

For example, a basis for 2-variate polynomials with total degree at most 3 is

$$[x]_3 := [1,\ x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2^2,\ x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3]^T.$$

Theorem

An n-variate polynomial p of total degree $2d$ is a sum-of-squares (SOS) if and only if

$$p(x) = [x]_d^T M [x]_d,$$

where M is a positive semidefinite matrix of size $\binom{n+d}{d} \times \binom{n+d}{d}$.

SDP reformulation

Notation:

$$\mathbb{N}_d^n := \left\{ \alpha \in \mathbb{N}_0^n \mid \sum_{i=1}^n \alpha_i \leq d \right\}.$$

SDP reformulation

Notation:

$$\mathbb{N}_d^n := \left\{ \alpha \in \mathbb{N}_0^n \mid \sum_{i=1}^n \alpha_i \leq d \right\}.$$

For $\alpha \in \mathbb{N}_d^n$ and $x \in \mathbb{R}^n$:

$$x^\alpha := x_1^{\alpha_1} \times \ldots \times x_n^{\alpha_n}.$$

SDP reformulation

Notation:

$$\mathbb{N}^n_d := \left\{ \alpha \in \mathbb{N}^n_0 \mid \sum_{i=1}^{n} \alpha_i \leq d \right\}.$$

For $\alpha \in \mathbb{N}^n_d$ and $x \in \mathbb{R}^n$:

$$x^\alpha := x_1^{\alpha_1} \times \ldots \times x_n^{\alpha_n}.$$

We consider: $p(x) = \sum_{\alpha \in \mathbb{N}^n_{2d}} p_\alpha x^\alpha$.

SDP reformulation

Notation:

$$\mathbb{N}^n_d := \left\{ \alpha \in \mathbb{N}^n_0 \mid \sum_{i=1}^{n} \alpha_i \le d \right\}.$$

For $\alpha \in \mathbb{N}^n_d$ and $x \in \mathbb{R}^n$:

$$x^\alpha := x_1^{\alpha_1} \times \ldots \times x_n^{\alpha_n}.$$

We consider: $p(x) = \sum_{\alpha \in \mathbb{N}^n_{2d}} p_\alpha x^\alpha$.

Semidefinite programming (SDP) formulation

$p(x) = \sum_{\alpha \in \mathbb{N}^n_{2d}} p_\alpha x^\alpha$ is SOS if and only if there exists a positive semidefinite matrix M of size $\binom{n+d}{d} \times \binom{n+d}{d}$, such that

$$\sum_{\gamma,\beta \in \mathbb{N}^n_d,\ \gamma+\beta=\alpha} M_{\gamma,\beta} = p_\alpha \quad \forall\, \alpha \in \mathbb{N}^n_{2d}.$$

Alternative SDP reformulation

Lemma

Let p_1, p_2 be n-variate polynomials of total degree d, and

$$\Delta(n,d) := \left\{ x \in \mathbb{R}^n \ : \ dx \in \mathbb{N}_0^n, \ \sum_{i=1}^{n} x_i \leq 1 \right\}.$$

Alternative SDP reformulation

Lemma

Let p_1, p_2 be n-variate polynomials of total degree d, and

$$\Delta(n,d) := \left\{ x \in \mathbb{R}^n \;:\; dx \in \mathbb{N}_0^n,\ \sum_{i=1}^n x_i \leq 1 \right\}.$$

One has $p_1 = p_2$ iff

$$p_1(x) = p_2(x) \quad \forall\, x \in \Delta(n,d).$$

Alternative SDP reformulation

Lemma

Let p_1, p_2 be n-variate polynomials of total degree d, and

$$\Delta(n,d) := \left\{ x \in \mathbb{R}^n \ : \ dx \in \mathbb{N}_0^n, \ \sum_{i=1}^n x_i \le 1 \right\}.$$

One has $p_1 = p_2$ iff

$$p_1(x) = p_2(x) \quad \forall\, x \in \Delta(n,d).$$

SDP formulation

$p(x) = \sum_{\alpha \in \mathbb{N}_{2d}^n} p_\alpha x^\alpha$ is SOS if and only if there exists a positive semidefinite matrix M of size $\binom{n+d}{d} \times \binom{n+d}{d}$, such that

$$p(x) = [x]_d^T M [x]_d \quad \forall\, x \in \Delta(n, 2d).$$

Example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

Example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

YES, because

$$p(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

Example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

YES, because

$$p(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

The 3×3 matrix (say M) is positive semidefinite and:

$$M = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

Example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

YES, because

$$p(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

The 3×3 matrix (say M) is positive semidefinite and:

$$M = L^TL, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

and consequently,

$$p(x) = \frac{1}{2}\left(2x_1^2 - 3x_2^2 + x_1x_2\right)^2 + \frac{1}{2}\left(x_2^2 + 3x_1x_2\right)^2.$$

Software: SOSTools

The SDP approach to sum-of square-decompositions is implemented in the free Matlab software *SOSTools*.

S. Prajna and A. Papachristodoulou and P. Seiler and P. A. Parrilo, SOSTOOLS: Sum of squares optimization toolbox for MATLAB, Available from `http://www.cds.caltech.edu/sostools`.

SOSTools requires an SDP solver, e.g. *SeDuMi* or *SDPT3*.

SOSTools code for Parrilo example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

SOSTools code for Parrilo example

Example (Parrilo)

Is $p(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

```
% Introduce the variables
mpvar(2,1,'x');

% Define the polynomial
p = 2*x(1)^4 + 2*x(1)^3*x(2) - x(1)^2*x(2)^2 + 5*x(2)^4;

% Test if p is SOS
[M,Z]=findsos(p)
```

SOSTools output in Matlab

```
M =

   5.000000000000336   0.000000000000001  -1.678812742759978
   0.000000000000001   2.357625485522271   0.999999999999674
  -1.678812742759978   0.999999999999674   2.000000000001323


Z =
  [      x_2_1^2 ]
  [ x_1_1*x_2_1 ]
  [      x_1_1^2 ]
```

SOSTools output in Matlab

```
M =

   5.000000000000336   0.000000000000001  -1.678812742759978
   0.000000000000001   2.357625485522271   0.999999999999674
  -1.678812742759978   0.999999999999674   2.000000000001323


Z =
  [      x_2_1^2 ]
  [ x_1_1*x_2_1 ]
  [      x_1_1^2 ]
```

One may verify that $M \succeq 0$ and $p(x) \approx Z^T MZ = \|M^{\frac{1}{2}} Z\|^2$.

SOSTools output in Matlab

```
M =

   5.000000000000336   0.000000000000001  -1.678812742759978
   0.000000000000001   2.357625485522271   0.999999999999674
  -1.678812742759978   0.999999999999674   2.000000000001323


Z =
  [       x_2_1^2 ]
  [ x_1_1*x_2_1 ]
  [       x_1_1^2 ]
```

One may verify that $M \succeq 0$ and $p(x) \approx Z^T M Z = \|M^{\frac{1}{2}} Z\|^2$. Note that M is different from before! (Not unique).

Lasserre hierarchy

Feasible set: $K = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0 \; (i \in \{1, \ldots, m\})\}$.

Lasserre hierarchy

Feasible set: $K = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0 \ (i \in \{1, \ldots, m\})\}$.

Truncated quadratic module:

$$Q(p_1, \ldots, p_m)_t := \{ \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \ \text{SOS}\}$$

Lasserre hierarchy

Feasible set: $K = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0 \; (i \in \{1, \ldots, m\})\}$.

Truncated quadratic module:

$$Q(p_1, \ldots, p_m)_t := \{ \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \;\; \text{SOS}\}$$

Replace

(P) $$p_{\min} = \inf_{x \in K} p(x) = \sup \lambda \;\text{ s.t. }\; p - \lambda \geq 0 \;\text{ on } K$$

Lasserre hierarchy

Feasible set: $K = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0 \ (i \in \{1, \ldots, m\})\}$.

Truncated quadratic module:

$$Q(p_1, \ldots, p_m)_t := \{ \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \ \text{SOS} \}$$

Replace

(P) $$p_{\min} = \inf_{x \in K} p(x) = \sup \lambda \ \text{ s.t. } \ p - \lambda \geq 0 \ \text{ on } K$$

by

(SOSt) $$p_t^{\text{sos}} = \sup \lambda \ \text{ s.t. } \ p - \lambda \in Q(p_1, \ldots, p_m)_t$$

Lasserre hierarchy

Feasible set: $K = \{x \in \mathbb{R}^n \mid p_i(x) \geq 0 \ (i \in \{1,\ldots,m\})\}$.

Truncated quadratic module:

$$Q(p_1,\ldots,p_m)_t := \{ \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \ \text{SOS}\}$$

Replace

(P) $$p_{\min} = \inf_{x \in K} p(x) = \sup \lambda \ \text{ s.t. } \ p - \lambda \geq 0 \ \text{ on } K$$

by

(SOSt) $$p_t^{\text{sos}} = \sup \lambda \ \text{ s.t. } \ p - \lambda \in Q(p_1,\ldots,p_m)_t$$

J.B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM Journal on Optimization **11** (2001) 796-817.

Relation to KKT conditions

KKT conditions for $\min_{x\in\mathbb{R}^n} \{p(x) \;:\; p_i(x) \geq 0 \; (i = 1, \ldots, m)\}$

We call (x^*, λ) a KKT point if

$$\begin{array}{rcl} \nabla p(x^*) & = & \sum_{j=1}^m \lambda_j \nabla p_j(x^*) \\ \lambda_j p_j(x^*) & = & 0 \qquad j = 1, \ldots, k \\ p_j(x^*) & \geq & 0 \qquad j = 1, \ldots, k \\ \lambda_j & \geq & 0 \qquad j = 1, \ldots, k. \end{array}$$

Relation to KKT conditions

KKT conditions for $\min_{x\in\mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}$

We call (x^*, λ) a KKT point if

$$\begin{array}{rcll} \nabla p(x^*) & = & \sum_{j=1}^{m} \lambda_j \nabla p_j(x^*) & \\ \lambda_j p_j(x^*) & = & 0 & j = 1, \ldots, k \\ p_j(x^*) & \geq & 0 & j = 1, \ldots, k \\ \lambda_j & \geq & 0 & j = 1, \ldots, k. \end{array}$$

If the Lasserre relaxation of order t is exact, and x^* optimal, then

$$p - p(x^*) = \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \ \text{SOS}$$

Relation to KKT conditions

KKT conditions for $\min_{x\in\mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}$

We call (x^*, λ) a KKT point if

$$\begin{array}{rcll} \nabla p(x^*) & = & \sum_{j=1}^m \lambda_j \nabla p_j(x^*) & \\ \lambda_j p_j(x^*) & = & 0 & j = 1, \ldots, k \\ p_j(x^*) & \geq & 0 & j = 1, \ldots, k \\ \lambda_j & \geq & 0 & j = 1, \ldots, k. \end{array}$$

If the Lasserre relaxation of order t is exact, and x^* optimal, then

$$p - p(x^*) = \underbrace{s_0}_{\deg \leq 2t} + \underbrace{s_1 p_1}_{\deg \leq 2t} + \ldots + \underbrace{s_m p_m}_{\deg \leq 2t} \mid s_j \ \text{ SOS}$$

Fact: In this case x^* and $\lambda_i = s_i(x^*)$ $(i = 1, \ldots, m)$ is a KKT point.

Software: Gloptipoly and YALMIP

Lasserre'a approach is implemented in the software *Gloptipoly* and *YALMIP*:

D. Henrion, J. B. Lasserre, J. Löfberg. GloptiPoly 3: moments, optimization and semidefinite programming. *Optimization Methods and Software*, **24**:4-5, 761–779, 2009,

J. Löfberg. YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In Proceedings of the CACSD Conference, Taipei, Taiwan, 2004.

Freely available at:

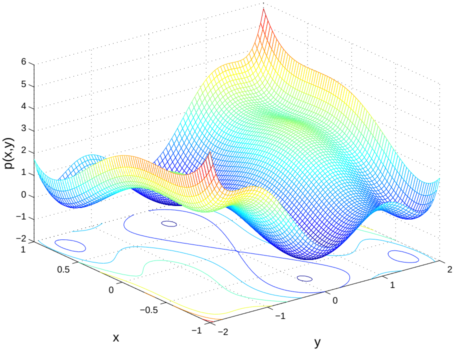
Gloptipoly: `http://homepages.laas.fr/henrion/software/gloptipoly3/`

YALMIP: `http://users.isy.liu.se/johanl/yalmip/`

Gloptipoly and YALMIP require an SDP solver, e.g. *SeDuMi* or *SDPT3*.

Example

$$-1.0316 = \min_{x^2+y^2\le 3} p(x,y) := x^2(4-2.1x^2+\frac{1}{3}x^4)+xy+y^2(-4+4y^2)$$



Two global minima: (0.0898 -0.7127) and (-0.0898 0.7127).

Example (ctd.)

$$-1.0316 \approx \min_{x^2+y^2 \leq 3} p(x,y) := x^2(4 - 2.1x^2 + \frac{1}{3}x^4) + xy + y^2(-4 + 4y^2)$$

Example (ctd.)

$$-1.0316 \approx \min_{x^2+y^2\leq 3} p(x,y) := x^2(4-2.1x^2+\frac{1}{3}x^4)+xy+y^2(-4+4y^2)$$

We obtain the decomposition

$$p(x,y)+1.0316 = s_0(x,y)+s_1(x,y)(3-x^2-y^2),$$

Example (ctd.)

$$-1.0316 \approx \min_{x^2+y^2\leq 3} p(x,y) := x^2(4-2.1x^2+\frac{1}{3}x^4)+xy+y^2(-4+4y^2)$$

We obtain the decomposition

$$p(x,y)+1.0316 = s_0(x,y)+s_1(x,y)(3-x^2-y^2),$$

where

$$s_0(x,y) = 0.18082+0.92289x^2+0.17596xy-0.66637y^2+0.31332x^4+0.64908y^4+\ldots$$

$$s_1(x,y) = 0.2836+1.1202x^2+0.27468xy-1.0167y^2-0.43103x^4+0.77808y^4+\ldots$$

are sums of squares, $\deg(s_0)=8$, $\deg(s_1)=6$.

Yalmip code

```
% First we define the variables
% and the polynomial to be minimized

sdpvar x1 x2
obj = 4*x1^2+x1*x2-4*x2^2-2.1*x1^4+4*x2^4+x1^6/3;

% Then we define the feasible set

F = [3-x1^2-x2^2 >0]

% Solve the SDP formulation using e.g. Sedumi

[DIAGNOSTIC,X,MOMENT,SOS] = solvemoment(F,obj,[],4);
```

Yalmip output in Matlab

```
SOS.t =

  -1.0316


X{1} =

   0.0898  -0.7126


X{2} =

  -0.0898   0.7126
```

Yalmip output in Matlab

```
SOS.t =

  -1.0316


X{1} =

    0.0898   -0.7126


X{2} =

   -0.0898    0.7126
```

Thus both global minimizers are found.

Part II

Research Directions & Open Problems

Recognizing convex problems

Theorem

Deciding convexity of n-variate polynomials of even degree at least 4 is NP-hard.

Recognizing convex problems

Theorem

Deciding convexity of n-variate polynomials of even degree at least 4 is NP-hard.

A. A. Ahmadi, A. Olshevsky, P. A. Parrilo, J. N. Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems. *Mathematical Programming*, 137(1-2), 453–476, 2013.

Recognizing convex problems

Theorem

Deciding convexity of n-variate polynomials of even degree at least 4 is NP-hard.

A. A. Ahmadi, A. Olshevsky, P. A. Parrilo, J. N. Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems. *Mathematical Programming*, 137(1-2), 453–476, 2013.

Theorem

There exist convex forms that are not sums of squares.

Recognizing convex problems

Theorem

Deciding convexity of n-variate polynomials of even degree at least 4 is NP-hard.

A. A. Ahmadi, A. Olshevsky, P. A. Parrilo, J. N. Tsitsiklis. NP-hardness of deciding convexity of quartic polynomials and related problems. *Mathematical Programming*, 137(1-2), 453–476, 2013.

Theorem

There exist convex forms that are not sums of squares.

G. Blekherman. Convex forms that are not sums of squares. arXiv:0910.0656v1, October 2009.

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\} .$$

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \;:\; p_i(x) \geq 0 \; (i = 1, \ldots, m)\}.$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{ p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m) \} .$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}.$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.
3. The quadratic module $Q(p_1, \ldots, p_m)$ is Archimedean.

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}.$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.
3. The quadratic module $Q(p_1, \ldots, p_m)$ is Archimedean.
4. The Hessian of p at x^* is positive definite if x^* is a global minimizer.

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}.$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.
3. The quadratic module $Q(p_1, \ldots, p_m)$ is Archimedean.
4. The Hessian of p at x^* is positive definite if x^* is a global minimizer.

Then the Lasserre hierarchy converges finitely ($p^{\text{sos}}_{t^*} = p_{\min}$ for some finite t^*) ...

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\}.$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.
3. The quadratic module $Q(p_1, \ldots, p_m)$ is Archimedean.
4. The Hessian of p at x^* is positive definite if x^* is a global minimizer.

Then the Lasserre hierarchy converges finitely ($p^{\text{sos}}_{t^*} = p_{\min}$ for some finite t^*) ... but t^* does not depend only on n, m and the polynomial degrees.

Lassere hierarchy: the convex case

Consider:

$$p_{\min} = \min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \geq 0 \ (i = 1, \ldots, m)\} .$$

Theorem [De Klerk-Laurent]

Assume:

1. The polynomials $p, -p_1, \ldots, -p_m$ are convex;
2. The Slater condition holds.
3. The quadratic module $Q(p_1, \ldots, p_m)$ is Archimedean.
4. The Hessian of p at x^* is positive definite if x^* is a global minimizer.

Then the Lasserre hierarchy converges finitely ($p^{\text{sos}}_{t^*} = p_{\min}$ for some finite t^*) ... but t^* does not depend only on n, m and the polynomial degrees.

E. de Klerk, M. Laurent. On the Lasserre hierarchy of semidefinite programming relaxations of convex polynomial optimization problems. *SIAM Journal on Optimization*, 21(3), 824–832, 2011.

Related result

Definition

A polynomial p is called SOS-convex if

$$\langle z, \nabla^2 p(x) z\rangle \text{ is a sum of squares in } (x, z).$$

Related result

Definition

A polynomial p is called SOS-convex if

$$\langle z, \nabla^2 p(x) z \rangle \text{ is a sum of squares in } (x, z).$$

Theorem (Lasserre (2009):

The Lasserre hierarchy converges in one step if $p, -p_1, \ldots, -p_m$ are SOS-convex.

Related result

Definition

A polynomial p is called SOS-convex if

$$\langle z, \nabla^2 p(x) z \rangle \text{ is a sum of squares in } (x, z).$$

Theorem (Lasserre (2009):

The Lasserre hierarchy converges in one step if $p, -p_1, \ldots, -p_m$ are SOS-convex.

J.B. Lasserre. Convexity in semialgebraic geometry and polynomial optimization. *SIOPT* **19**, 1995–2014, 2009.

Related result

Definition

A polynomial p is called SOS-convex if

$$\langle z, \nabla^2 p(x) z \rangle \text{ is a sum of squares in } (x, z).$$

Theorem (Lasserre (2009):

The Lasserre hierarchy converges in one step if $p, -p_1, \ldots, -p_m$ are SOS-convex.

J.B. Lasserre. Convexity in semialgebraic geometry and polynomial optimization. *SIOPT* **19**, 1995–2014, 2009.

Theorem (Helton and Nie)

SOS-convex forms are sums of squares.

Related result

Definition

A polynomial p is called SOS-convex if

$$\langle z, \nabla^2 p(x) z\rangle \text{ is a sum of squares in } (x,z).$$

Theorem (Lasserre (2009):

The Lasserre hierarchy converges in one step if $p, -p_1, \ldots, -p_m$ are SOS-convex.

J.B. Lasserre. Convexity in semialgebraic geometry and polynomial optimization. *SIOPT* **19**, 1995–2014, 2009.

Theorem (Helton and Nie)

SOS-convex forms are sums of squares.

J. W. Helton and J. Nie. Semidefinite representation of convex sets, *Mathematical Programming*, 122(1), 21–64, 2010.

Error bounds for the Lasserre hierarchy

Consider minimizing a polynomial on the sphere:

$$\min_{x \in \mathbb{R}^n} \left\{ p(x) \ : \ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

Error bounds for the Lasserre hierarchy

Consider minimizing a polynomial on the sphere:

$$\min_{x \in \mathbb{R}^n} \left\{ p(x) \ : \ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

Theorem

If p is of degree $2d$ with only even exponents, then

$$p_{\min} - p_t^{sos} \leq \frac{C_d}{t} \qquad (t \geq d),$$

where C_d depends on d only.

Error bounds for the Lasserre hierarchy

Consider minimizing a polynomial on the sphere:

$$\min_{x \in \mathbb{R}^n} \left\{ p(x) \ : \ \sum_{i=1}^{n} x_i^2 = 1 \right\}$$

Theorem

If p is of degree $2d$ with only even exponents, then

$$p_{\min} - p_t^{sos} \leq \frac{C_d}{t} \qquad (t \geq d),$$

where C_d depends on d only.

E. de Klerk, E., M. Laurent, P. Parrilo, A PTAS for the minimization of polynomials of fixed degree over the simplex. *Theoret. Comput. Sci.* 361(2–3), 210–225 (2006)

Error bounds for the Lasserre hierarchy: the general case

$$\min_{x \in \mathbb{R}^n} \{p(x) \ : \ p_i(x) \ge 0 \ (i = 1, \ldots, m)\}.$$

Error bounds for the Lasserre hierarchy: the general case

$$\min_{x\in\mathbb{R}^n} \{p(x) \;:\; p_i(x) \geq 0 \; (i = 1, \ldots, m)\}.$$

Theorem

For the general polynomial optimization problem

$$p_{\min} - p_t^{sos} = O\left(\frac{1}{\sqrt[c]{\frac{\log(t)}{c}}}\right) \textit{ for some } c > 0.$$

Error bounds for the Lasserre hierarchy: the general case

$$\min_{x\in\mathbb{R}^n} \left\{p(x) \ : \ p_i(x) \geq 0 \ (i = 1,\ldots,m)\right\}.$$

Theorem

For the general polynomial optimization problem

$$p_{\min} - p_t^{sos} = O\left(\frac{1}{\sqrt[c]{\frac{\log(t)}{c}}}\right) \text{ for some } c > 0.$$

Nie, J., Schweighofer, M.: On the complexity of Putinar's Positivstellensatz. *Journal of Complexity*, 23(1), 135–150 (2007)

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t.

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)
- Interesting classes of convex polynomials that may be identified in polynomial time?

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)
- Interesting classes of convex polynomials that may be identified in polynomial time?
- Example of a convex form that is not SOS?

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)
- Interesting classes of convex polynomials that may be identified in polynomial time?
- Example of a convex form that is not SOS?
- A bound for the number of steps of the Lasserre hierarchy for the convex case in terms of a condition number of the problem?

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)
- Interesting classes of convex polynomials that may be identified in polynomial time?
- Example of a convex form that is not SOS?
- A bound for the number of steps of the Lasserre hierarchy for the convex case in terms of a condition number of the problem?

Computation questions:

- A self-concordant barrier for the cone of SOS polynomials of degree $2t$?

Open questions

Theory questions:

- Error bounds for the Lasserre hierarchy: give upper bounds for $p_{\min} - p_t^{\text{sos}}$ in terms of t. (Very little is known.)
- Interesting classes of convex polynomials that may be identified in polynomial time?
- Example of a convex form that is not SOS?
- A bound for the number of steps of the Lasserre hierarchy for the convex case in terms of a condition number of the problem?

Computation questions:

- A self-concordant barrier for the cone of SOS polynomials of degree $2t$?
- More efficient computation of p_t^{sos} by exploiting sparsity, symmetry, etc.

The End

Further reading:

M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In *Emerging Applications of Algebraic Geometry*, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds.), Springer, pages 157-270, 2009
http://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf

THANK YOU!