Robust to Dynamics Optimization

Amir Ali Ahmadi
Princeton, ORFE

Oktay Gunluk
IBM Research

INFORMS 2014
Session on Optimization in Dynamics and Control

PRINCETON UNIVERSITY ORFE

1

Robust to Dynamics Optimization (RDO)

An RDO is describe by two pieces of input:

1) An optimization problem: $\min_{x}\{f(x) : x \in \Omega\}$

2) A dynamical system: $x_{k+1} = g(x_k)$ or $\dot{x} = g(x)$

RDO is then the following problem:

(discrete time) $$\min_{x_0}\{f(x_0) : x_k \in \Omega, k = 0, 1, 2, \ldots\}$$

(continuous time) $$\min_{x_0}\{f(x_0) : x(t; x_0) \in \Omega, \forall t \geq 0\}$$

PRINCETON UNIVERSITY ORFE

2

Various RDO problems to study…

$$\min_{x_0}\{f(x_0): x_k \in \Omega, \forall k;\ x_{k+1} = g(x_k)\}$$

This talk:

Optimization Problem	**Dynamics**
Linear Program	Linear
Integer Program	Nonlinear
Semidefinite Program	Uncertain
Polynomial Program	Time-varying
Robust Linear Program	Discrete/continuous/hybrid of both
⋮	⋮

PRINCETON UNIVERSITY ORFE

R-LD-LP

Robust to linear dynamics linear programming (R-LD-LP)

Classical LP:

$$\min_{x}\{c^T x: Ax \leq b\}$$

Robust LP:

$$\min_{x}\{c^T x: Ax \leq b, \forall A \in \mathbb{A}, b \in \mathbb{B}\}$$

R-LD-LP:

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots ; x_{k+1} = Gx_k\}$$

PRINCETON UNIVERSITY ORFE

4

R-LD-LP

Robust to linear dynamics linear programming (R-LD-LP)

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots ; x_{k+1} = Gx_k\}$$

Input data: A, b, c, G

Alternative form:

$$\min_{x}\{c^T x: Ax \leq b, AGx \leq b, AG^2 x \leq b, AG^3 x \leq b, \ldots\}$$

(An infinite LP)

Feasible set of R-LD-LP:

$$\mathcal{S} := \bigcap_{k=0}^{\infty}\{x|\ AG^k x \leq b\}$$

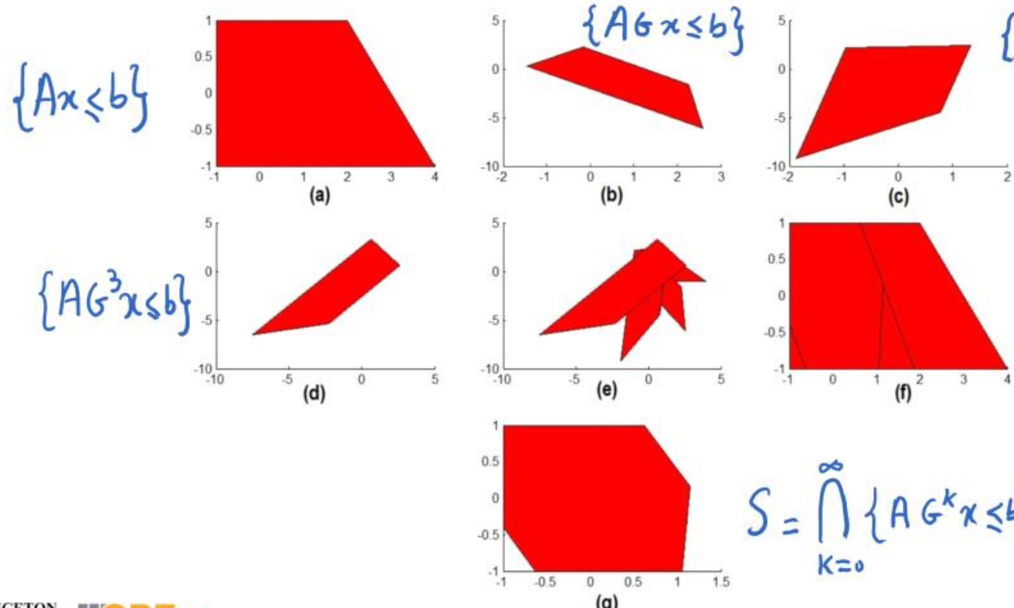
PRINCETON UNIVERSITY ORFE

5

An example...

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \dots; x_{k+1} = Gx_k\}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0.6 & -0.4 \\ 0.8 & 0.5 \end{bmatrix}$$



$$S = \bigcap_{k=0}^{\infty} \{AG^k x \leq b\} = \bigcap_{k=0}^{2} \{AG^k x \leq b\}$$

PRINCETON UNIVERSITY ORFE

Obvious way to get lower bounds

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots; x_{k+1} = Gx_k\}$$

Can get a sequence of lower bounds by solving finite LPs:

$$\min_{x}\{c^T x : Ax \leq b, AGx \leq b, AG^2 x \leq b, \ldots, AG^r x \leq b\}$$

Natural questions:

- Is the optimal value of R-LD-LP achieved in a finite number of steps?
- Is the feasible set of R-LD-LP always a polytope?
- When it is, how large are the number of facets?
- How to get **upper bounds**?!
  - (We'll see soon: from **semidefinite programming**)

PRINCETON UNIVERSITY ORFE

Feasible set of R-LD-LP

Lemma. The feasible set of R-LD-LP is **closed, convex, and invariant**.

Proof. Easy.

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x \mid AG^k x \leq b\}$$

- But it may not be polyhedral.
- Even if it is, it may not be achieved at a finite level.

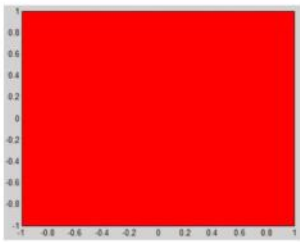
PRINCETON UNIVERSITY ORFE

Feasible set of R-LD-LP

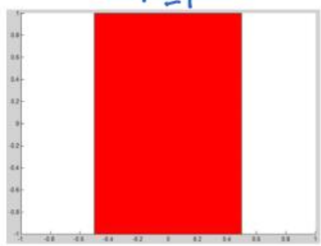
$$G = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\bigcap_{k=0}^{r} \{AG^k x \leq b\}$$

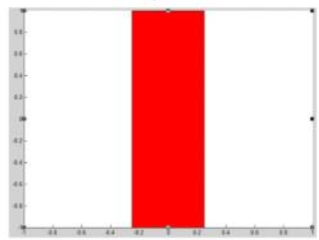
$r=0$



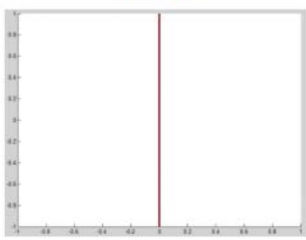
$r=1$



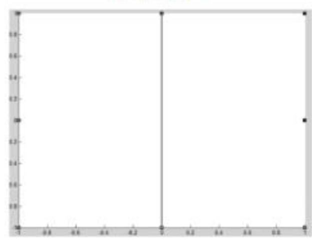
$r=2$



$r=8$



$r=\infty$



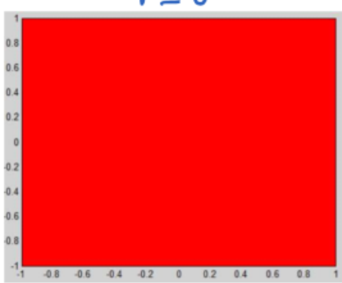
PRINCETON UNIVERSITY ORFE

Feasible set of R-LD-LP

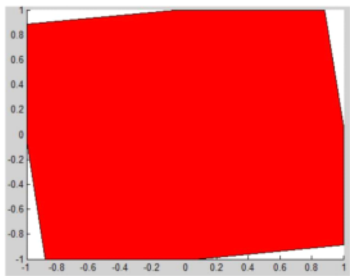
$$G = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \theta = 7^\circ$$

$$\bigcap_{k=0}^{r} \{AG^k x \leq b\}$$

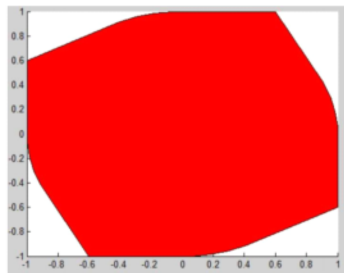
$r=0$



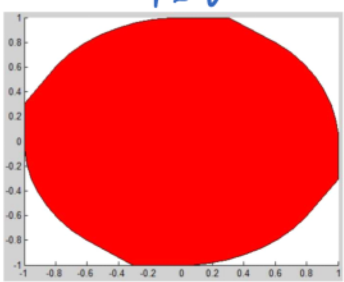
$r=1$



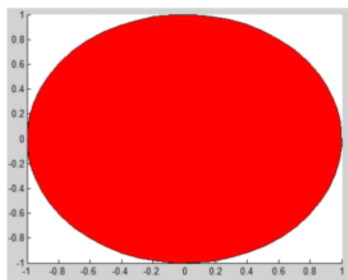
$r=4$



$r=8$



$r=\infty$



PRINCETON UNIVERSITY ORFE

10

Detecting finiteness

Lemma. Suppose for some r we have

$$\bigcap_{k=0}^{r}\{x|\ AG^k x \leq b\} = \bigcap_{k=0}^{r+1}\{x|\ AG^k x \leq b\}$$

Then,

$$\bigcap_{k=0}^{\infty}\{x|\ AG^k x \leq b\} = \bigcap_{k=0}^{r}\{x|\ AG^k x \leq b\}.$$

Proof. Equivalence of the r-th and $r+1$-th iterations implies invariance of the r-th iteration polytope.

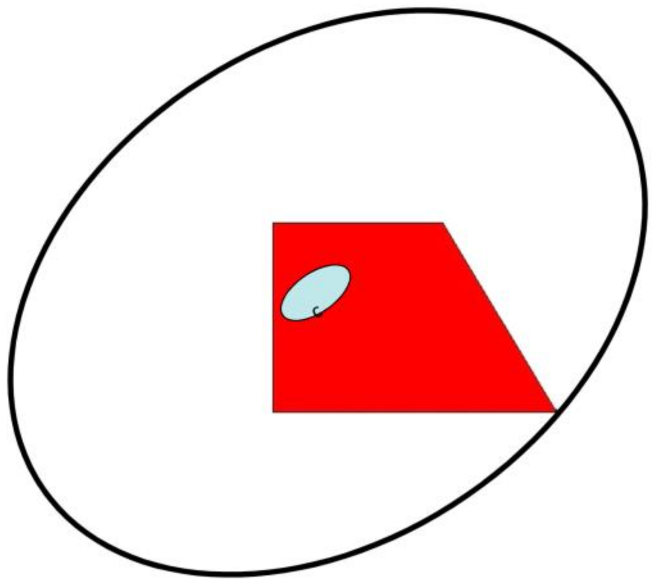
Note. This condition can be efficiently checked.

PRINCETON UNIVERSITY ORFE

Solving R-LD-LP exactly via LP

Theorem. If $\rho(G) < 1$, then convergence is finite. Moreover, the number of steps needed is polynomial in the size of the input $(A, b, c\ , G)$.

Proof idea.



Invariant ellipsoid: $\{x^T P x \leq 1\}$

PRINCETON UNIVERSITY ORFE

12

Upper bound on the number of iterations

- Find an invariant ellipsoid defined by a positive definite matrix P
- Find a shrinkage factor $\gamma \in (0,1)$; i.e., a scalar satisfying $G^T P G \preceq \gamma P$
- Find a scalar $\alpha_2 > 0$ such that

$$\{Ax \leq b\} \subseteq \{x^T P x \leq \alpha_2\}$$

- Find a scalar $\alpha_1 > 0$ such that

$$\{x^T P x \leq \alpha_1\} \subseteq \{Ax \leq b\}$$

- Let

$$r = \lceil \frac{\log \frac{\alpha_1}{\alpha_2}}{\log \gamma} \rceil$$

PRINCETON UNIVERSITY ORFE

Finding an invariant ellipsoid

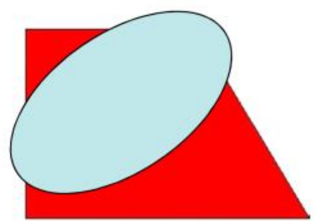
- **Computation of P.**

To find an invariant ellipsoid for G, we solve the linear system

$$G^T PG - P = -I,$$

where I is the $n \times n$ identity matrix. This is called the Lyapunov equation.

The matrix P will automatically turn out to be positive definite.

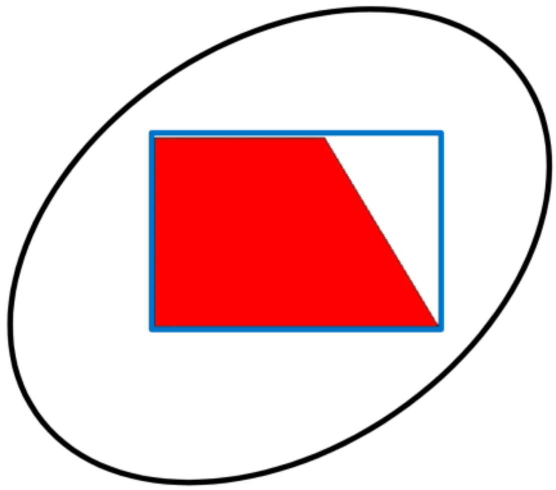


PRINCETON UNIVERSITY ORFE

14

Finding the outer ellipsoid

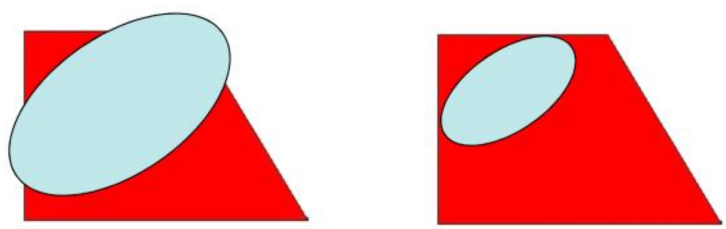
- **Computation of α_2.** By solving, e.g., n LPs, we can place our polytope $\{Ax \leq b\}$ in a box; i.e., compute $2n$ scalars l_i, u_i such that
$$\{Ax \leq b\} \subseteq \{l_i \leq x_i \leq u_i\}.$$
We then bound $x^T Px = \sum_{i,j} P_{i,j} x_i x_j$ term by term to get α_2:
$$\alpha_2 = \sum_{i,j} \max\{P_{i,j}u_i u_j, P_{i,j}l_i l_j, P_{i,j}u_i l_j, P_{i,j}l_i u_j\}.$$
This ensures that $\{l_i \leq x_i \leq u_i\} \subseteq \{x^T Px \leq \alpha_2\}$. Hence, $\{Ax \leq b\} \subseteq \{x^T Px \leq \alpha_2\}$.



PRINCETON UNIVERSITY ORFE

Finding the inner ellipsoid

- **Computation of α_1.** For $i = 1, \ldots, m$, we compute a scalar η_i by solving the convex program
$$\eta_i := \min_x \{a_i^T x : x^T P x \leq 1\},$$
where a_i is the i-th row of the constraint matrix A. This problem has a closed form solution:
$$\eta_i = -\sqrt{a_i^T P^{-1} a_i}.$$
Note that P^{-1} exists since $P \succ 0$. We then let
$$\alpha_1 = \min_i \{\frac{b_i^2}{\eta_i^2}\}.$$



PRINCETON UNIVERSITY ORFE

Upper bounds on R-LD-LP via SDP

- Goal: Find the best invariant ellipsoid inside the original polytope and optimize over that.

$$\min_{x,P} \quad c^T x$$

$$P \succ 0$$

$$G^T P G \preceq P$$

$$x^T P x \leq 1$$

$$\left[\forall z, \ z^T P z \leq 1 \Rightarrow Az \leq b\right] \overset{\text{S-Lemma}}{\longleftrightarrow} b - a_i^T z \geq \gamma_i (1 - z^T P z) \quad \forall z$$

$$\gamma_i \geq 0$$

Non-convex formulation

Upper bounds on R-LD-LP via SDP

- If we parameterize in terms of P^{-1} instead, then it becomes convex!

$$\min_{x,P} \; c^T x$$
$$P \succ 0$$
$$G^T P G \preceq P$$
$$x^T P x \leq 1$$

⟵ Dynamics Duality ⟶

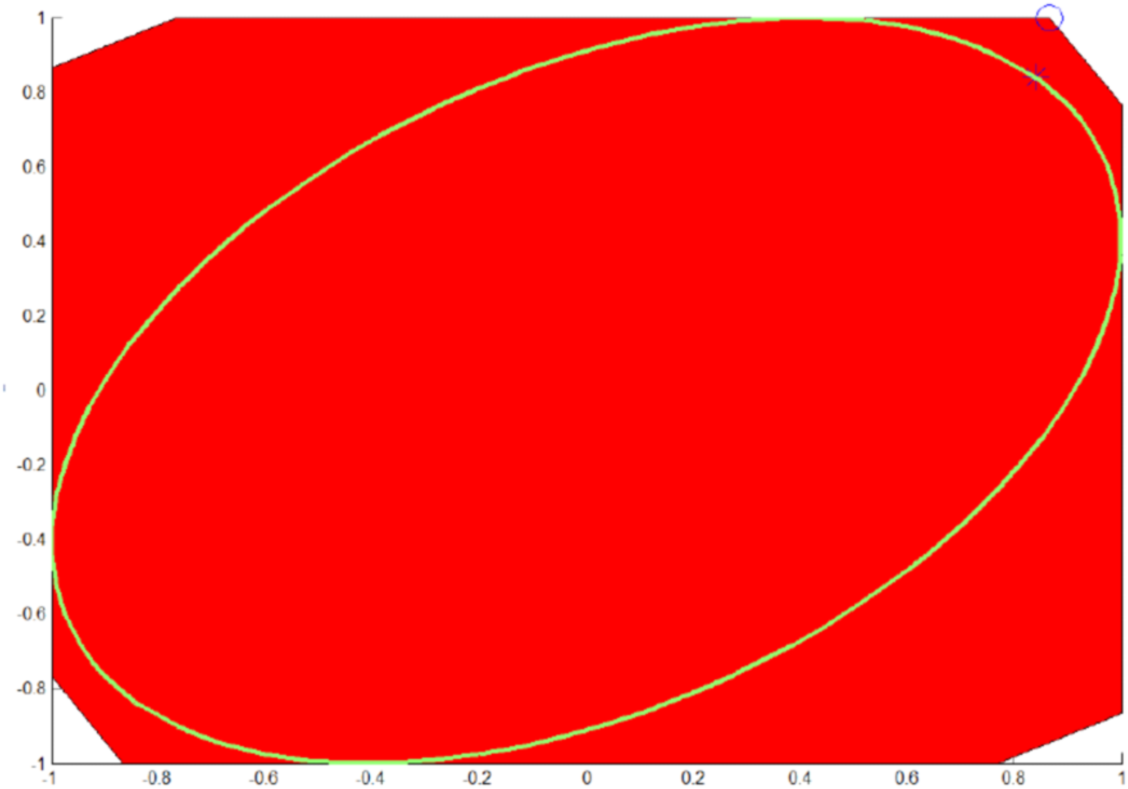
⟵ Schur Complement ⟶

$$\min_{x,Q} \; c^T x$$
$$Q \succ 0$$
$$G Q G^T \preceq Q$$
$$\begin{bmatrix} Q & x \\ x^T & 1 \end{bmatrix} \succeq 0$$
$$a_i^T Q a_i \leq 1$$

$$\left[\forall z, \; z^T P z \leq 1 \Rightarrow Az \leq b\right]$$

ellipsoid $\subseteq$ polytope ⟵ Polar duality ⟶ (polytope)* $\subseteq$ (ellipsoid)*

Upper bounds on R-LD-LP via SDP



PRINCETON UNIVERSITY ORFE

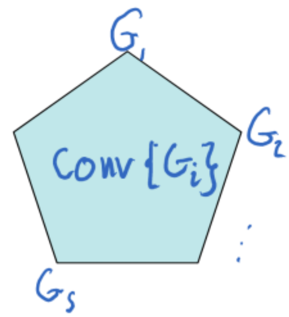
Robust to *switched* linear dynamics linear programming (R-SLD-LP)

PRINCETON UNIVERSITY ORFE

R-SLD-LP

Robust to switched linear dynamics linear programming (R-SLD-LP)

$$x_k = G_i x_k, i = 1, \dots, s$$



Models **uncertainty** and **variations with time** in the dynamics

$$\min_x \{c^T x : AGx \leq b, \forall G \in \mathbb{G}^*\}$$ (An infinite LP)

$\mathbb{G}^*$: set of all finite products of $G_1, \dots, G_s$

PRINCETON UNIVERSITY ORFE

Input data: $A, b, c, G_1, \dots, G_s$

21

Feasible set of R-SLD-LP

It's still convex, closed, and invariant (but typically much more nasty).

Theorem. If the *joint spectral radius* of $G_1, \ldots, G_s$ is less than one, then the feasible set of R-SLD-LP is a polytope.

Joint spectral radius (JSR):

$$\rho(G_1, \ldots, G_s) = \lim_{k \to \infty} \max_{\sigma \in \{1,\ldots,s\}^m} ||G_{\sigma_1} \cdots G_{\sigma_k}||^{\frac{1}{k}}$$

- Computation the JSR is a major topic in controls.
- Testing whether the JSR of two 47x47 matrices is is undecidable! [Blondel, Tsitsiklis], [Blondel, Canterini]

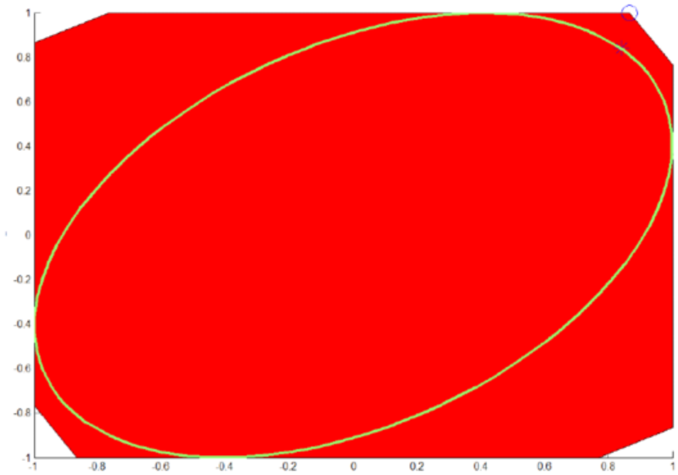
PRINCETON UNIVERSITY ORFE

Lower and upper bounds for R-SLD-LP

To get lower bounds, truncate the sequence and solve an LP.
For example,

$$\min_x \{c^T x : Ax \leq b, AG_1 x \leq b, AG_1 G_2 x \leq b, \ldots, AG_1 G_2 G_1 x \leq b\}$$

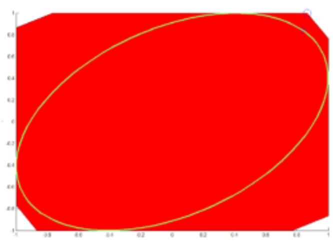
To get upper bounds, same SDP idea works. Just require invariance with respect to all matrices G_i.

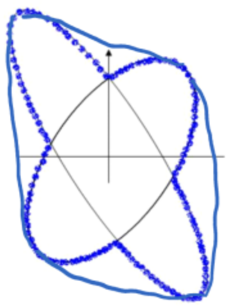


PRINCETON UNIVERSITY ORFE

SDP upper bounds for R-SLD-LP

- Unlike the linear case, there may be no invariant ellipsoid!
- But, we have a way to parameterize the **convex hull of the union of ellipsoids** as invariant sets for R-SLD-LP using again SDP.
- If the JSR is less than one, there will always be a finite union of ellipsoids that is invariant. The convex hull of this set is also invariant.





PRINCETON UNIVERSITY ORFE

Recap

Robust to dynamics optimization (RDO):

$$\min_{x_0}\{f(x_0): x_k \in \Omega, \forall k;\ x_{k+1} = g(x_k)\}$$

- LP + linear dynamical system:
  - Polynomial time solvable (in the most interesting case)
- LP + switched linear system:
  - Lower bounds via LP
  - Upper bounds via SDP
- Numerous other cases remain open

PRINCETON UNIVERSITY ORFE

Preprint in preparation: http://aaa.princeton.edu