Robust to Dynamics Optimization

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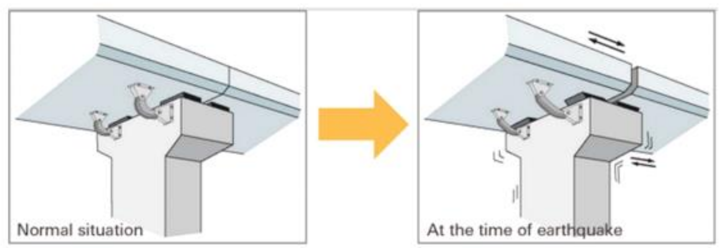
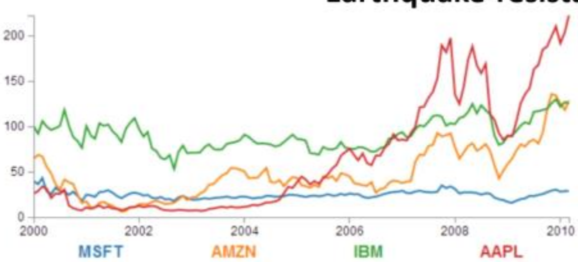
The setting & motivating applications

- You solve a constrained optimization problem today
- An external dynamical system may move your optimal point in the future and make it infeasible
- You want your initial decision to be "safe enough" to not let this happen

Drug design



Earthquake-resistant structures



Robust investments

Robust to Dynamics Optimization (RDO)

An RDO is describe by two pieces of input:

1) An optimization problem: $\min_{x}\{f(x) : x \in \Omega\}$

2) A dynamical system: $x_{k+1} = g(x_k)$ or $\dot{x} = g(x)$

RDO is then the following problem:

(discrete time) $$\min_{x_0}\{f(x_0) : x_k \in \Omega, k = 0, 1, 2, \ldots\}$$

(continuous time) $$\min_{x_0}\{f(x_0) : x(t; x_0) \in \Omega, \forall t \geq 0\}$$

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Various RDO problems to study…

$$\min_{x_0}\{f(x_0): x_k \in \Omega, \forall k;\ x_{k+1} = g(x_k)\}$$

This talk:

Optimization Problem	**Dynamics**
Linear Program	Linear
Integer Program	Nonlinear
Semidefinite Program	Uncertain
Polynomial Program	Time-varying
Robust Linear Program	Discrete/continuous/hybrid of both
⋮	⋮

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R-LD-LP

Robust to linear dynamics linear programming (R-LD-LP)

Classical LP:

$$\min_{x}\{c^T x : Ax \leq b\}$$

Robust LP:

$$\min_{x}\{c^T x : Ax \leq b, \forall A \in \mathbb{A}, b \in \mathbb{B}\}$$

R-LD-LP:

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots ; x_{k+1} = Gx_k\}$$

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R-LD-LP

Robust to linear dynamics linear programming (R-LD-LP)

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots; x_{k+1} = Gx_k\}$$

Input data: A, b, c, G

Alternative form:

$$\min_{x}\{c^T x: Ax \leq b, AGx \leq b, AG^2x \leq b, AG^3x \leq b, \ldots\}$$

(An infinite LP)

Feasible set of R-LD-LP:

$$\mathcal{S} := \bigcap_{k=0}^{\infty}\{x|\ AG^k x \leq b\}$$

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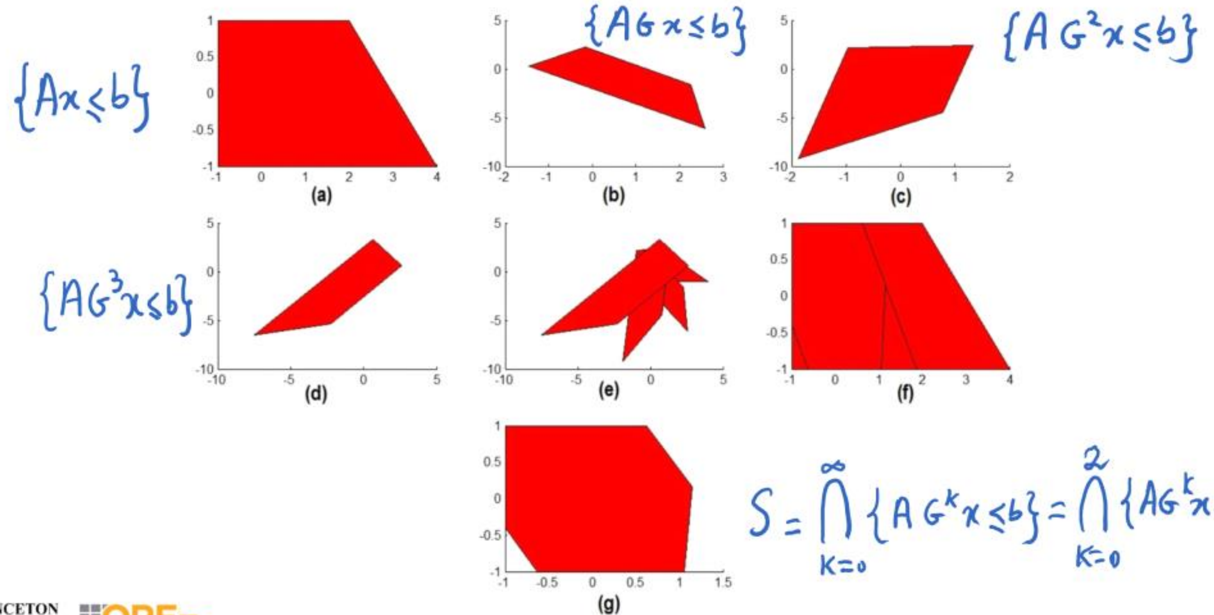
An example...

$$\min_{x_0}\{c^T x_0 : Ax_k \le b, k = 0, 1, 2, \ldots; x_{k+1} = Gx_k\}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 0.6 & -0.4 \\ 0.8 & 0.5 \end{bmatrix}$$



$S = \bigcap_{k=0}^{\infty} \{AG^k x \le b\} = \bigcap_{k=0}^{2} \{AG^k x \le b\}$

Obvious way to get lower bounds

$$\min_{x_0}\{c^T x_0 : Ax_k \leq b, k = 0, 1, 2, \ldots ; x_{k+1} = Gx_k\}$$

Can get a sequence of lower bounds by solving finite LPs:

$$\min_{x}\{c^T x: Ax \leq b, AGx \leq b, AG^2x \leq b, \ldots, AG^r x \leq b\}$$

Natural questions:

- Is the optimal value of R-LD-LP achieved in a finite number of steps?
- Is the feasible set of R-LD-LP always a polytope?
- When it is, how large are the number of facets?
- How to get **upper bounds**?!
  - (We'll see soon: from **semidefinite programming**)

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Feasible set of R-LD-LP

Lemma. The feasible set of R-LD-LP is **closed, convex, and invariant**.

Proof. Easy.

$$\mathcal{S} := \bigcap_{k=0}^{\infty} \{x| \; AG^k x \leq b\}$$

- But it may not be polyhedral.
- Even if it is, it may not be achieved at a finite level.

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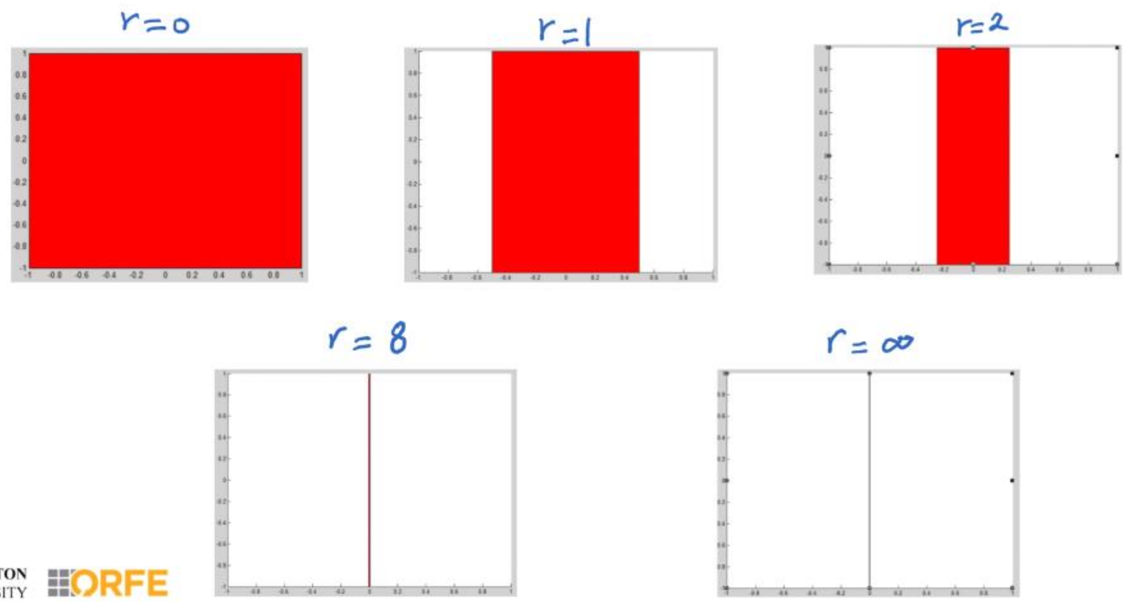
Feasible set of R-LD-LP

$$G = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\bigcap_{k=0}^{r} \{AG^k x \leq b\}$$



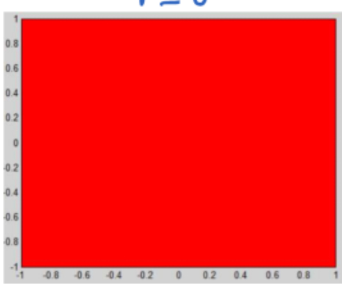
Feasible set of R-LD-LP

$$\bigcap_{k=0}^{r} \{AG^k x \leq b\}$$

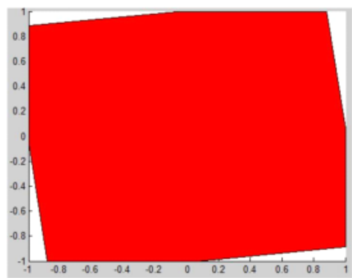
$$G = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Rotation by irrational degree

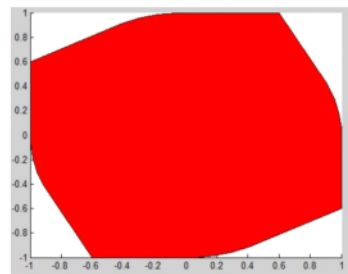
$r=0$



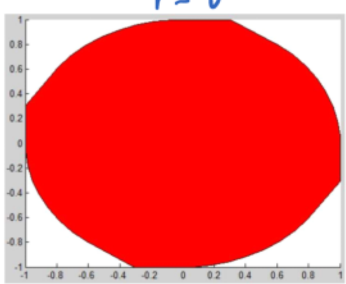
$r=1$



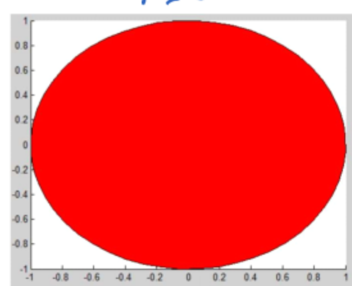
$r=4$



$r=8$



$r=\infty$



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Detecting termination

Lemma. Let $S_r := \bigcap_{k=0}^{r} \{x \mid AG^k x \leq b\}$.

Then, $S_r = S_{r+1} \Rightarrow S_r = S_\infty$.

Proof. $x \in S_r \Rightarrow x \in S_{r+1} \Rightarrow Gx \in S_r$, Repeat.

Note. The condition $S_r = S_{r+1}$ can be efficiently checked.

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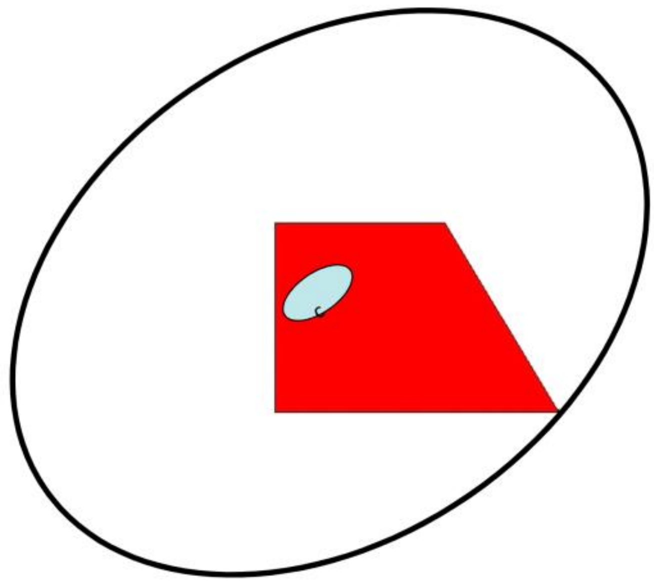
Solving R-LD-LP exactly via LP

Theorem. If $\rho(G) < 1$, then convergence is finite. Moreover, the number of steps needed is polynomial in the size of the input $(A, b, c\ , G)$.

(Recall: $\rho(A) = \lim_{k \to \infty} \|A^k\|^{\frac{1}{k}}$)

Proof idea.



Invariant ellipsoid: $\{x^T P x \leq 1\}$

Upper bound on the number of iterations

- Find an invariant ellipsoid defined by a positive definite matrix P
- Find a shrinkage factor $\gamma \in (0, 1)$; i.e., a scalar satisfying $G^T P G \preceq \gamma P$
- Find a scalar $\alpha_2 > 0$ such that

$$\{Ax \leq b\} \subseteq \{x^T P x \leq \alpha_2\}$$

- Find a scalar $\alpha_1 > 0$ such that

$$\{x^T P x \leq \alpha_1\} \subseteq \{Ax \leq b\}$$

- Let

$$r = \lceil \frac{\log \frac{\alpha_1}{\alpha_2}}{\log \gamma} \rceil$$

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Finding an invariant ellipsoid

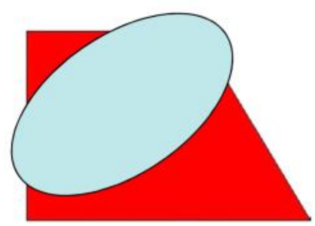
- **Computation of P.**

To find an invariant ellipsoid for G, we solve the linear system

$$G^T P G - P = -I,$$

where I is the $n \times n$ identity matrix. This is called the Lyapunov equation.

The matrix P will automatically turn out to be positive definite.



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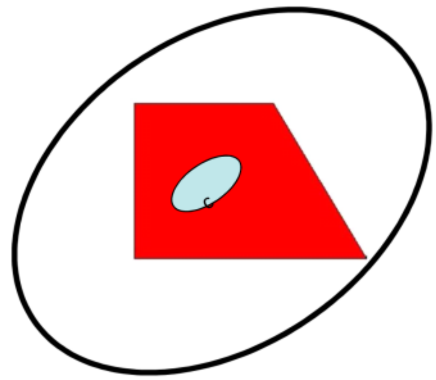
Finding the shrinkage factor

- **Computation of γ.**

$$\gamma = 1 - \frac{1}{\max_i\{P_{ii} + \sum_{j\neq i} |P_{i,j}|\}}.$$



Proof idea.

$$\begin{aligned} x^T G^T P G x &= x^T P x - x^T x \\ &\leq x^T P x (1-\eta) \end{aligned}$$

where η is any number such that

$$\eta x^T P x \leq x^T x$$

Shrinkage is at least $1 - \frac{1}{\lambda_{max}(P)}$

$$\lambda_{max}(P) \leq \max_i\{P_{ii} + \sum_{j\neq i} |P_{i,j}|\}.$$

(Bound from Greshgorin's circle theorem)

Finding the outer ellipsoid

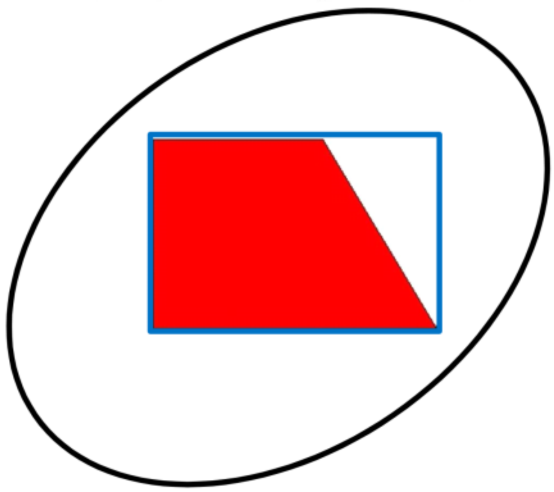
- **Computation of α_2.** By solving, e.g., n LPs, we can place our polytope $\{Ax \leq b\}$ in a box; i.e., compute $2n$ scalars l_i, u_i such that

$$\{Ax \leq b\} \subseteq \{l_i \leq x_i \leq u_i\}.$$

We then bound $x^T Px = \sum_{i,j} P_{i,j} x_i x_j$ term by term to get α_2:

$$\alpha_2 = \sum_{i,j} \max\{P_{i,j} u_i u_j, P_{i,j} l_i l_j, P_{i,j} u_i l_j, P_{i,j} l_i u_j\}.$$

This ensures that $\{l_i \leq x_i \leq u_i\} \subseteq \{x^T Px \leq \alpha_2\}$. Hence, $\{Ax \leq b\} \subseteq \{x^T Px \leq \alpha_2\}$.



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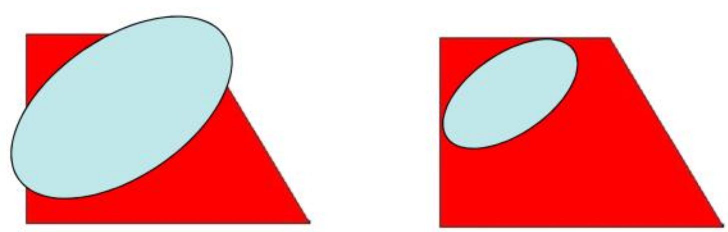
Finding the inner ellipsoid

- **Computation of α_1.** For $i = 1, \ldots, m$, we compute a scalar η_i by solving the convex program
$$\eta_i := \min_x \{a_i^T x : x^T P x \leq 1\},$$
where a_i is the i-th row of the constraint matrix A. This problem has a closed form solution:
$$\eta_i = -\sqrt{a_i^T P^{-1} a_i}.$$
Note that P^{-1} exists since $P \succ 0$. We then let
$$\alpha_1 = \min_i \{\frac{b_i^2}{\eta_i^2}\}.$$



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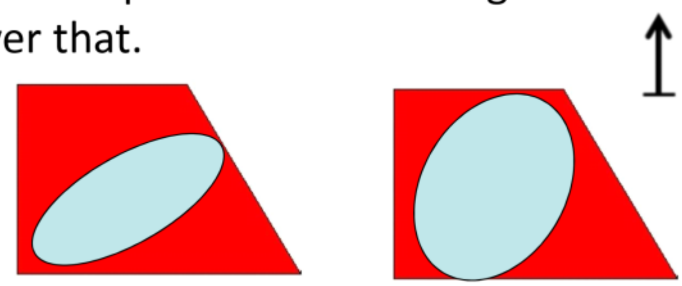
Upper bounds on R-LD-LP via SDP

- Goal: Find the best invariant ellipsoid inside the original polytope and optimize over that.



$$\min_{x,P} \quad c^T x$$

$$P \succ 0$$

$$G^T P G \preceq P$$

$$x^T P x \leq 1$$

$$\left[\forall z, \; z^T P z \leq 1 \Rightarrow Az \leq b\right] \xleftrightarrow{\text{S-Lemma}} b_i - a_i^T z \geq \gamma_i \left(1 - z^T P z\right) \quad \forall z$$

$$\gamma_i \geq 0$$

Non-convex formulation

Upper bounds on R-LD-LP via SDP

- If we parameterize in terms of P^{-1} instead, then it becomes convex!

$$\min_{x,P} \quad c^T x$$

$$P \succ 0$$

$$G^T P G \preceq P$$

$$x^T P x \leq 1$$

Dynamics Duality ⟷

Schur Complement ⟷

$$\min_{x,Q} \quad c^T x$$

$$Q \succ 0$$

$$G Q G^T \preceq Q$$

$$\begin{bmatrix} Q & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

$$a_i^T Q a_i \leq 1$$

$$\left[\forall z, \quad z^T P z \leq 1 \Rightarrow Az \leq b\right]$$

ellipsoid ⊆ polytope ⟷ (Polar duality) ⟷ $(\text{polytope})^* \subseteq (\text{ellipsoid})^*$

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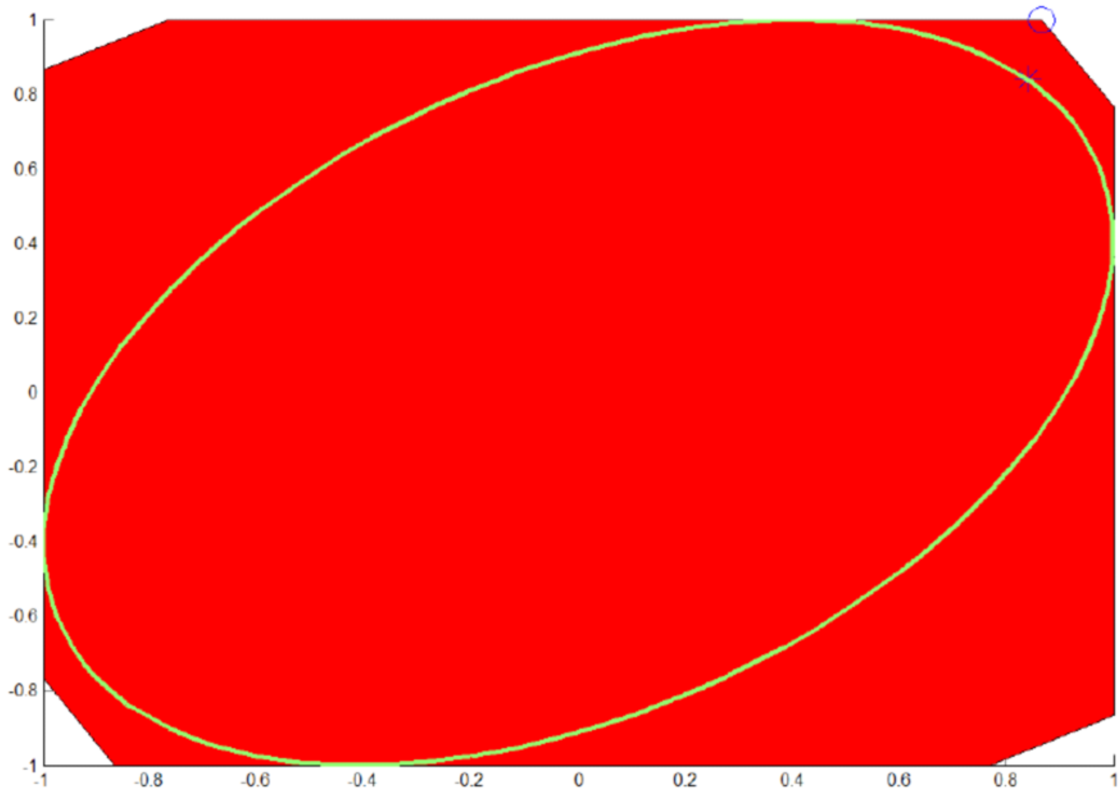
Upper bounds on R-LD-LP via SDP



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LP
+
Uncertain & time-varying linear systems

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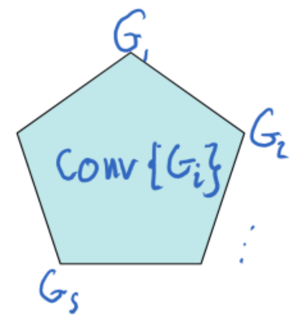
R-ULD-LP

Robust to uncertain linear dynamics linear programming (R-ULD-LP)

$$x_{k+1} \in \operatorname{conv}\{G_1, \dots, G_s\}\, x_k$$



Models **uncertainty** and **variations with time** in the dynamics

$$\min_x \{c^T x : AGx \le b, \forall G \in \mathbb{G}^*\}$$ (An infinite LP)

$\mathbb{G}^*$: set of all finite products of $G_1, \dots, G_s$

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Input data: $A, b, c, G_1, \dots, G_s$

Feasible set of R-ULD-LP

It's still convex, closed, and invariant (but typically much more nasty).

Theorem. If the *joint spectral radius* of $G_1, \ldots, G_s$ is less than one, then the feasible set is a polytope.

Joint spectral radius (JSR):

$$\rho(G_1, \ldots, G_s) = \lim_{k\to\infty} \max_{\sigma\in\{1,\ldots,s\}^m} ||G_{\sigma_1} \cdots G_{\sigma_k}||^{\frac{1}{k}}$$

- Unlike the spectral radius, computation of the JSR is difficult (Testing if JSR<=1is undecidable already for two 47x47 matrices [Blondel, Tsitsiklis], [Blondel, Canterini])

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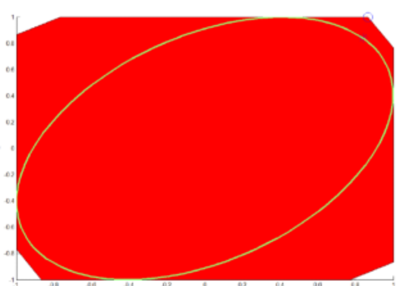
Lower and upper bounds for R-ULD-LP

- To get **lower bounds**, truncate the sequence and solve an LP. For example,

$$\min_x \{c^T x : Ax \le b, AG_1 x \le b, AG_1 G_2 x \le b, \dots, AG_1 G_2 G_1 x \le b\}$$

- What about **upper bounds?**



Invariant ellipsoid may not exist even when JSR<1

$$\min_{x,Q} \; c^T x$$

$$Q \succ 0$$

$$G Q G^T \preceq Q$$

$$\begin{bmatrix} Q & x \\ x^T & 1 \end{bmatrix} \succeq 0$$

$$a_i^T Q a_i \le 1$$

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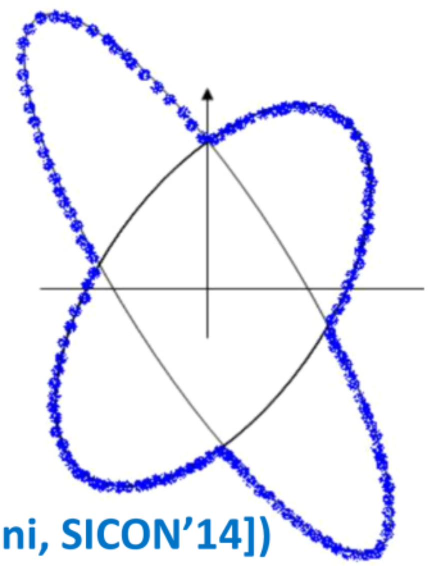
Idea: search instead for union of ellipsoids



Theorem ([AAA, Jungers, Parrilo, Roozbehani, SICON'14])

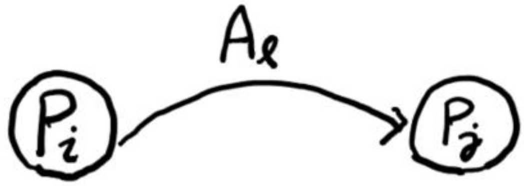
If JSR<1,

then there exists an invariant set which is a union of *k* ellipsoids.

Moreover,

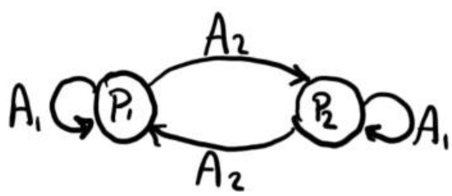
- We give you a bound on *k*.
- We tell you how to find the *k* ellipsoids by SDP.

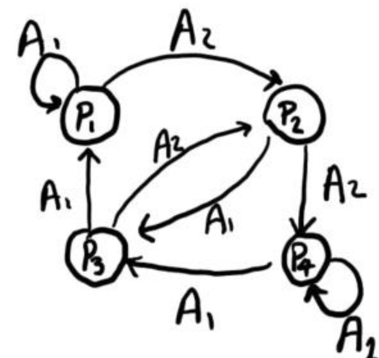
The SDPs have a recipe!



$$\Longleftrightarrow A_\ell^T P_j A_\ell \preceq P_i$$

De Bruijn graphs





A numerical example

$$\{Ax \leq b\} = \square_{(-1,-1)}^{(1,1)}, \quad c = \begin{pmatrix}1\\2\end{pmatrix}, \quad G_1 = 0.254\begin{bmatrix}-1 & -1\\-4 & 0\end{bmatrix}, \quad G_2 = 0.254\begin{bmatrix}3 & 3\\-2 & 1\end{bmatrix}$$

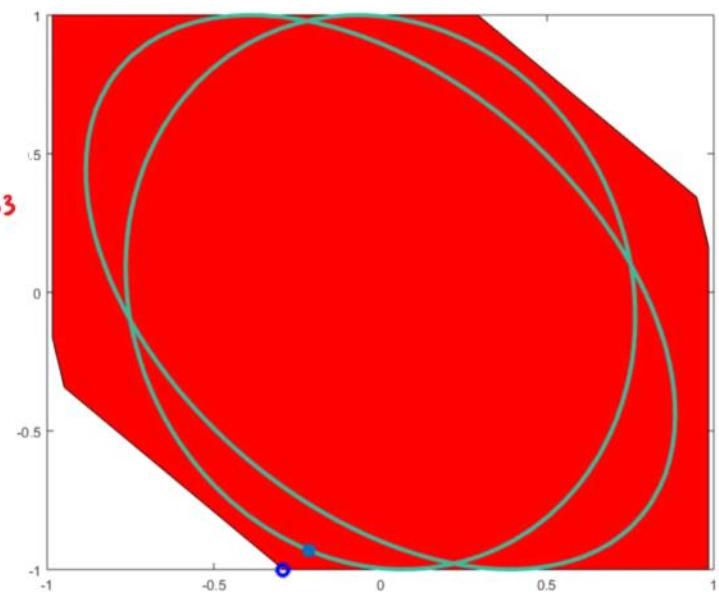
Red (LP)

$\{x \mid AG_\sigma x \leq b\}$

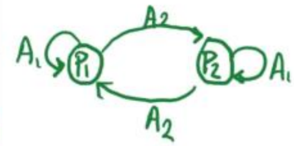
↓

Products of length 0,1,2,3

(15 total)



Green (SDP)



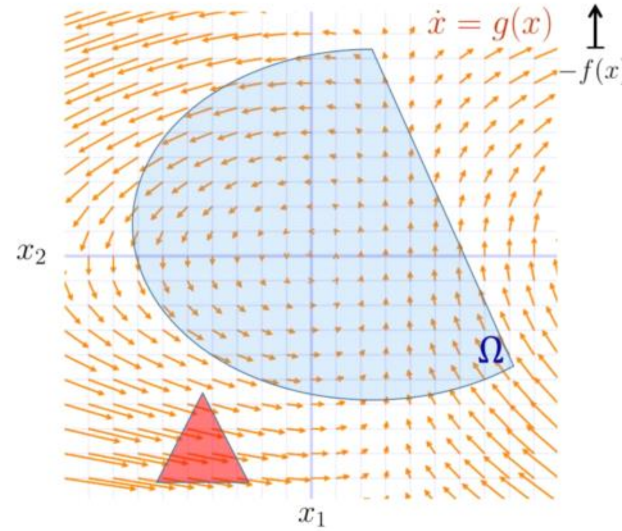
Union of ellipses invariant

$$-2.292 \leq c^T x^* \leq -2.076$$

The broader perspective

Optimization problems with dynamical systems (DS) constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \Omega \cap \Omega_{DS}. \end{array}$$



Optimization Problem "f, Ω"	**Type of Dynamical System "g"**	**DS Constraint "Ω_{DS}"**
Linear program*	Linear*	Invariance*
Convex quadratic program*	Linear and uncertain/stochastic	Inclusion in region of attraction
Semidefinite program	Linear and time-varying*	Collision avoidance
Robust linear program	Nonlinear (polynomial)	Reachability
Polynomial program	Nonlinear and time-varying	Orbital stability
Integer program	Discrete/continuous/hybrid of both	Stochastic stability
⋮	⋮	⋮