Time-Varying SDPs

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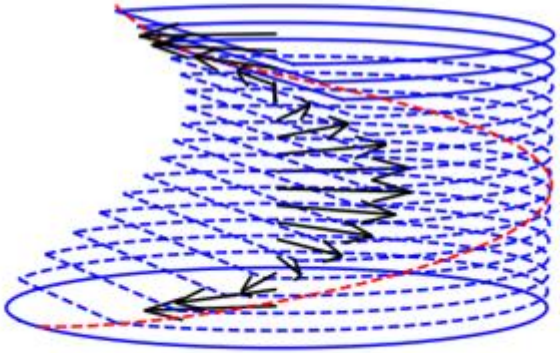

Joint work with Amir Ali Ahmadi

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Toy example

Problem:

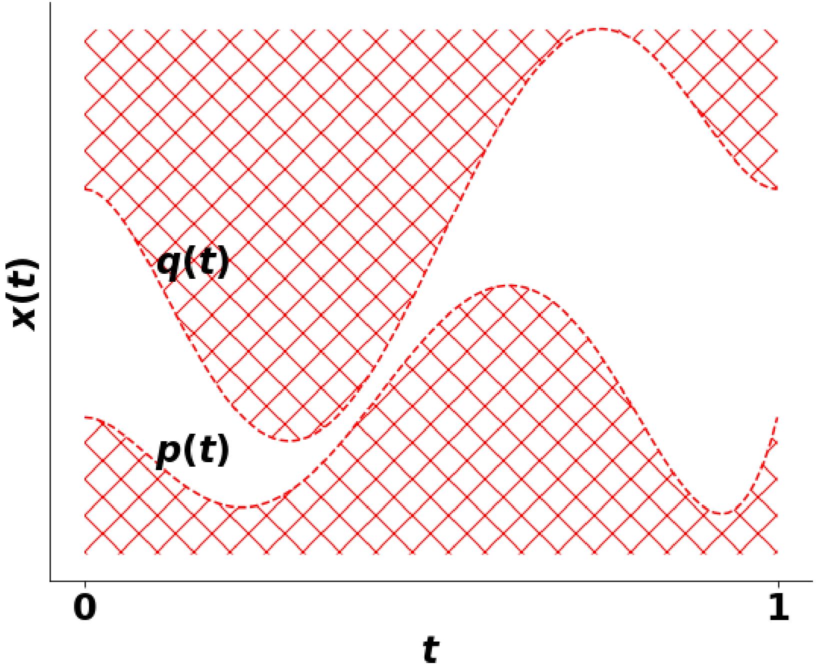
Find a path $x : [0,1] \to \mathbb{R}$

with minimum length

$$\int_0^1 \sqrt{1 + x'(t)^2}\, \mathrm{d}t$$

Such that

$$\left.\begin{aligned} p(t) \le x(t) \le q(t) \\ -\gamma \le x''(t) \le \gamma \end{aligned}\right\} \forall t \in [0,1]$$



Framework

SDP: Find $x \in \mathbb{R}^n$ that minimizes $\langle c, x \rangle$ such that

$$A_0 + \sum A_i x_i \succeq 0 \qquad A_i \in \mathbb{R}^{m \times m}$$

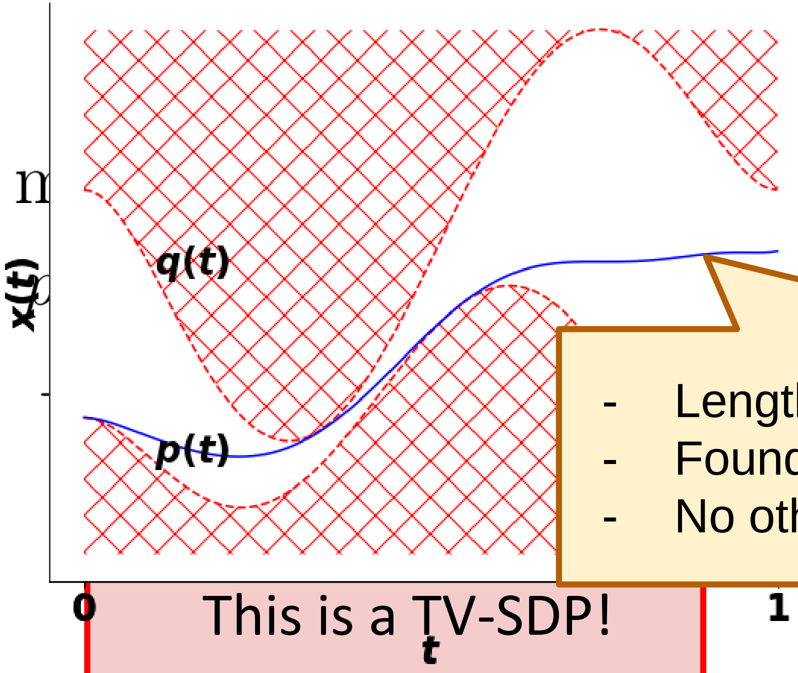
TV-SDP: Find $x : [0,1] \to \mathbb{R}^n$ that minimizes $\int_0^1 \langle c(t), x(t) \rangle dt$
such that $\forall t \in [0,1]$

$$Fx(t) := A_0(t) + \sum A_i(t) x_i(t) + \sum \int_0^t D_i(t,s) x_i(s) \mathrm{d}s \succeq 0$$

F is linear operator that takes a vector-valued function $x(t)$ and outputs a matrix-valued function $Fx(t)$

Data: c, A_i , D_i polynomials.
Polynomials are general enough.

Toy example - continued



$$\min \int_0^1 y(t)\, dt$$

$$p(t) \leq x(t) \leq q(t)$$

$$\forall t \in [0, 1]$$

$$\begin{pmatrix} y(t) & 1 & x'(t) \\ 1 & y(t) & 0 \\ x'(t) & 0 & y(t) \end{pmatrix} \succeq 0$$

- Length: 1.64
- Found via a small SDP
- No other curve has length < 1.63

This is a TV-SDP!

- How to get good solutions?
- How to get lower bounds?

Plan

Primal approach: Getting upper bounds.

Any feasible solution will do!

Focus on polynomials:
- Smooth
- Tractable

?

Optimal value of TV-SDP

?

When are polynomial solutions optimal?

When are these lower bounds tight?

Dual approach: Getting lower bounds.

Finite dimensional outer-approxmiation!

Primal approach

Our Constraint : $Fx(t) \succeq 0 \quad \forall t \in [0,1]$

Idea: Restrict the search space to polynomials of a given degree.

$$x(t) \in \mathbb{R}^n_{\hat{d}}[t] \implies Fx(t) \in \mathbb{R}^{m\times m}_d[t]$$

SOS polynomials:

$p(x_1, \ldots, x_n)$ SOS if $p(x) = \sum q_i(x)^2$

- SOS implies nonnegative.
- Search over SOS is an SDP.

Positivstellensatz [Dette&Studden 02]

For $X(t) \in \mathbb{R}^{m\times m}_d[t] \quad X(t) \succeq 0 \; \forall t \in [0,1]$,

if and only if

$$y^T X(t) y = p_0(t,y) + t p_1(t,y) + (1-t)p_2(t,y) + t(1-t)p_3(t,y)$$

with $p_0, \ldots, p_3$ SOS of deg. 2 in y and d in t.

LP case studied in [Bampou&Kuhn 12]

Takeaway: (Small) SDP can find the best polynomial solution to a TV-SDP

Optimality of polynomial solutions (1/2)

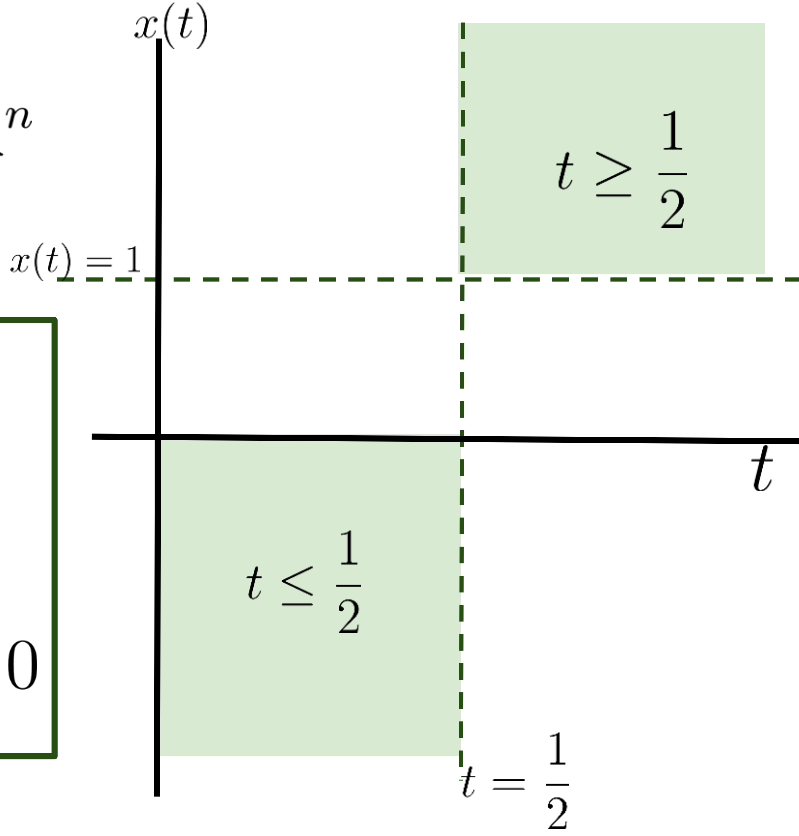
Strategy:

- Take any feasible solution $x : [0,1] \to \mathbb{R}^n$
- Approximate it with a polynomial

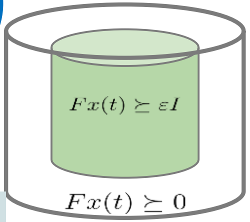
What can go wrong?

A "discontinuous" TV-SDP

$$(t - \frac{1}{2})x(t) \geq 0, (t - \frac{1}{2})(x(t) - 1) \geq 0$$



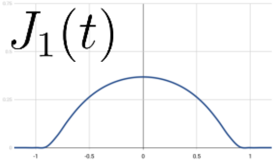
Optimality of polynomial solutions (2/2)



Definition (Strict feasiblity) $\exists \varepsilon > 0 \;\; \exists x : [0,1] \to \mathbb{R}^n \;\; Fx(t) \succeq \varepsilon I$

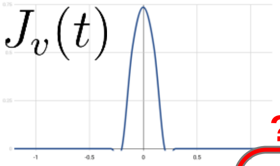
Mollification | Continuous func. | Weierstrass

$x(t)$ strictly feasible $\circledast$



$\Longrightarrow \mathcal{M}_1 x(t)$

⋮



$\Longrightarrow \mathcal{M}_v x(t)$ Strictly Feasible! ☺

Uniform approximation $\Longrightarrow$ **Polynomial Solution** ☺

?

1) $Fx(t) \succeq 0 \implies \mathcal{M}_v Fx(t) \succeq 0$

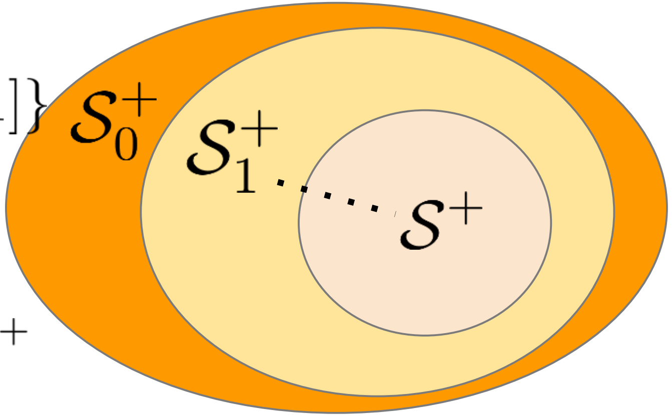
2) $\sup_t \|\mathcal{M}_v Fx(t) - F\mathcal{M}_v x(t)\| \underset{v\to\infty}{\to} 0$

Theorem (Ahmadi, BEK)

Strict feasibility $\implies$ Optimality of polynomials

Dual approach (1/3)

Our constraint $Fx \in \mathcal{S}^+ := \{X \mid X(t) \succeq 0 \ \forall t \in [0,1]\}$



Inner product $\langle X, Y \rangle := \int_0^1 Tr(X(t)Y(t)) \, \mathrm{d}t$

$\mathcal{S}^+$ is self dual $X \in \mathcal{S}^+ \iff \langle X, Y \rangle \geq 0 \ \forall Y \in \mathcal{S}^+$

Idea:

Restrict Y to be a polynomial of degree $\leq d$

$$\mathcal{S}_d^+ := \{X \mid \langle X, Y \rangle \geq 0 \ \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t]\}$$

Dual approach (2/3)

$$\mathcal{S}^+ := \{X \mid \langle X, Y\rangle \geq 0 \; \forall Y \in \mathcal{S}^+\}$$ Initial feasible set

$$\mathcal{S}_d^+ := \{X \mid \langle X, Y\rangle \geq 0 \; \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m\times m}[t]\}$$ Outer-approximation

$$Fx \in \mathcal{S}_d^+ \iff \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m\times m}[t] \qquad \langle Fx, Y\rangle \geq 0$$

$$\langle x, F^*Y\rangle \geq 0$$

$$= \int_0^1 \langle x(t), F^*Y(t)\rangle \mathrm{d}t$$

Depends only on the first moments of x

$$\iff x \in \text{dual of } F^*(\mathcal{S}^+ \cap \mathbb{R}_d^{m\times m}[t])$$

Semidefinite representable

Takeaway: Lower bound at level d can be obtained via SDP

Dual Approach (3/3)

Level	d	$\Longrightarrow$	∞
Optimal value	v^d	$\overset{?}{\Longrightarrow}$	v^∞
Optimal solution	x^d	$\overset{?}{\Longrightarrow}$	x^∞ ☺
Constraint	$\langle Fx, Y\rangle \geq 0$ $\forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m\times m}[t]$	$\Longrightarrow$	$\langle Fx, Y\rangle \geq 0$ $\forall Y \in S^+$ $\Longleftrightarrow Fx(t) \succeq 0$

Weak convergence

$$\langle x^d, \cdot\rangle \to \langle x^\infty, \cdot\rangle$$

Boundedness

$$\forall p \in \mathbb{R}_d^n[t] \qquad p(t) \geq 0$$

$$|\langle x^d, p\rangle| \leq M\langle 1, p\rangle$$

Theorem (Ahmadi, BEK)

Boundedness of TV-SDP $\Longrightarrow$ Lower bounds tight

Compactness [Lasserre]

A bounded sequence has a weakly convergent subsequence

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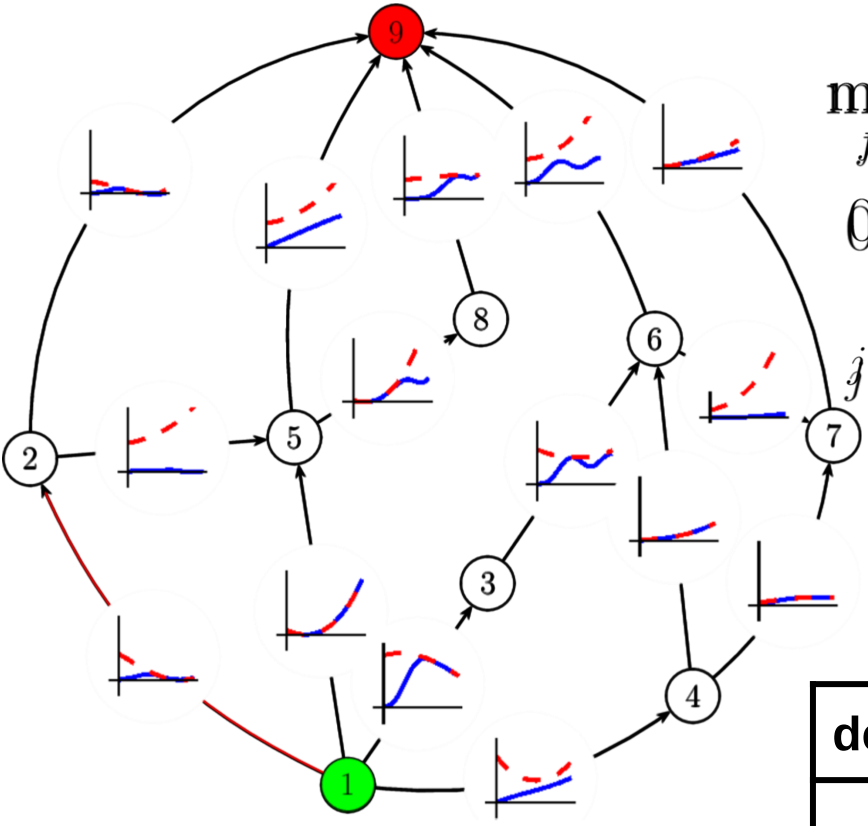
TV-maxflow



$$\max_{f_{ij}} \int_0^1 \sum_{(1,j)\in E} f_{1j}(t)\,dt$$

$$0 \le f_{ij}(t) \le b_{ij}(t) \qquad \forall (i,j)$$

$$\sum_{j:(i,j)\in E} f_{ij}(t) - \sum_{j:(j,i)\in E} f_{ji}(t) = 0 \quad \forall i \neq 1, 9$$

$$|f'_{14}(t)| \le \frac{1}{2} \qquad |f'_{59}(t)| \le \frac{1}{2}$$

$$\int_0^t \sum_{j=1}^n f_{1j}(t) \le t^2$$

Obtained from dual problem

degree	2	3	...	10	∞
	0.61	0.81		0.90	< 0.91

Pareto Curve approximation

$$\min \quad \langle c_1, x\rangle \text{ and } \langle c_2, x\rangle \quad \text{s.t.} \quad x \in \mathcal{C}$$

[Gorissen, Hertog '12]

[Magron, Henrion, Lasserre '14]

Pareto Curve:

$$\{(\langle c_1, x(t)\rangle, \langle c_2, x(t)\rangle) \mid t \in [0,1]\}$$

$$x(t) := \arg\min \quad \langle c_1, x(t)\rangle$$

$$\langle c_2, x(t)\rangle \leq t \quad x(t) \in \mathcal{C}$$

Idea:

$$\min \quad \int_0^1 \langle c_1, x(t)\rangle \mathrm{d}t$$

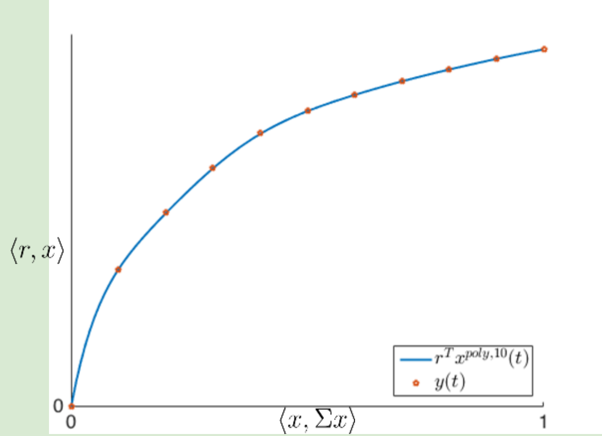
$$\langle c_2, x(t)\rangle \leq t \quad x(t) \in \mathcal{C}$$

Takeaway: This is a TV-SDP!

Markowitz Portfolio Theory

$$\max \ \langle r, x\rangle \quad \min \langle x, \Sigma x\rangle$$

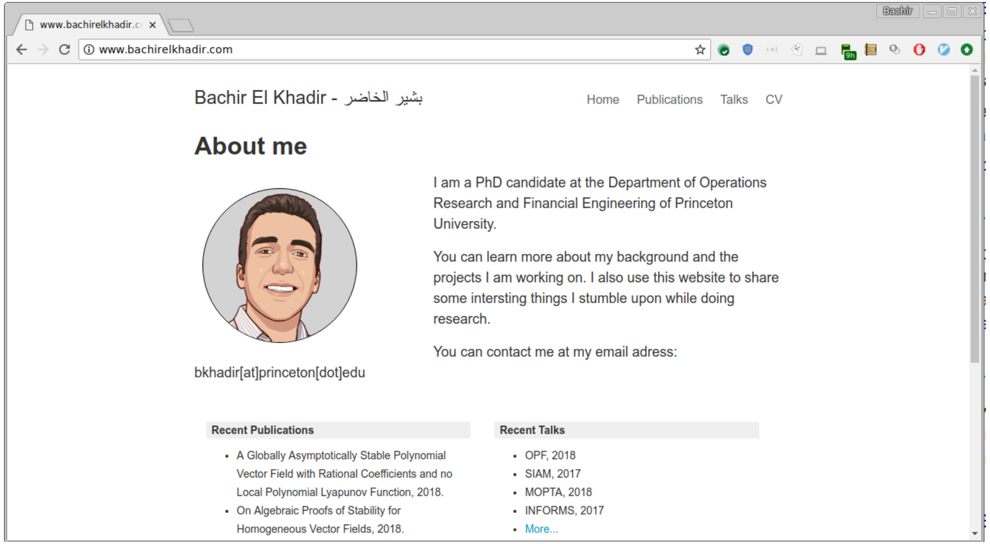
$$\sum x_i = 1 \qquad x \geq 0$$



Thanks!

Want to know more?

bachirelkhadir.com



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