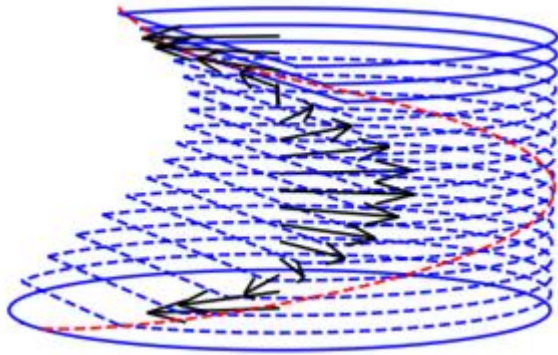


# Time-Varying SDPs

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Joint work with Amir Ali Ahmadi



# Toy example

## Problem:

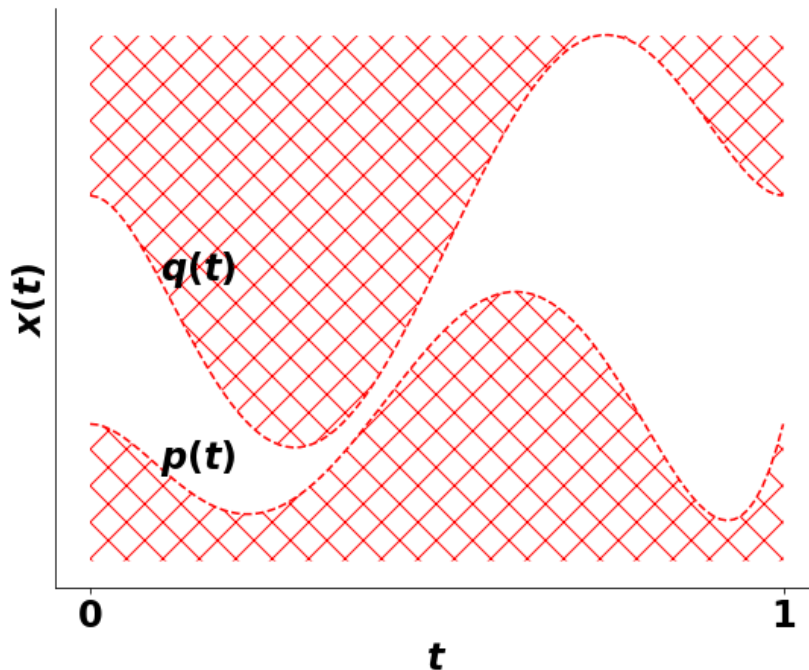
Find a path  $x : [0, 1] \rightarrow \mathbb{R}$

with minimum length

$$\int_0^1 \sqrt{1 + x'(t)^2} dt$$

Such that

$$\left. \begin{array}{l} p(t) \leq x(t) \leq q(t) \\ -\gamma \leq x''(t) \leq \gamma \end{array} \right\} \forall t \in [0, 1]$$



# Framework

**SDP:** Find  $x \in \mathbb{R}^n$  that minimizes  $\langle c, x \rangle$  such that

$$A_0 + \sum A_i x_i \succeq 0 \quad A_i \in \mathbb{R}^{m \times m}$$

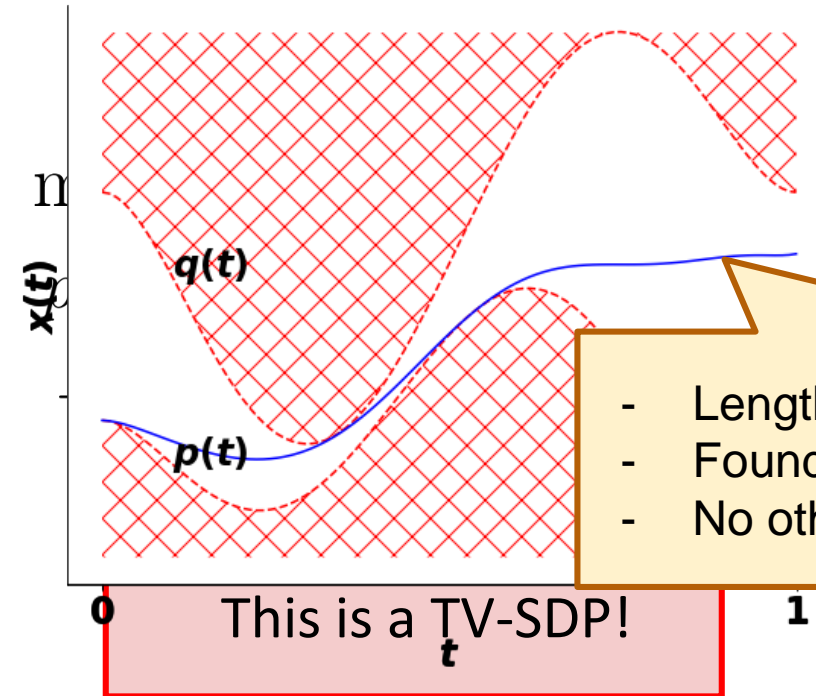
**TV-SDP:** Find  $x : [0, 1] \rightarrow \mathbb{R}^n$  that minimizes  $\int_0^1 \langle c(t), x(t) \rangle dt$   
such that  $\forall t \in [0, 1]$

$$Fx(t) := A_0(t) + \sum A_i(t)x_i(t) - \sum_0^t D_i(t,s)x_i(s)ds \succeq 0$$

$F$  is linear operator that takes a vector-valued function  $x(t)$  and outputs a matrix-valued function  $Fx(t)$

Data:  $c, A_i, D_i$  polynomials.  
Polynomials are general enough.

# Toy example - continued



$$\min \int_0^1 y(t) dt$$

$$p(t) \leq x(t) \leq q(t)$$

$\forall t \in [0, 1]$

- Length: 1.64
- Found via a small SDP
- No other curve has length  $< 1.63$

$$\begin{pmatrix} y(t) & 1 & x'(t) \\ 1 & y(t) & 0 \\ x'(t) & 0 & y(t) \end{pmatrix} \succeq 0$$

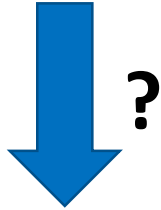
- How to get good solutions?
- How to get lower bounds?

# Plan

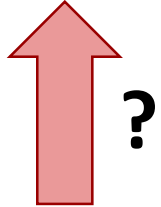
**Primal approach:** Getting upper bounds.  
Any feasible solution will do!

Focus on polynomials:

- Smooth
- Tractable



**Optimal value of TV-SDP**



**When are polynomial solutions optimal?**

**When are these lower bounds tight?**

**Dual approach:** Getting lower bounds.  
Finite dimensional outer-approximation!

# Primal approach

**Our Constraint :**  $Fx(t) \succeq 0 \quad \forall t \in [0, 1]$

**Idea:** Restrict the search space to polynomials of a given degree.

$$x(t) \in \mathbb{R}_d^n[t] \implies Fx(t) \in \mathbb{R}_d^{m \times m}[t]$$

## SOS polynomials:

$$p(x_1, \dots, x_n) \text{ SOS if } p(x) = \sum q_i(x)^2$$

- SOS implies nonnegative.
- Search over SOS is an SDP.

## Positivstellensatz [Dette&Studden 02]

For  $X(t) \in \mathbb{R}_d^{m \times m}[t] \quad X(t) \succeq 0 \quad \forall t \in [0, 1]$   
if and only if

$$y^T X(t) y = p_0(t, y) + tp_1(t, y) \\ + (1 - t)p_2(t, y) + t(1 - t)p_3(t, y)$$

with  $p_0, \dots, p_3$  SOS of deg. 2 in  $y$  and  $d$  in  $t$ .

LP case studied in [Bampou&Kuhn 12]

**Takeaway:** (Small) SDP can find the best polynomial solution to a TV-SDP

# Optimality of polynomial solutions (1/2)

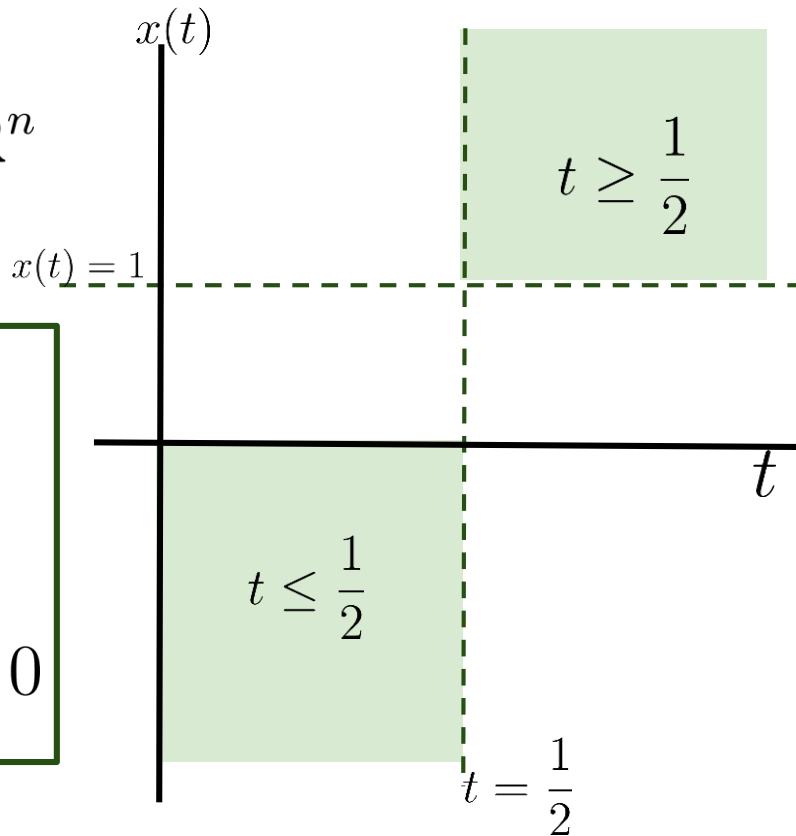
## Strategy:

- Take any feasible solution  $x : [0, 1] \rightarrow \mathbb{R}^n$
- Approximate it with a polynomial

## What can go wrong?

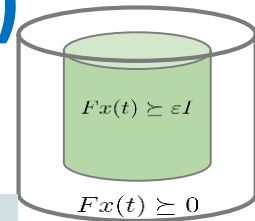
A “discontinuous” TV-SDP

$$\left(t - \frac{1}{2}\right)x(t) \geq 0, \left(t - \frac{1}{2}\right)(x(t) - 1) \geq 0$$



# Optimality of polynomial solutions (2/2)

**Definition (Strict feasibility)**  $\exists \varepsilon > 0 \exists x : [0, 1] \rightarrow \mathbb{R}^n \ Fx(t) \succeq \varepsilon I$



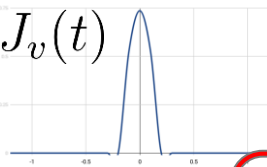
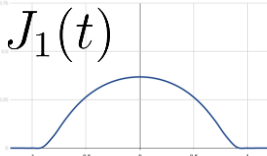
Mollification

Continuous func.

Weierstrass

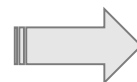
$x(t)$   
strictly  
feasible

(\*)



$$\mathcal{M}_1 x(t)$$

⋮



$$\mathcal{M}_v x(t)$$

Strictly  
Feasible! 😊

Uniform  
approximation

**Polynomial  
Solution**



$$1) Fx(t) \succeq 0 \implies \mathcal{M}_v Fx(t) \succeq 0$$

$$2) \sup_t \|\mathcal{M}_v Fx(t) - F\mathcal{M}_v x(t)\| \xrightarrow{v \rightarrow \infty} 0$$

**Theorem (Ahmadi, BEK)**

Strict feasibility  $\implies$  Optimality of polynomials

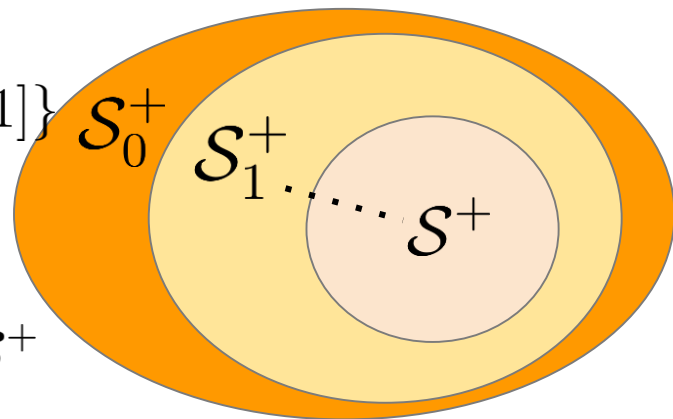


# Dual approach (1/3)

Our constraint  $Fx \in \mathcal{S}^+ := \{X \mid X(t) \succeq 0 \forall t \in [0, 1]\}$

Inner product  $\langle X, Y \rangle := \int_0^1 \text{Tr}(X(t)Y(t)) dt$

$\mathcal{S}^+$  is self dual  $X \in \mathcal{S}^+ \iff \langle X, Y \rangle \geq 0 \forall Y \in \mathcal{S}^+$



## Idea:

Restrict  $Y$  to be a polynomial of degree  $\leq d$

$$\mathcal{S}_d^+ := \{X \mid \langle X, Y \rangle \geq 0 \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t]\}$$

# Dual approach (2/3)

$\mathcal{S}^+ := \{X \mid \langle X, Y \rangle \geq 0 \forall Y \in \mathcal{S}^+\}$  Initial feasible set

$\mathcal{S}_d^+ := \{X \mid \langle X, Y \rangle \geq 0 \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t]\}$  Outer-approximation

$$Fx \in \mathcal{S}_d^+ \iff \forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t] \quad \langle Fx, Y \rangle \geq 0$$

$$\langle x, F^*Y \rangle \geq 0$$

$$= \int_0^1 \langle x(t), F^*Y(t) \rangle dt$$

Depends only on the first moments of  $x$

$$\iff x \in \text{dual of } F^*(\mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t])$$

Semidefinite representable

**Takeaway:** Lower bound at level  $d$  can be obtained via SDP

# Dual Approach (3/3)

Level	$d$	$\longrightarrow$	$\infty$
Optimal value	$v^d$	$\overset{?}{\longrightarrow}$	$v^\infty$
Optimal solution	$x^d$	$\overset{?}{\longrightarrow}$	$x^\infty$ 😊
Constraint	$\langle Fx, Y \rangle \geq 0$ $\forall Y \in \mathcal{S}^+ \cap \mathbb{R}_d^{m \times m}[t]$	$\longrightarrow$	$\langle Fx, Y \rangle \geq 0 \iff Fx(t) \succeq 0$ $\forall Y \in \mathcal{S}^+$

## Weak convergence

$$\langle x^d, \cdot \rangle \rightarrow \langle x^\infty, \cdot \rangle$$

## Boundedness

$$\forall p \in \mathbb{R}_d^n[t] \quad p(t) \geq 0$$

$$|\langle x^d, p \rangle| \leq M \langle 1, p \rangle$$

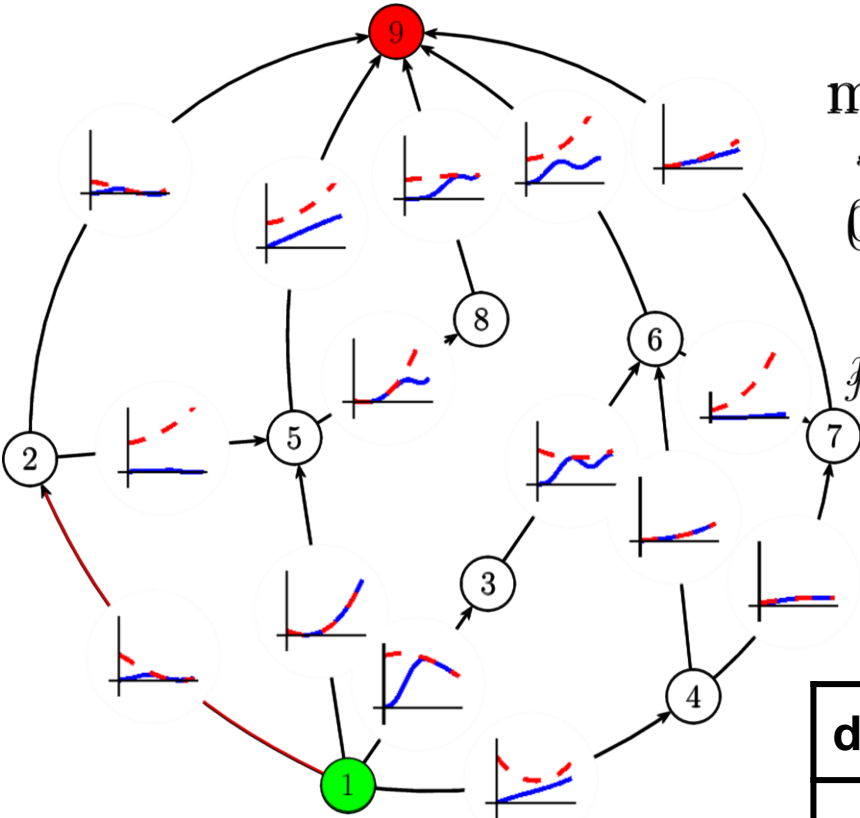
## Theorem (Ahmadi, BEK)

Boundedness of TV-SDP  $\implies$  Lower bounds tight

## Compactness [Lasserre]

A bounded sequence has a weakly convergent subsequence

# TV-maxflow



$$\max_{f_{ij}} \int_0^1 \sum_{(1,j) \in E} f_{1j} f_{1j}(t) dt$$

$$0 \leq f_{ij}(t) \leq b_{ij}(t) \quad \forall (i,j) \in E$$

$$\sum_{j:(i,j) \in E} f_{ij}(t) - \sum_{j:(j,i) \in E} f_{ji}(t) = 0 \quad \forall i \neq 1, 9$$

$$|f'_{14}(t)| \leq \frac{1}{2} \quad |f'_{59}(t)| \leq \frac{1}{2}$$

$$\int_0^t \sum_{j=1}^n f_{1j}(t) \leq t^2$$

Obtained from dual problem

<b>degree</b>	2	3	...	10	$\infty$
	0.61	0.81		0.90	< 0.91

# Pareto Curve approximation

$$\min \langle c_1, x \rangle \text{ and } \langle c_2, x \rangle \text{ s.t. } x \in \mathcal{C}$$

## Pareto Curve:

$$\{(\langle c_1, x(t) \rangle, \langle c_2, x(t) \rangle) \mid t \in [0, 1]\}$$
$$x(t) := \arg \min_{\langle c_2, x(t) \rangle \leq t} \langle c_1, x(t) \rangle \quad x(t) \in \mathcal{C}$$

**Idea:**

$$\min \int_0^1 \langle c_1, x(t) \rangle dt$$
$$\langle c_2, x(t) \rangle \leq t \quad x(t) \in \mathcal{C}$$

**Takeaway:** This is a TV-SDP!

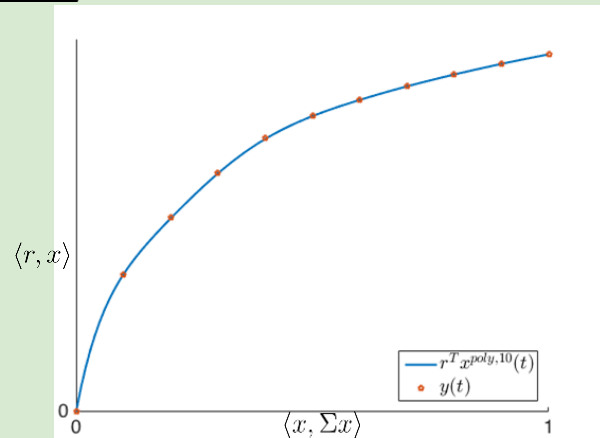
[Gorissen, Hertog '12]

[Magron, Henrion, Lasserre '14]

## Markowitz Portfolio Theory

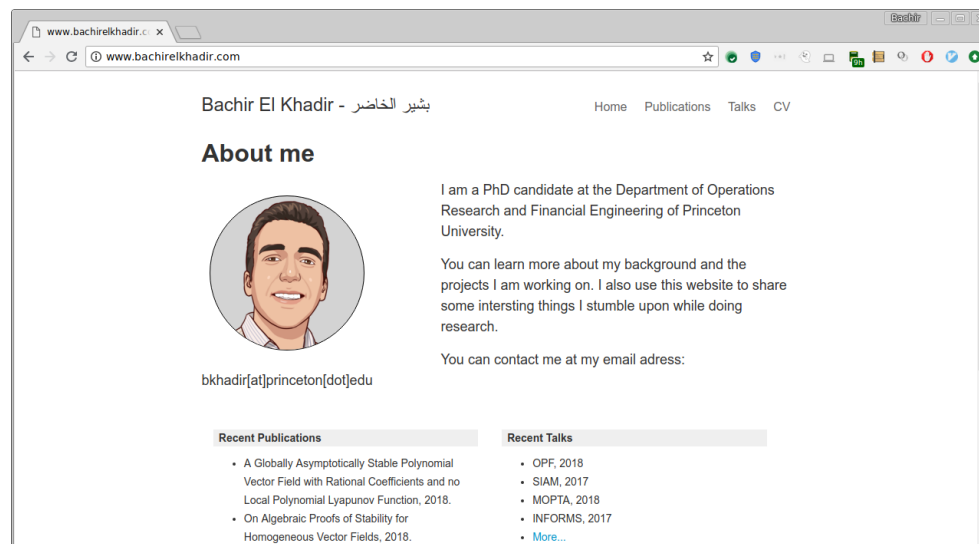
$$\max \langle r, x \rangle \quad \min \langle x, \Sigma x \rangle$$

$$\sum x_i = 1 \quad x \geq 0$$



# Thanks!

## Want to know more? [bachirelkhadir.com](http://bachirelkhadir.com)



The screenshot shows a web browser displaying the homepage of Bachir El Khadir. The browser's address bar shows the URL [www.bachirelkhadir.com](http://www.bachirelkhadir.com). The page content includes a navigation menu with links for Home, Publications, Talks, and CV. The main heading is "Bachir El Khadir - بشير الخاضر". Below this is an "About me" section featuring a circular portrait of Bachir El Khadir. To the right of the portrait, there is a paragraph of text: "I am a PhD candidate at the Department of Operations Research and Financial Engineering of Princeton University. You can learn more about my background and the projects I am working on. I also use this website to share some interesting things I stumble upon while doing research. You can contact me at my email address:". Below the portrait, the email address `bkhadir[at]princeton[dot]edu` is displayed. At the bottom of the page, there are two columns: "Recent Publications" and "Recent Talks". The "Recent Publications" column lists three items: "A Globally Asymptotically Stable Polynomial Vector Field with Rational Coefficients and no Local Polynomial Lyapunov Function, 2018.", "On Algebraic Proofs of Stability for Homogeneous Vector Fields, 2018.", and "OPF, 2018". The "Recent Talks" column lists three items: "SIAM, 2017", "MOPTA, 2018", and "INFORMS, 2017". A "More..." link is also present under the "Recent Talks" section.