A Working Knowledge of Computational Complexity for an Optimizer

ORF 363/COS 323

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Why computational complexity?

- **What is computational complexity theory?**

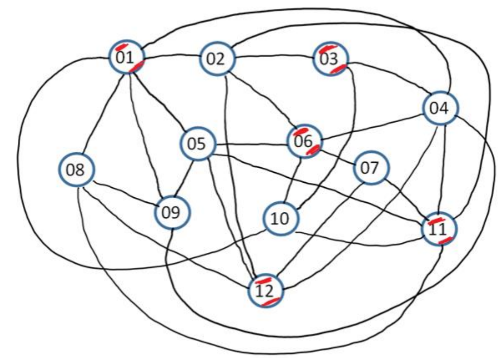
  It's a branch of mathematics that provides a formal framework for studying how efficiently one can solve problems on a computer.

- This is absolutely crucial to optimization and many other computational sciences.
  - In optimization, we are constantly looking for algorithms to solve various problems as fast as possible. So it is of immediate interest to understand the fundamental limitations of efficient algorithms.
- So far in this class we've had a rule of thumb for checking if an optimization problem is "easy":
  - See if it's ***convex***!
- But this only scratches the surface. Are all nonconvex problems hard? Are some of them hard? Are there even convex problems that are hard?
- What does it even mean to be hard?!

- Let's begin by understanding what it means to have a "problem"!

Optimization problems/Decision problems/Search problems

- Let's introduce these concepts using an example we know well: stable set (aka independent set) of a graph.



- Recall that a stable set in a graph G is a subset of the nodes with no edges among them.

Optimization problem:

- Given a graph G, find its largest stable set.

Decision problem:

- Given a graph G and an integer b, decide if there exists a stable set of size $\geq b$?

(answer to a decision question is just YES or NO)

Search problem:

- Given a graph G and an integer b, find a stable set of size $\geq b$ or declare that none exists.

- It turns out that all three problems are equivalent, in the sense that if you could solve one efficiently, you could also solve the other two. See Ex. 8.1,8.2 of [DPV].

- We will focus on decision problems, since it's a bit cleaner to develop the theory there.

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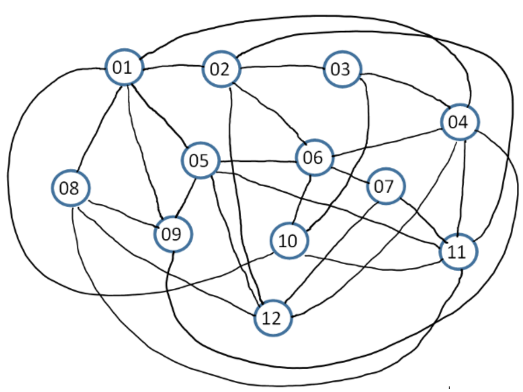
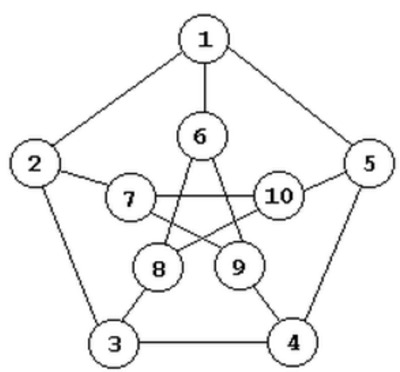
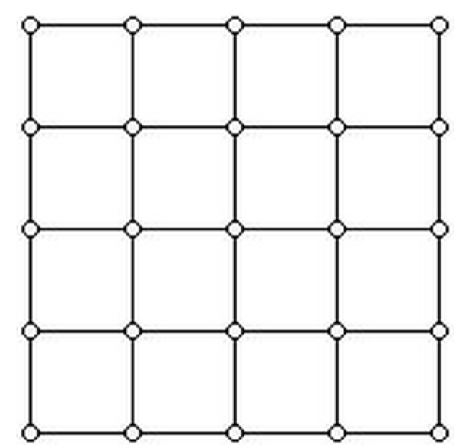
A “problem” versus a “problem instance”

- A (decision) problem is a general description of a problem to be answered with yes or no.
- Every decision problem has a *finite input* that needs to be specified for us to choose a yes/no answer.
- Each such input defines an **instance** of the problem.
- A decision problem has an infinite number of instances.
(Why doesn’t it make sense to study problems with a finite number of instances?)

- Different instances of the STABLE SET problem:
(It is common to use capital letters for the name of a decision problem.)







Examples of decision problems

- **LINEQ**
- **Input:** An $m \times n$ matrix A and an $m \times 1$ vector b, both with rational entries.
- **Question:** Is there a solution to the linear system $Ax = b$?

An instance of LINEQ:

$$2x_1 + 7x_2 = 6$$
$$\tfrac{1}{2}x_1 - x_2 = -\tfrac{1}{3}$$

$$A = \begin{pmatrix} 2 & 7 \\ \tfrac{1}{2} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ -\tfrac{1}{3} \end{pmatrix}$$

- **ZOLINEQ**
- **Input:** An $m \times n$ matrix A and an $m \times 1$ vector b, both with rational entries.
- **Question:** Is there a 0/1 solution x to the linear system $Ax = b$?

An instance of ZOLINEQ:

$$2x_1 + 7x_2 = 6$$
$$\tfrac{1}{2}x_1 - x_2 = -\tfrac{1}{3}$$

$$A = \begin{pmatrix} 2 & 7 \\ \tfrac{1}{2} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ -\tfrac{1}{3} \end{pmatrix}$$

- **Remark.** Input is rational so we can represent it with a finite number of bits. This is the so-called *"bit model of computation"*, aka the *"Turing model."*

Examples of decision problems

- **LP**
  - **Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational
    a rational number k
  - **Question:** Is the optimal value of the LP (in standard form) $\leq k$?

(This is equivalent to testing LP feasibility (why?).)

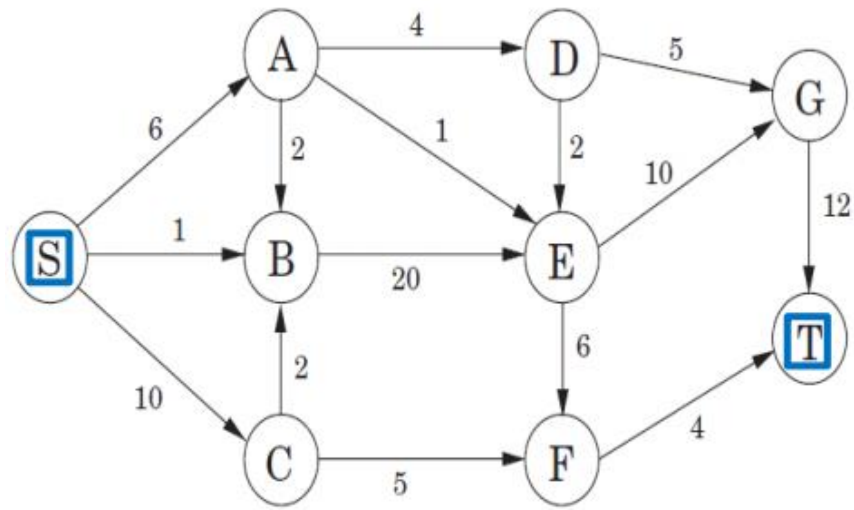
An instance of LP: $A = \begin{pmatrix} 4 & 7 & \frac{1}{2} \\ 3 & -1 & 3 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; k = 5.$

- **IP**
  - **Input:** same as above
  - **Question:** Is there an *integer* feasible solution to the LP with objective value $\leq k$?

Examples of decision problems

Let's look at a problem we have seen…

An instance of MAXFLOW:



Can you formulate the decision problem?

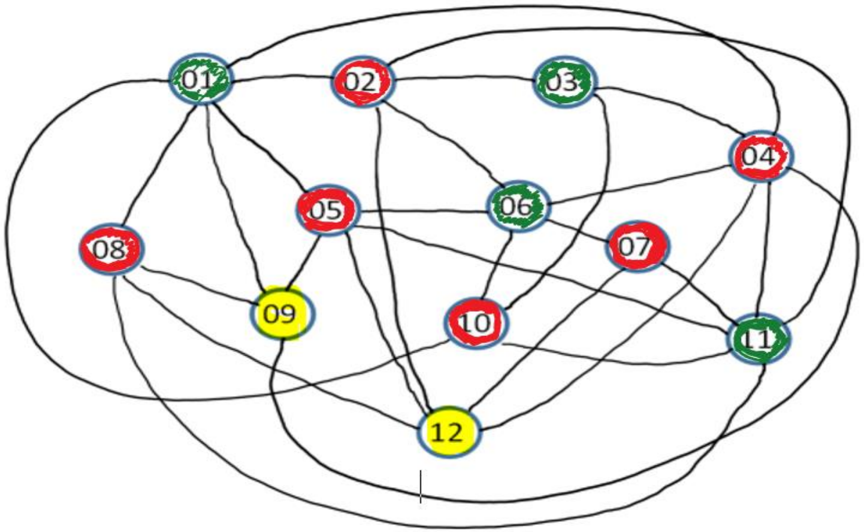
- **MAXFLOW**
- **Input:** A directed graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a designated node S, a designated node T, a rational number k.
- **Question:** Is there a flow of value $\geq k$ from S to T that respetcs the edge cost constraints and the conservation of flow constraints?

Examples of decision problems

- A graph is said to be *k-colorable* if there is a way to color its nodes with k colors such that no two adjacent nodes get the same color.

- For example, the following graph is 3-colorable.

- Graph coloring has important applications in job scheduling.



- **COLORING**

- **Input:** An undirected graph G and a positive integer k.

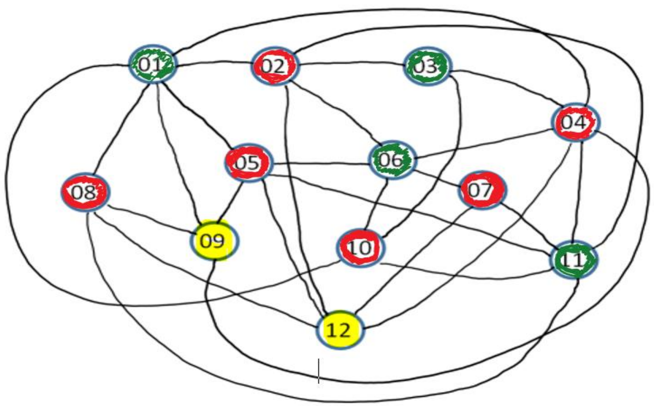
- **Question:** Is the graph k-colorable?

- We want to understand how fast can all these problems be solved?

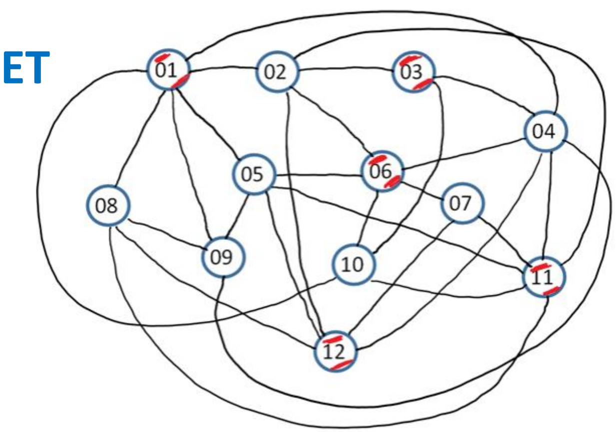
Size of an instance

- To talk about the running time of an algorithm, we need to have a notion of the "size of the input".
- Of course, an algorithm is allowed to take longer on larger instances.

- **COLORING**



- **STABLE SET**



- Reasonable candidates for input size:
  - Number of nodes n
  - Number of nodes + number of edges (number of edges can at most be n(n-1)/2)
  - Number of bits required to store the adjacency matrix of the graph

A =

0	1	0	1	1	0	0	1	1	1	0	0
1	0	1	0	0	1	0	0	0	0	1	1
0	1	0	1	0	0	0	0	0	1	0	0
1	0	1	0	0	1	0	0	1	0	1	1
1	0	0	0	0	1	0	0	1	0	1	1
0	1	0	1	1	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	1	1
1	0	0	0	0	0	0	0	1	0	1	1
1	0	0	1	1	0	0	1	0	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
0	1	0	1	1	0	1	1	0	1	0	0
0	1	0	1	1	0	1	1	0	0	0	0

Size of an instance

- In general, can think of input size as the total number of bits required to represent the input.

- For example, consider our LP problem:

- **LP**

- **Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational
  a rational number k

- **Question:** Is the optimal value of the LP (in standard form) $\leq k$?

- Input size is bounded by $2(mn + m + n + 1)\log L$, where L is the largest integer appearing in the numerator or denominator of any entry of A, b, c, k.
- Same idea holds for all other decision problems we introduced.

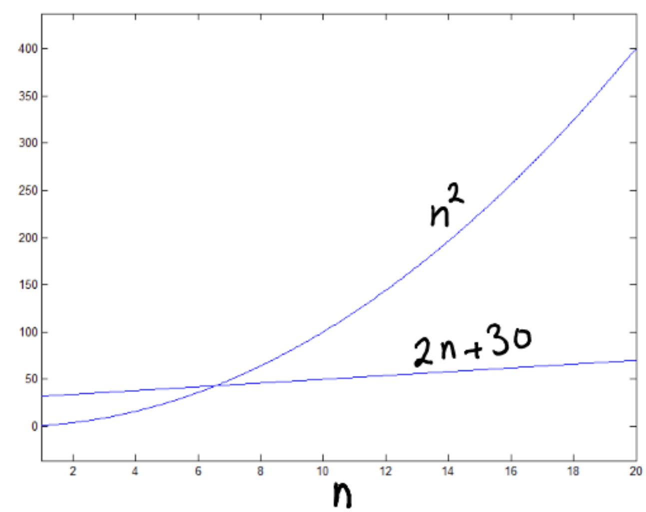
Useful notation for referring to running times

Definition. Let $f, g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We write

- $f(n) = O\big(g(n)\big)$, if $\exists n_0, c > 0$, such that
$$f(n) \leq cg(n), \forall n \geq n_0.$$
- $f(n) = \Omega\big(g(n)\big)$, if $\exists n_0, c > 0$, such that
$$f(n) \geq cg(n), \forall n \geq n_0.$$
- $f(n) = \Theta\big(g(n)\big)$, if we have both $f(n) = O\big(g(n)\big)$ and $f(n) = \Omega\big(g(n)\big)$.

Examples.

- $5n^3 + 2n^2 + 40 = \Theta(n^3)$.
- $n \log n = O(n^2)$.
- $n \log n = \Omega(n)$.
- $\forall c, k > 0, 2^{cn} = \Omega\big(n^k\big)$.



$2n+30 = O(n^2)$

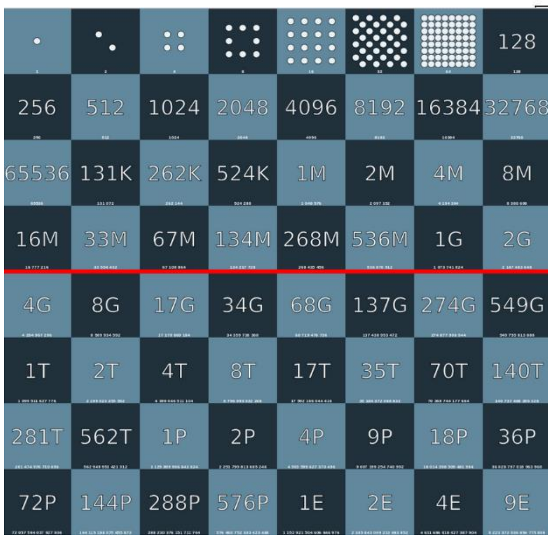
Polynomial-time and exponential-time algorithms

- A polynomial-time algorithm is an algorithm whose running time as a function of the input size is $O(p(n))$ for some polynomial function p.
- Equivalent definition: Running time is $O(n^k)$ for some positive integer k.
- Note: this is the **worst-case running time** over all inputs of size n.

- An exponential-time algorithm is an algorithm whose running time as a function of the input size is $\Omega(2^{cn})$ for some positive constant c.
- Once again, when we talk about running time for a given input size n, we mean the worst-case running time over all inputs of size n.
- There are also algorithms with running time in between (e.g., $O(n^{\log n})$), but these also are perceived as slow.

Something you all know: Poly-time:  Exp-time: 

On the awfulness of 2^n



2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7
2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}
2^{16}	2^{17}	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}
2^{24}	2^{25}	2^{26}	2^{27}	2^{28}	2^{29}	2^{30}	2^{31}
2^{32}	2^{33}	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}	2^{39}
2^{40}	2^{41}	2^{42}	2^{43}	2^{44}	2^{45}	2^{46}	2^{47}
2^{48}	2^{49}	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}
2^{56}	2^{57}	2^{58}	2^{59}	2^{60}	2^{61}	2^{62}	2^{63}

See page 233 of [DPV] for the story.

Sissa
(credited for creating the game of chess)

grains of rice on the board: $2^{64} - 1 = 18,446,744,073,709,551,615$

Comparison of running times

Time complexity function	Size n					
	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.0001 second	.0004 second	.0009 second	.0016 second	.0025 second	.0036 second
n^3	.001 second	.008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minutes	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	2×10^8 centuries	1.3×10^{13} centuries

Image credit: [GJ79]

Can Moore's law come to rescue?

Size of Largest Problem Instance
Solvable in 1 Hour

Time complexity function	With present computer	With computer 100 times faster	With computer 1000 times faster
n	N_1	$100\ N_1$	$1000\ N_1$
n^2	N_2	$10\ N_2$	$31.6\ N_2$
n^3	N_3	$4.64\ N_3$	$10\ N_3$
n^5	N_4	$2.5\ N_4$	$3.98\ N_4$
2^n	N_5	$N_5+6.64$	$N_5+9.97$
3^n	N_6	$N_6+4.19$	$N_6+6.29$

Effect of improved technology on several polynomial and exponential time algorithms.

Image credit: [GJ79]

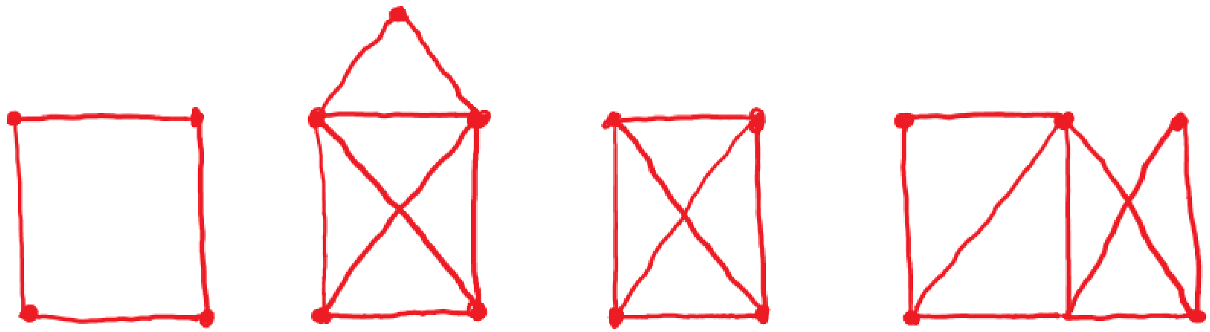
The complexity class P

- The class of all decision problems that admit a polynomial-time algorithm.

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_{\epsilon}$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,…

Example of a problem in P

- **PENONPAPER**
- **Input:** A connected undirected graph.
- **Question:** Can you draw it without lifting your pen from the paper?



- **Euler:** answer to PENONPAPER is YES if and only if “every node, with the possible exception of two nodes, has even degree.”
- This condition can obviously be checked in polynomial time.
- Hence PENONPAPER∈P.

- **Peek ahead:** this problem is asking if there is a path that visits **every *edge* exactly once**.
- If we were to ask for a path that instead visits **every *node* exactly once**, we would have a completely different story in terms of complexity!

How to prove a problem is in P?

- Develop a poly-time algorithm from scratch! Can be **far from trivial** (examples below).
- Much easier: use a poly-time hammer somebody else has developed. (Reductions!)

- **LINEQ** (solve a system of linear equations)
  - Gaussian elimination -- $O(n^3)$
  - Can also use, e.g., the conjugate gradient algorithm -- $O(n^3)$
  - (Faster algorithms known: Google Strassen)
- **LP** (solve a system of linear inequalities)
  - Was open for a long time – simplex doesn't do it (at least, we don't know how to modify it so it does)
  - The ellipsoid algorithm (Khachiyan-1979)
  - Interior point algorithms (Karmarkar-1984)

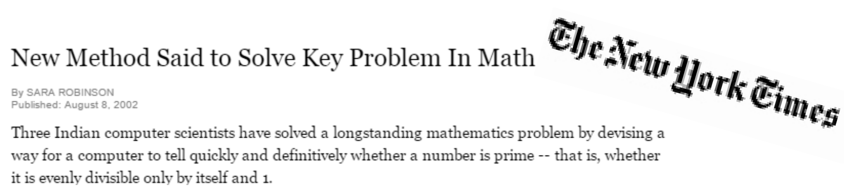
The New York Times

BREAKTHROUGH IN PROBLEM SOLVING

By JAMES GLEICK
Published: November 19, 1984

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

- **PRIMES** (decide if a given integer is prime)
  - Was open for a long time -- Proved to be in P by Agrawal-Kayal-Saxena in 2002.
    - Kayal and Saxena were undergraduates!
  - Why doesn't the naïve algorithm work? "Given n, check all candidate divisors up to $\sqrt{n}$."



New Method Said to Solve Key Problem In Math

The New York Times

By SARA ROBINSON
Published: August 8, 2002

Three Indian computer scientists have solved a longstanding mathematics problem by devising a way for a computer to tell quickly and definitively whether a number is prime -- that is, whether it is evenly divisible only by itself and 1.

An aside: Factoring

- Despite knowing that PRIMES is in P, it is a major open problem to determine whether we can *factor* an integer in polynomial time.

```
RSA-1024 = 1350664108659952233496032162788059699388814756056670275244851438515265106048595338339402871505719094417982072821644715513736804197039641917430464965
           8927425623934102086438320211037295872576235850964311056407350150818751067659462920556368552947521350085287941637732853390610975054433499981115005697
           7236890927563
```

$100,000 prize money by RSA

```
RSA-2048 = 2519590847565789349402718324004839857142928212620403202777713783604366202070
           7595556264018525880784406918290641249515082189298559149176184502808489120072
           8449926873928072877767359714183472702618963750149718246911650776133798590957
           0009733045974880842840179742910064245869181719511874612151517265463228221686
           9987549182422433637259085141865462043576798423387184774447920739934236584823
           8242811981638150106748104516603773060562016196762561338441436038339044149526
           3443219011465754445417842402092461651572335077870774981712577246796292638635
           6373289912154831438167899885040445364023527381951378636564391212010397122822
           120720357
```

$200,000 prize money by RSA

- The RSA challenge is no longer active (as of 2007), but factoring these numbers will result in an automatic A+ in this class!

- Got some free time over the winter break?

Reductions

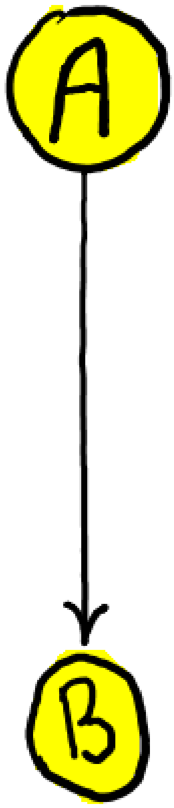
- Many new problems are shown to be in P via **a reduction to a problem that is already known to be in P.**
- What is a reduction?
  - Very intuitive idea -- **A reduces to B means: “If we could do B, then we could do A.”**
  - Being happy in life reduces to finding a good partner.
  - Landing a good job reduces to graduating from Princeton.
  - Getting an A+ in ORF 363 reduces to factoring RSA-2048.
  - …
- Well-known joke - mathematician versus engineer boiling water:

- Day 1: 

- Day 2: 

Reductions

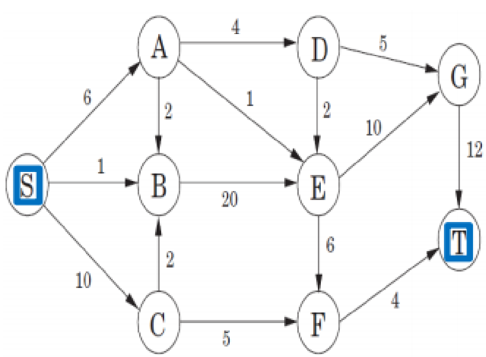
- A reduction from a decision problem A to a decision problem B is
  - **a "general recipe" (aka an algorithm)** for taking any instance of A and explicitly producing an instance of B, such that
  - the answer to the instance of A is YES if and only if the answer to the produced instance of B is YES.
    (OK for our purposes also if the YES/NO answer gets flipped.)
- This enables us to answer A by answering B.



MAXFLOW→LP

- **MAXFLOW**
- **Input:** A directed graph $G(V,E)$, nonnegative rational numbers c_i on each edge, a designated node S, a designated node T, a rational number k.
- **Question:** Is there a flow of value $\geq k$ from S to T that respetcs the edge cost constraints and the conservation of flow constraints?

- **LP**
- **Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational
  a rational number k.
- **Question:** Is the optimal value of the LP $\geq k$?



Poly-time reduction

(shown on once instance)

max. $x_{SA} + x_{SB} + x_{SC}$

s.t.

$x_{SA}, x_{AD}, x_{BE}, \dots, x_{GT} \geq 0$

$x_{SA} \leq 6, x_{AB} \leq 2, x_{EG} \leq 10, \dots, x_{GT} \leq 12$

$$\begin{cases} x_{SA} = x_{AD} + x_{AB} + x_{AE} \\ x_{SC} = x_{CB} + x_{CF} \\ \vdots \\ x_{CF} + x_{EF} = x_{FT}. \end{cases}$$

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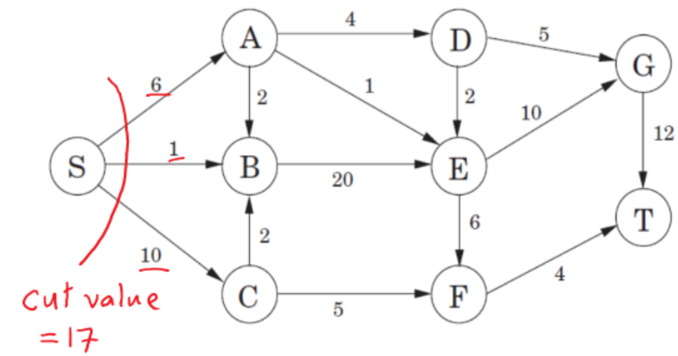
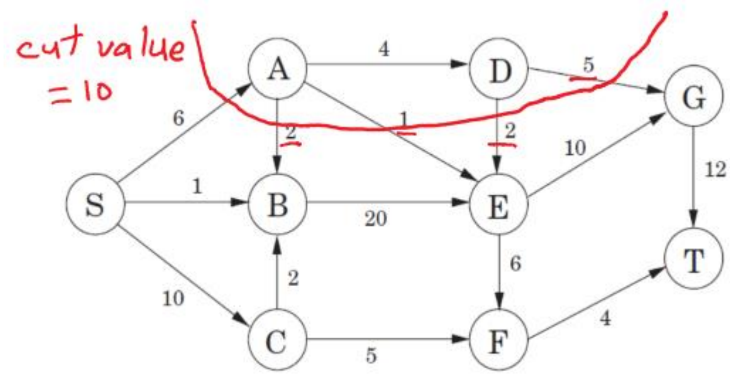
Polynomial time reductions

- So we say that "MAXFLOW reduces to LP". (Notation: MAXFLOW→LP.)
- Since we know how to solve LP in polynomial time (e.g., via interior point methods), now we know how to solve MAXFLOW in polynomial time. So MAXFLOW∈P.
- This argument relies crucially on the fact that **the reduction is polynomial in length.**
  - Before we even solve the LP, we need to make sure its size is not too big (e.g., it doesn't have too many decision variables, too many constraints, or data that takes an exponential number of bits to write down.)
  - What does "not too big" mean? The size needs to be polynomial in the size of the instance of the original problem (in this case MAXFLOW).
  - Without this constraint, one could give, e.g., a simple reduction from STABLE SET to LP (do you see how)? This should not happen (we'll see why soon).
- In your HW problem you need to argue that a certain problem about scheduling appointments is in P by giving a reduction. Don't forget to argue that the length of the reduction is polynomial.

MINCUT

- A cut is a partition of the nodes of a graph into two (non-empty) sets U and $\overline{U}$.
- The value of a cut is the sum of edge weights going from U to $\overline{U}$.





- **MINCUT**

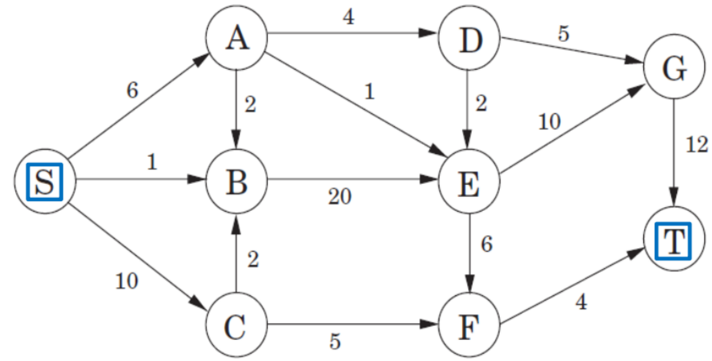
- **Input:** A directed graph $G(V,E)$, nonnegative rational numbers c_i on each edge, a rational number k.
- **Question:** Is there a cut of value $\leq k$?

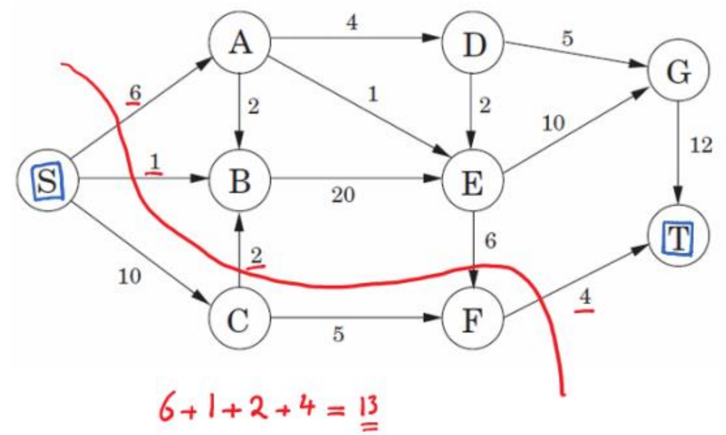
- Is MINCUT in P?

- Yes! We'll reduce it to LP.

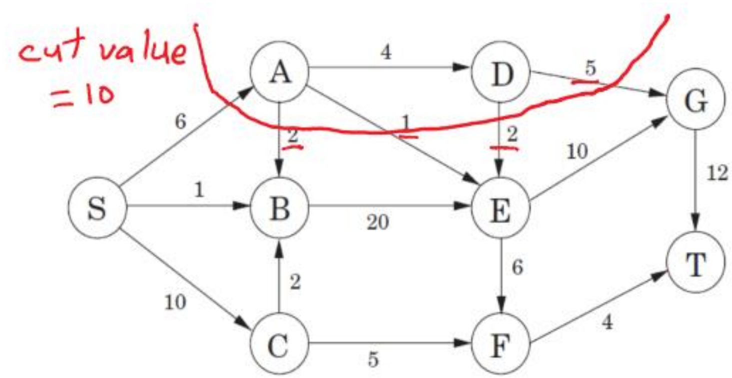
MIN S-T CUT

- **MIN S-T CUT**
- **Input:** A directed graph $G(V,E)$, nonnegative rational numbers c_i on each edge, a rational number k, two designated nodes S and T.
- **Question:** Is there a cut of value $\leq k$?





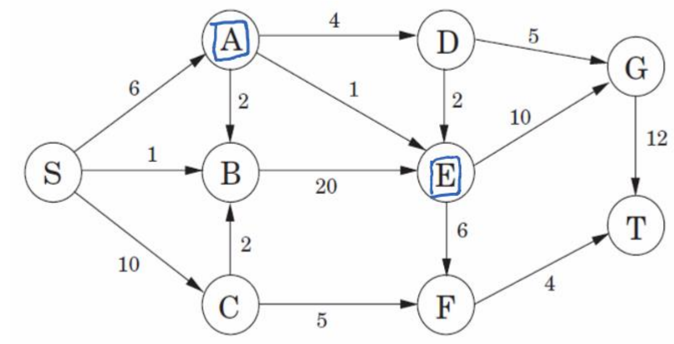
- Strong duality of linear programming implies the minimum S-T cut of a graph is exactly equal to the maximum flow that can be sent from S to T.
- Hence, MIN S-T CUT→MAXFLOW
- We have already seen that MAXFLOW→ LP.

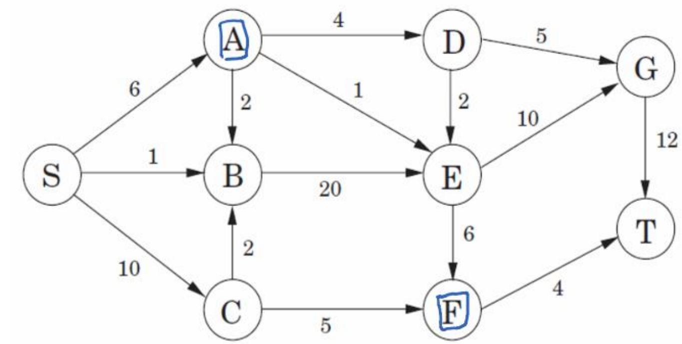


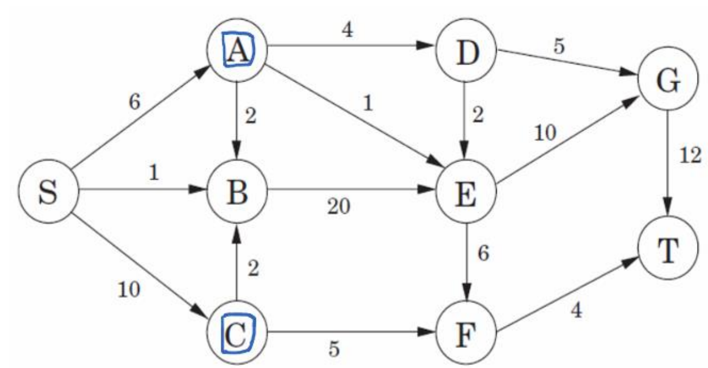
- But what about MINCUT? (without designated S and T)

MINCUT→MIN S-T CUT

- Pick a node (say, node A)
- Compute MIN S-T CUT from A to every other node
- Compute MIN S-T CUT from every other node to A
- Take the minimum over all these 2(|V|-1) numbers
- That's your MINCUT!
- The reduction is polynomial in length.







Overall reduction

- We have shown the following:

MINCUT→MIN S-T CUT→MAXFLOW→LP

- Polynomial time reductions compose (why?):

MINCUT→LP

- MINCUT∈P

- Unfortunately, we are not so lucky with all decision problems…
- Now comes the bad stuff…

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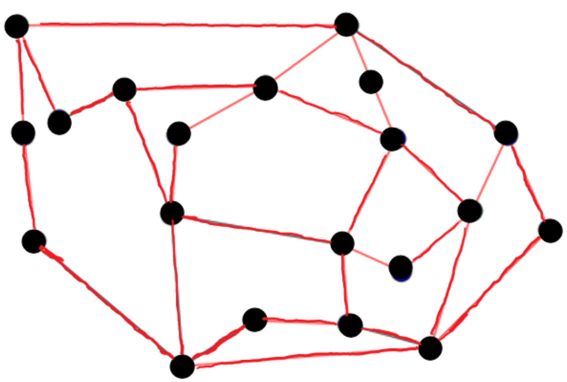
MAXCUT

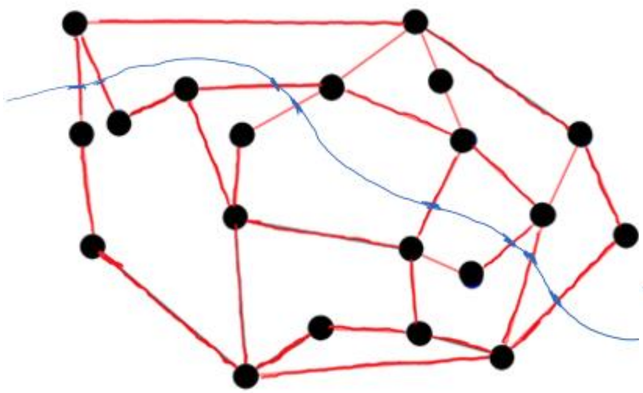
- **MAXCUT**

- **Input:** A graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a rational number k.

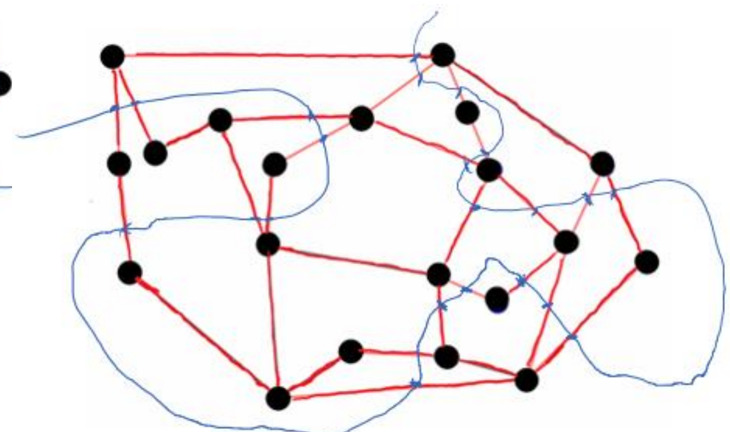
- **Question:** Is there a cut of value $\geq k$?

- Examples with edge costs equal to 1:





- Cut value=8



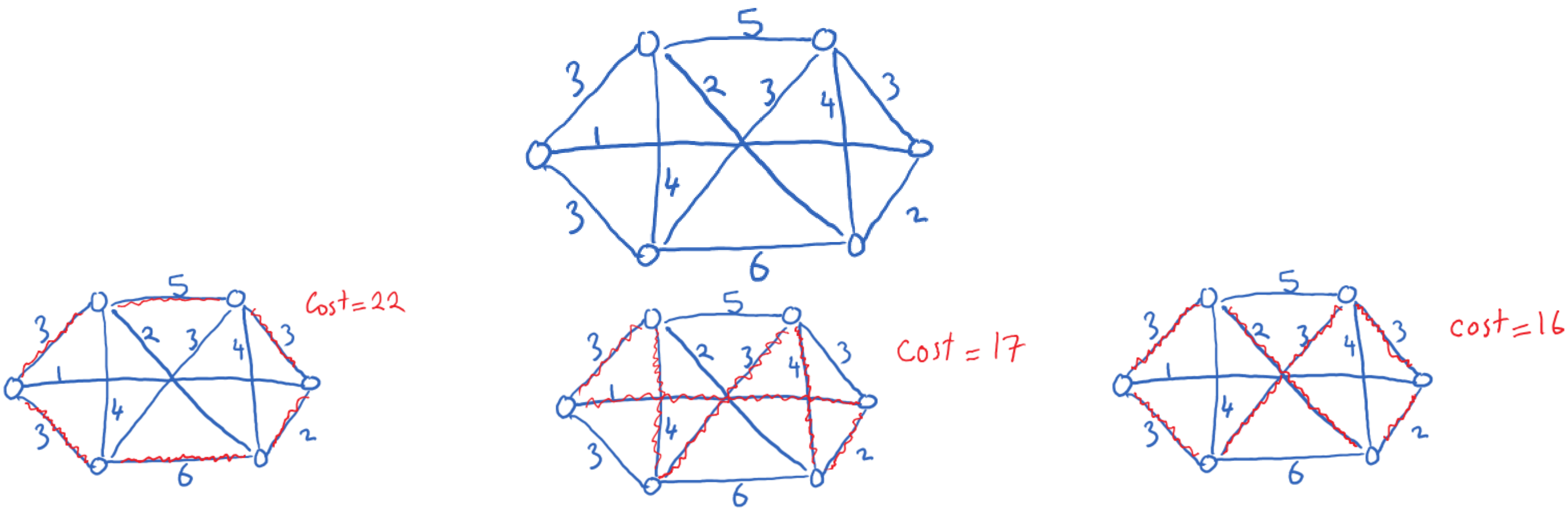
- Cut value=23 (optimal)

- To date, no one has come up with a polynomial time algorithm for MAXCUT.

- We want to understand why that is…

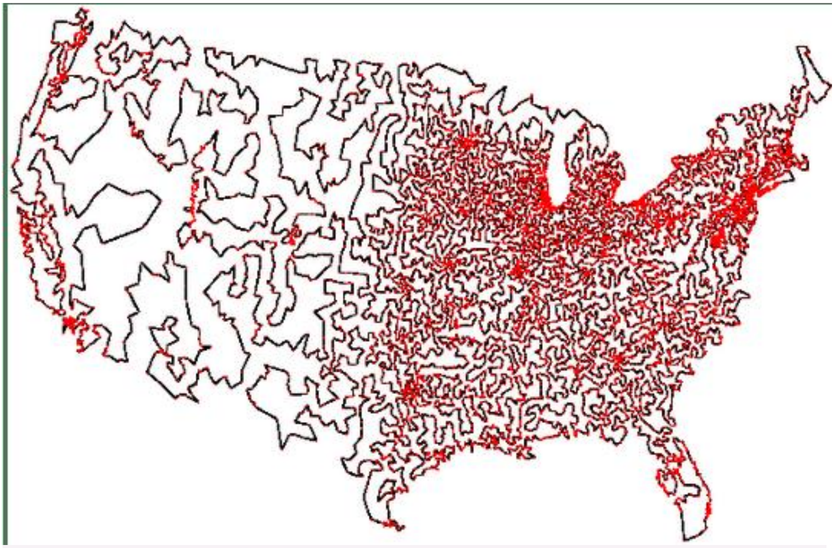
The traveling salesman problem (TSP)

- **TSP**
- **Input:** A graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a rational number k.
- **Question:** Is there a tour of cost $\leq k$ that visits each node exactly once?



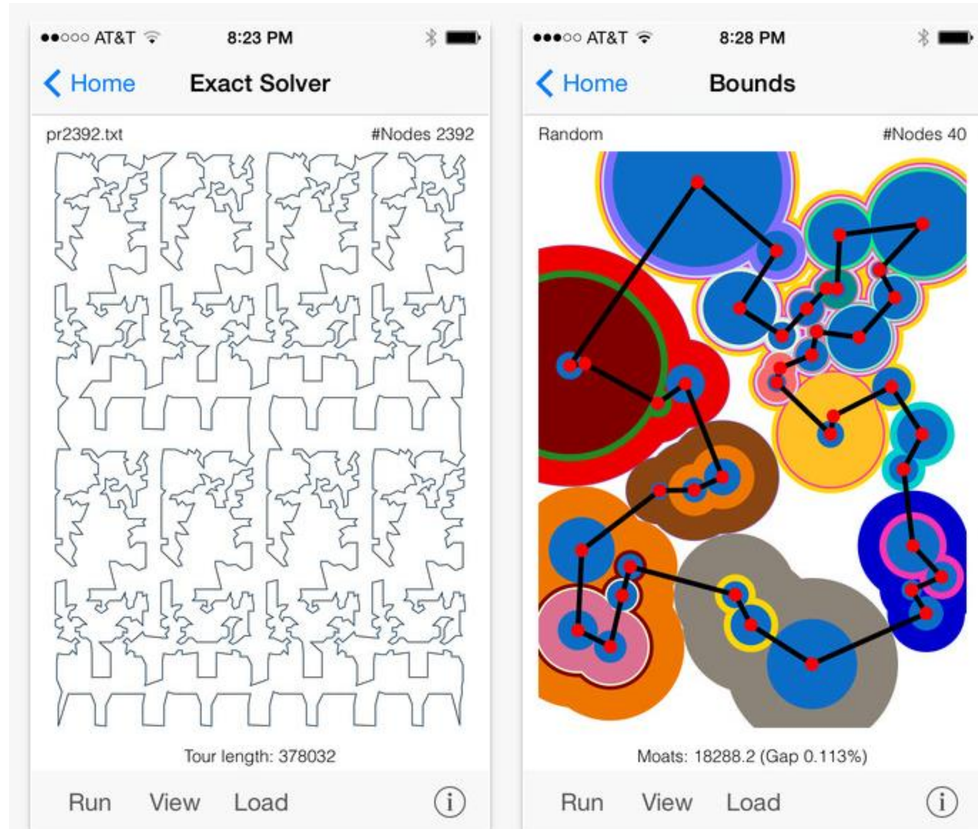
- Again, nobody knows how to solve this efficiently (over all instances).
- Note the sharp contrast with PENONPAPER.
- Amazingly, MAXCUT and TSP are in a precise sense “equivalent”: there is a polynomial time reduction between them in either direction.

TSP



24,978 Cities in Sweden
Solved in 2004

15,112 Cities in Germany
Solved in 2001



The complexity class NP

- A decision problem belongs to the class **NP (Nondeterministic Polynomial time)** if “the YES answer to any instance is easily verifiable.”
    - More precisely, every YES instance has a “certificate” of its correctness that can be verified in polynomial time.

- Examples: TSP, MAXCUT, PENONPAPER....what’s the certificate in each case?

Remarks.

- A nondeterministic computer is a machine that can “guess” an answer and then verify it. It’s a very unrealistic computer.
- NP does not mean “not polynomial”! There are many easy problems in NP (e.g., ADDITION, LINEQ).
- P⊆NP. (The poly-time algorithm itself is a certificate.)
- Note that for a given decision problem, it’s not at all clear that a short certificate for the YES answer also implies a short certificate for the NO answer. (Think, e.g., of TSP.)

The complexity class NP

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,…

- TSP
- MAXCUT
- STABLE SET
- SAT
- 3SAT
- PARTITION
- KNAPSACK
- IP
- COLORING
- VERTEXCOVER
- 3DMATCHING
- SUDOKU,…

NP-hard and NP-complete problems

Definition.

- A decision problem is said to be NP-hard if every problem in NP reduces to it via a polynomial-time reduction.
(roughly means “harder than all problems in NP.”)

Definition.

- A decision problem is said to be NP-complete if

(i) It is NP-hard

(ii) It is in NP.

(roughly means “the hardest problems in NP.”)

Remarks.

- NP-hardness is shown by a reduction from a problem that’s already known to be NP-hard.

- Membership in NP is shown by presenting an easily checkable certificate of the YES answer.

- NP-hard problems may not be in NP (or may not be known to be in NP as is often the case.)

The complexity class NP

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,…

NP

- TSP
- MAXCUT
- STABLE SET
- SAT
- 3SAT
- PARTITION
- KNAPSACK
- IP
- COLORING
- VERTEXCOVER
- 3DMATCHING
- SUDOKU,…

$\in$P

NP-complete

The satisfiability problem (SAT)

- **SAT** (one of the most fundamental NP-complete problems.)

- **Input:** A Boolean formula in conjunctive normal form (CNF).

- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

$$\varphi = \underbrace{(x \vee y \vee z)} \wedge \underbrace{(x \vee \bar{y})} \wedge \underbrace{(y \vee \bar{z})} \wedge \underbrace{(\bar{x} \vee \bar{y} \vee \bar{z})}$$

Formula

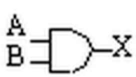
Clauses

Variables: x, y, z

$\vee$: OR, $\wedge$: AND, $\bar{x}$: NOT x

Literal: a variable or its complement.

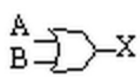
AND



$X = A \cdot B$

A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

OR



$X = A + B$

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

The satisfiability problem (SAT)

- **SAT**

- **Input:** A Boolean formula in conjunctive normal form (CNF).
- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

$$(x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

YES $x=1, y=1, z=0.$

$$(x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee z)$$

NO

3SAT

- **3SAT**

- **Input:** A Boolean formula in conjunctive normal form (CNF), where *each clause has exactly three literals.*

- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

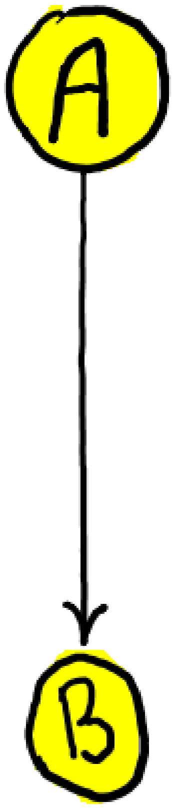
$$(x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee w \vee \bar{z}) \wedge (\bar{y} \vee w \vee z)$$

- There is a simple reduction from SAT to 3SAT. (See, e.g., [DPV, Chap. 8]).

- Hence, since SAT is NP-hard, then so is 3SAT. Moreover, 3SAT is clearly in NP (why?), so 3SAT is NP-complete.

Reductions (again)

- A reduction from a decision problem A to a decision problem B is
  - **a "general recipe" (aka an algorithm)** for taking any instance of A and explicitly producing an instance of B, such that
  - the answer to the instance of A is YES if and only if the answer to the produced instance of B is YES.
    (OK for our purposes also if the YES/NO answer gets flipped.)
- This enables us to answer A by answering B.



- This time we use the reduction for a different purpose:
  - If A is known to be hard, then B must also be hard.

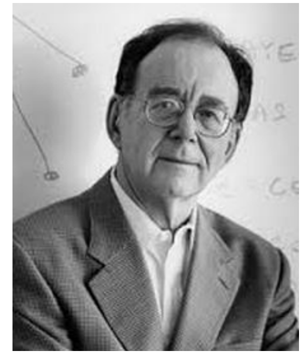
The first 21 (official) reductions

REDUCIBILITY AMONG COMBINATORIAL PROBLEMS†

Richard M. Karp

University of California at Berkeley

Abstract: A large class of computational problems involve the determination of properties of graphs, digraphs, integers, arrays of integers, finite families of finite sets, boolean formulas and elements of other countable domains. Through simple encodings from such domains into the set of words over a finite alphabet



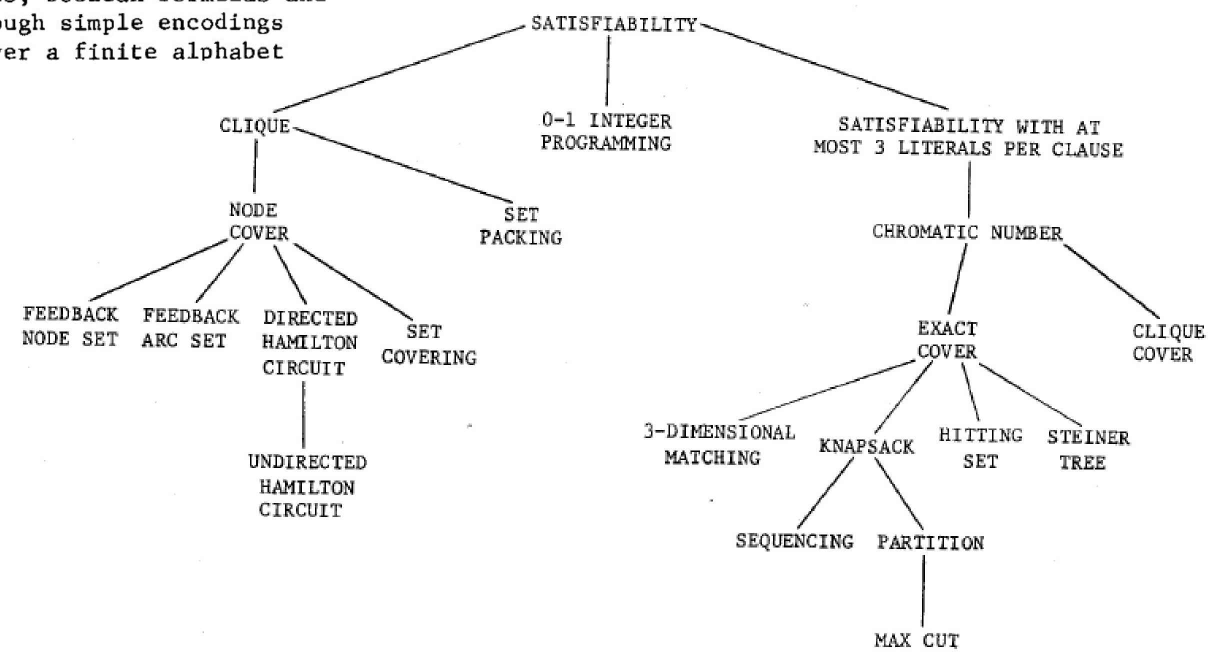
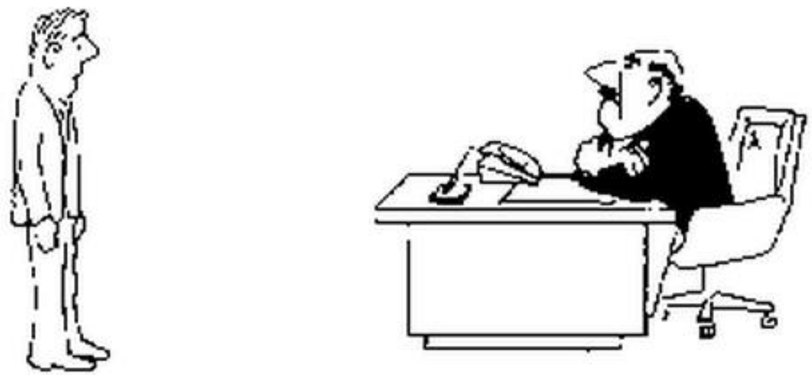


FIGURE 1 - Complete Problems

- Today we have thousands of NP-complete problems. In all areas of science and engineering.

The value of reductions



I can't find an efficient algorithm, I guess I'm just too dumb.

I can't find an efficient algorithm, because no such algorithm is possible

I can't find an efficient algorithm, but neither can all these famous people.

[Garey, Johnson]

Practice with reductions

I’ll do a few reductions for you:

- **3SAT→STABLE SET** (also in [DPV, Chap 8, p. 249])
- **STABLE SET→ 0/1 IP** (you already know this from lecture 1)
- **3SAT→POLYPOS (degree 6)**

In your homework you have to do:

- **PARTITION→POLYPOS (degree 4)**
- **STABLE SET→CHEAPHOST**

More practice: try to prove NP-hardness of problems on the following slides. Read [DPV, Chap. 8] for many more.

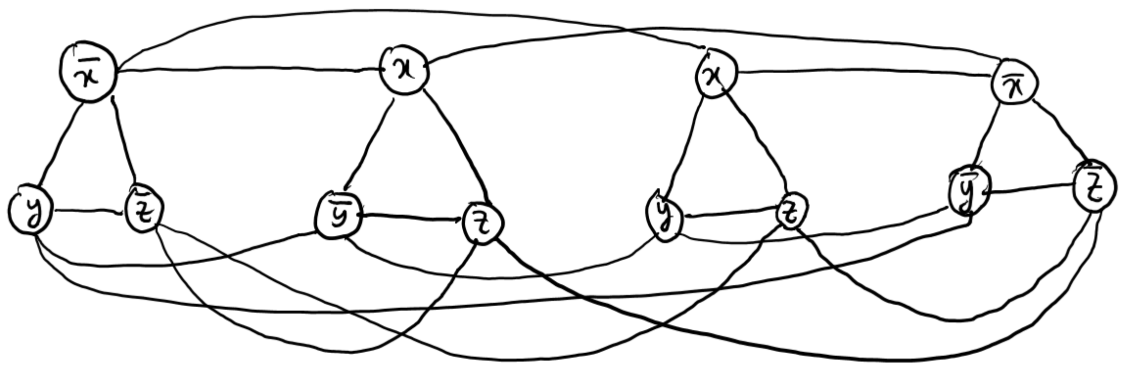
3SAT→STABLE SET

We show the reduction on an instance only. The pattern should be clear.

$$\varphi = (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

(k clauses)

G:



Construction: For each clause create a "triangle". Across triangles connect each variable to its complement.

Claim: φ is satisfiable $\iff$ $\alpha(G) \geq k$.

STABLE SET → 0/1 Integer Programming

Given $G(V,E)$

$$\alpha(G) \geq k$$

$$\Updownarrow$$

$$\begin{cases} \sum_{i=1}^{n} x_i \geq k \\ x_i + x_j \leq 1 & \text{if } ij \in E \\ x_i \in \{0,1\} & i=1,\dots,n \end{cases}$$

is feasible.

STABLE SET → Feasibility of Quadratic Equations

Given $G(V,E)$

$$\alpha(G) \geq K$$

$$\Updownarrow$$

$$\begin{cases} \sum_{i=1}^{n} x_i - K = s^2 \\ x_i x_j = 0 \quad \text{if } ij \in E \\ x_i(1-x_i) = 0 \quad i=1,\dots,n \end{cases}$$

is feasible.

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3SAT→POLYPOS (degree 6)

We show the reduction on an instance only. The pattern should be clear.

Start with any instance of 3SAT, such as:

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4)$$

Construct p as

$$p(x) = \underbrace{\sum_{i=1}^{4} \left(x_i(1-x_i)\right)^2}_{\text{encoding 0/1 requirement}} + \Big[\overbrace{(x_1+(1-x_2)+x_3-1)(x_1+(1-x_2)+x_3-2)(x_1+(1-x_2)+x_3-3)}^{\text{encoding clause 1}}\Big]^2$$

$$+\Big[\overbrace{((1-x_1)+(1-x_2)+x_3-1)((1-x_1)+(1-x_2)+x_3-2)((1-x_1)+(1-x_2)+x_3-3)}^{\text{clause 2}}\Big]^2$$

$$+\Big[\overbrace{(x_1+x_2+x_4-1)(x_1+x_2+x_4-2)(x_1+x_2+x_4-3)}^{\text{clause 3}}\Big]^2.$$

Observe that the reduction is polynomial in length.

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3SAT→POLYPOS (degree 6)

Claim : A general instance φ of 3SAT will be satisfiable

$$\Updownarrow$$

$\exists \bar{x} \in \mathbb{R}^n$ such that $p(\bar{x}) \leq 0$ (in fact $p(\bar{x}) = 0$), where p is constructed as above.

Pf. ($\Downarrow$) Take $\bar{x}$ to be the satisfying assignment of 3SAT.

All the terms of p vanish (why?)

($\Uparrow$) Suppose φ not satisfiable. Claim: $p(x) > 0 \;\; \forall x \in \mathbb{R}^n$.

p is a sum of squares $\Rightarrow p(x) \geq 0 \;\; \forall x \in \mathbb{R}^n$.

- If $x \notin \{0,1\}^n$, then $\sum (x_i(1-x_i))^2 > 0$ (why?)
- If $x \in \{0,1\}^n$, then at least one term out of the terms encoding the clauses will be positive. $\square$

The knapsack problem

- **KNAPSACK**

- **Input:** A list of item values $p_1, \dots, p_n$, a list of weights on the same items $w_1, \dots, w_n$, two rational numbers P, W.

- **Question:** Can the thief steal a set of items of total value $\geq P$ that fit in his knapsack of total weight W?

The partition problem

- **PARTITION**

- **Input:** A list of positive integers $a_1, \dots, a_n$.

- **Question:** Can you split them into to bags such that the sum in one equals the sum in the other?

$\{5,2,1,6,3,8,5,4,1,1,10\}$

$\{5,2,1,6,3,8,5,4,1,1,10\}$

- Note that the YES answer is easily verifiable.

- How would you efficiently verify a NO answer? (no one knows)

Testing polynomial positivity

- **POLYPOS**

- **Input:** A multivariate polynomial $p(x) := p(x_1, \dots, x_n)$ of degree four.
- **Question:** Is there an $x \in \mathbb{R}^n$ for which $p(x) \leq 0$?

- Example:

$$p(x) = x_1^4 + 2x_1^2x_2^2 - 3x_1x_3 + 5x_2^4 + 6x_1^2x_2 - x_1x_2x_3 + 4x_3^4 + 100.$$

- A reduction from PARTITION to POLYPOS is on your homework.

- Is there an easy certificate of the NO answer? (the answer is believed to be negative)

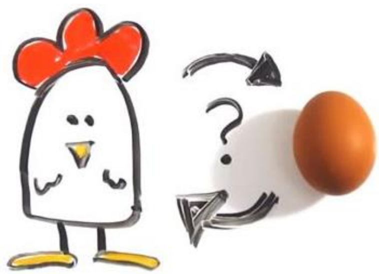
- Is there an easy certificate of the YES answer? We don't know; the obvious approach doesn't work:

$$p(x) = (x_1 - 2)^2 + (x_2 - x_1^2)^2 + (x_3 - x_2^2)^2 + \dots + (x_n - x_{n-1}^2)^2$$

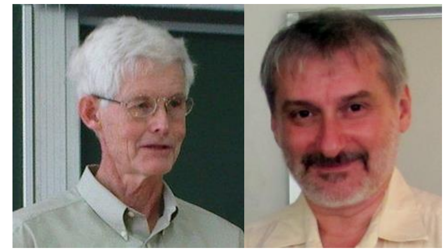
$$p(x) = 0 \Rightarrow x_n = 2^{2^n}.$$

- An aside: Testing convexity of polynomials is also NP-hard! [AOPT13]

But what about the first NP-complete problem?!!

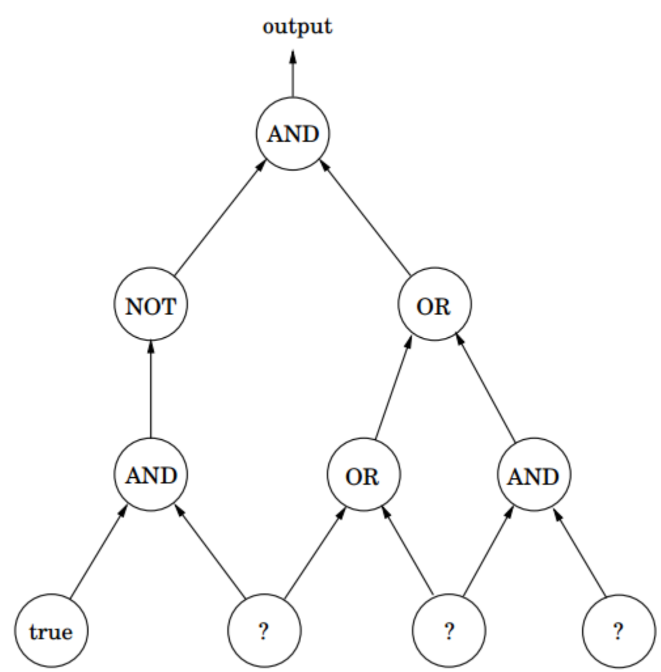


- The Cook-Levin theorem.



- Every problem in NP reduces to CIRCUIT SAT.
- In a way a very deep theorem.
- At the same time almost a tautology.
- See page 260 of [DPV] for a short explanation.

An instance of CIRCUIT SAT.

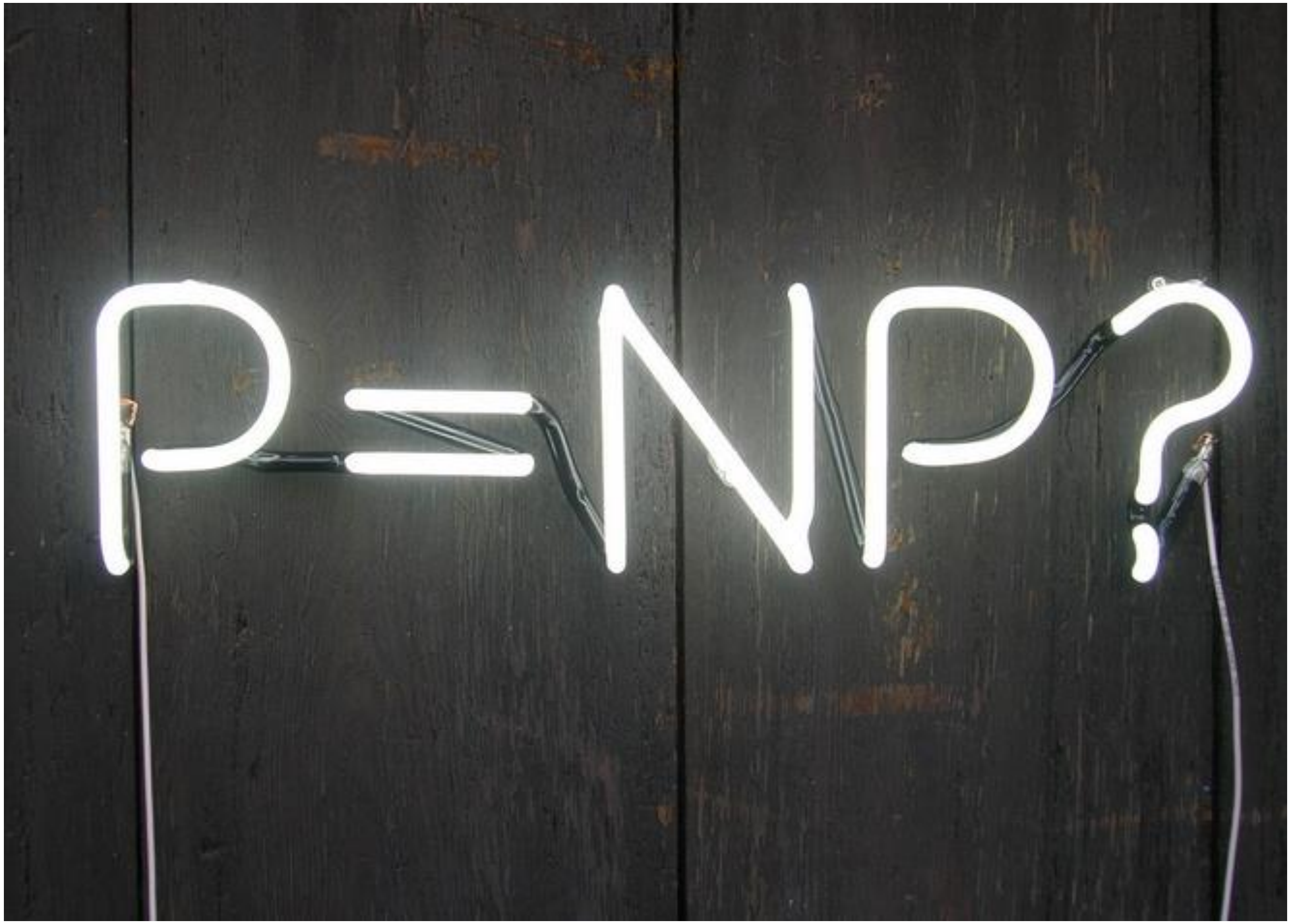


- CIRCUIT SAT→ SAT→ 3SAT (easy reductions; see [DPV]).

The domino effect

- All NP-complete problems reduce to each other!
- If you solve one in polynomial time, you solve ALL in polynomial time!

The $1M question!



- Most people believe the answer is NO!
- Philosophical reason: If a proof of the Goldbach conjecture (or any other longstanding open problem in mathematics) were to fly from the sky, we could efficiently verify it. But should this imply that we can *find* this proof efficiently? P=NP would imply that the answer is yes.

Nevertheless, there are believers too…

- Over 100 wrong proofs have appeared so far (in both directions)! See http://www.win.tue.nl/~gwoegi/P-versus-NP.htm

Main messages...

- Computational complexity theory beautifully classifies many problems of optimization theory as easy or hard
    - At the most basic level, easy means "in P", hard means "NP-hard."
- The boundary between the two is very delicate:
    - MINCUT vs. MAXCUT, PENONPAPER vs. TSP, LP vs. IP, ...
- Important: When a problem is shown to be NP-hard, it doesn't mean that we should give up all hope. NP-hard problems arise in applications all the time. There are good strategies for dealing with them.
    - Solving special cases exactly
    - Heuristics that work well in practice
    - Using convex optimization to find bounds and near optimal solutions
    - Approximation algorithms – suboptimal solutions with worst-case guarantees
- P=NP?
    - Maybe one of you guys will tell us one day.

Notes & References

Notes:

- Relevant reading for this lecture is Chapter 8 of [DPV08].

References:

- [DPV08] S. Dasgupta, C. Papadimitriou, and U. Vazirani. Algorithms. McGraw Hill, 2008.
- [GJ79] D.S. Johnson and M. Garey. Computers and Intractability: a guide to the theory of NP-completeness, 1979.
- [BT00] V.D. Blondel and J.N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 2000.
- [AOPT13] NP-hardness of testing convexity: http://web.mit.edu/~a_a_a/Public/Publications/convexity_nphard.pdf