Discrete Optimization

(at IBM's Mathematical Sciences Department)

Sanjeeb Dash
IBM Research

Lecture, ORF 363
Princeton University, Dec 13, 2016

Outline

▷ Real-world optimization (and at IBM)
- Application areas
- Recent job postings

▷ Discrete Optimization basics
- Problems
- Computational Complexity
- Formulations

▷ Solution techniques: integer programming
- Branch-and-bound
- Cutting planes

▷ Applications
- Steel industry
- Pipelines + more

Real-world Optimization

IBM Research

IBM: 377,757 employees (end of 2015)
IBM Research: 12 labs, 3000+ researchers

IBM's Math. Sciences Dept.

IBM Mathematical Sciences Department:

$\diamond$ 50+ years old

$\diamond$ 50+ people

$\diamond$ 50 % funding from contracts, 50% from IBM grants

- 40% of time spent on applied work $\equiv$ need to publish 2-3 papers (or perish)
- 100% of time spent on applied work $\equiv$ need to publish 0 papers

Discrete Optimization

Discrete optimization is the study of problems where the goal is to select a minimum cost alternative from a finite (or countable) set of alternatives.

Application areas

Airlines	route planning, crew scheduling	American, United
	revenue management	Air New Zealand, British Airways
Package Delivery	vehicle routing	UPS, Fedex, USPS
Trucking	route planning, vehicle routing	Schnieder
Transportation	network optimization	Amazon
Telecommunication	network design	AT&T
Shipping	route planning	Maersk
Pipelines	batch scheduling, network flows	CLC, CNPC
Steel Industry	cutting stock	Posco
Paper Industry	cutting stock	GSE mbH
Finance	portfolio management	Axioma
Oil & Gas	pooling	ExxonMobil
Petrochemicals		SK Innovation
Power generation	unit commitment, resource management	BC Hydro
Railways	Timetabling, crew-scheduling	BNSF, CSX, Belgian Railways, Deutsche Bahn, Trenitalia

Recent jobs in optimization

2016

Amazon - *Operations Research Scientist*

Network optimization, Linear and Integer programming - AMPL, Mosel, R, Matlab (Ph.D.)

Amazon - *Applied Scientist- Operations Research, Devices*

Linear Programming, Combinatorial optimization, Integer programming - CPLEX, Gurobi, XPRESS (M.S.)

BAE Systems - *Research Engineer Planning and Control*

Vehicle Routing, Network Optimization - Mathematical Programming, Control Theory (B.S./M.S./Ph.D.)

BNSF -*OR & Advanced Analytics Specialist I*

Railroad logistics - CPLEX, Gurobi, ProModel, ARENA, Frontline Solver (M.S./Ph.D.)

Facebook -*Operations Research Scientist*

Supply chain optimization, Inventory planning - Mathematical Programming (M.S./Ph.D.)

FedEx - *Operations Research Analyst*

Schedule optimization, network planning, truck routing, crew planning, facility location (M.S.)

GE - *Operations Research Specialist*

Optimize railroad operations - CPLEX or Gurobi (M.S./Ph.D.)

LG Electronics USA - *Applied Operations Research Lead*

Demand/Inventory planning, Logistics optimization - AMPL, CPLEX (B.S.)

Modular mining - *Operations Researcher*

"Optimization of operational aspects of mining" - CPLEX, Gurobi, GAMS, AMPL (M.S./Ph.D.)

Walt Disney World Resort - *Decision Science Consultant*

Mixed-integer, non-linear, stochastic optimization or Simulation (M.S./Ph.D.)

Uber - *Operations Research Scientist*

Optimize matching of riders to drivers, schedule optimization (Ph.D.)

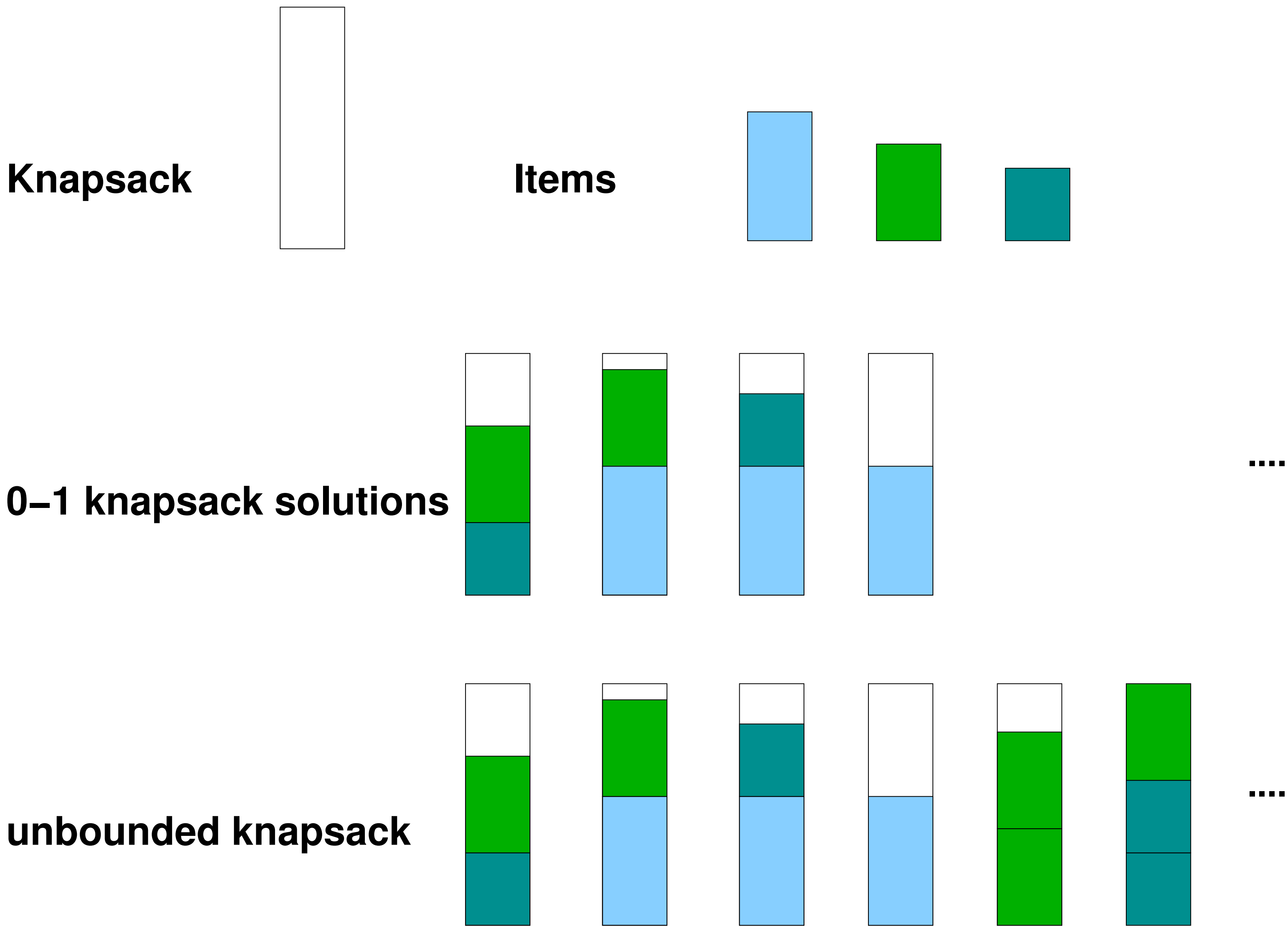
IBM, SAS, Gurobi, Mosek, ORTEC

Problems

Knapsack Problem

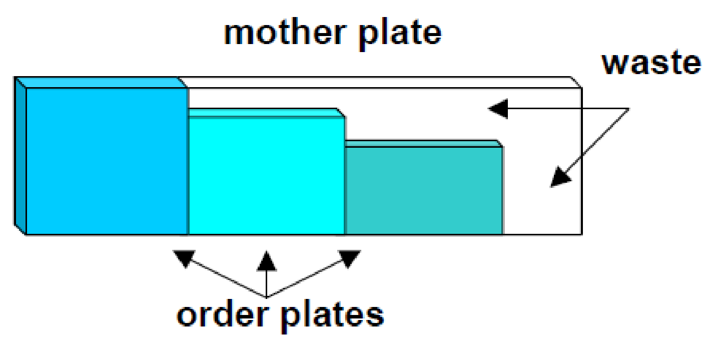
Maximize the value of items packed in a knapsack while not exceeding its capacity

Knapsack Problem



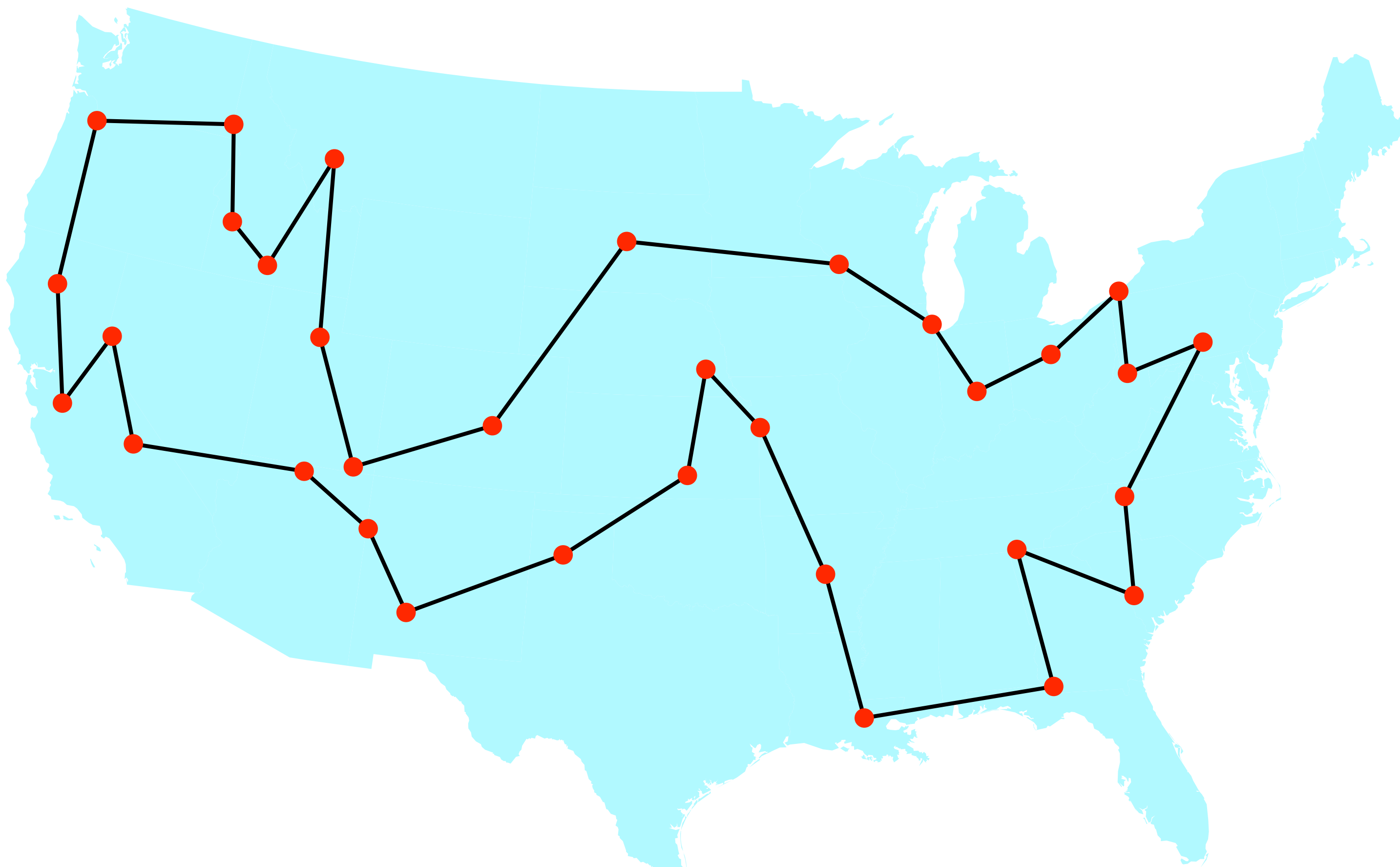
Cutting stock

Pack items into as few identical knapsacks as possible:
Used in steel, paper industry)



Traveling Salesman Problem

TSP: Minimize distance traveled while visiting a collection of cities and returning to the starting point.



33-city TSP instance from a 1962 Procter and Gamble competition ($10,000 prize won by Gerald Thompson of CMU)

HELP! WE'RE LOST!
HELP "CAR 54"...AND WIN CASH
54...$1,000 PRIZES
ONE...$10,000 GRAND PRIZE
START and FINISH
Lewiston
Portland
Butte
Boise
Twin Falls
Blunt
La Crosse
Chicago
Marion
Erie
Carlisle
Redding
Reno
Salt Lake City
Lincoln
Indianapolis
Wana
Kansas City
Gustine
Lone Pine
Mexican Hat
Colorado Springs
Wichita
Chattanooga
Wilkesboro
Marble Canyon
Manuelito
Amarillo
Little Rock
Barnwell
Truth or Consequences
Baton Rouge
Bainbridge
Map by Rand McNally
Help Toody and Muldoon find the shortest round trip route to visit all 33 locations shown on the map.
All you do is draw connecting straight lines from location to location to show the shortest round trip route.
HERE'S THE CORRECT START...
Begin at Chicago, Illinois. From there, lines show correct route as far as Erie, Pennsylvania. Next, do you go to Carlisle, Pennsylvania or Wana, West Virginia? Check the easy instructions on back of this entry blank for details.

10-city instance

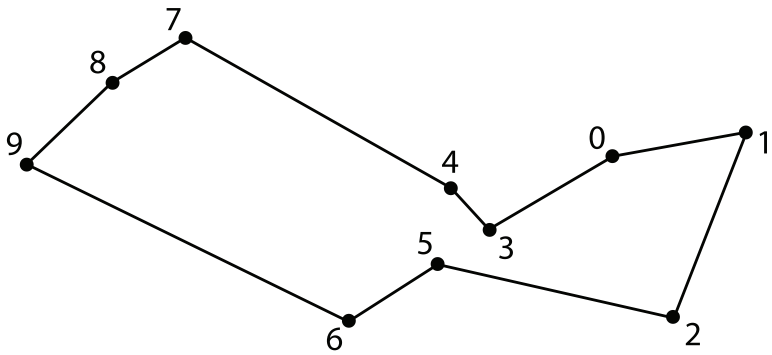
$(n-1)! = 362,880$ possible tours

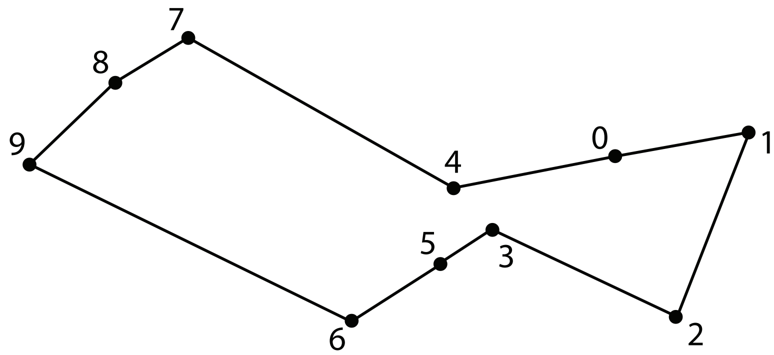
10-city instance

	0	1	2	3	4	5	6	7	8	9
0 Chicago	0									
1 Erie	449	0								
2 Chattanooga	618	787	0							
3 Kansas City	504	937	722	0						
4 Lincoln	529	1004	950	219	0					
5 Wichita	805	1132	842	195	256	0				
6 Amarillo	1181	1441	1080	563	624	368	0			
7 Butte	1538	2045	2078	1378	1229	1382	1319	0		
8 Boise	1716	2165	2217	1422	1244	1375	1262	483	0	
9 Reno	2065	2514	2355	1673	1570	1507	1320	842	432	0

10-city instance: solutions

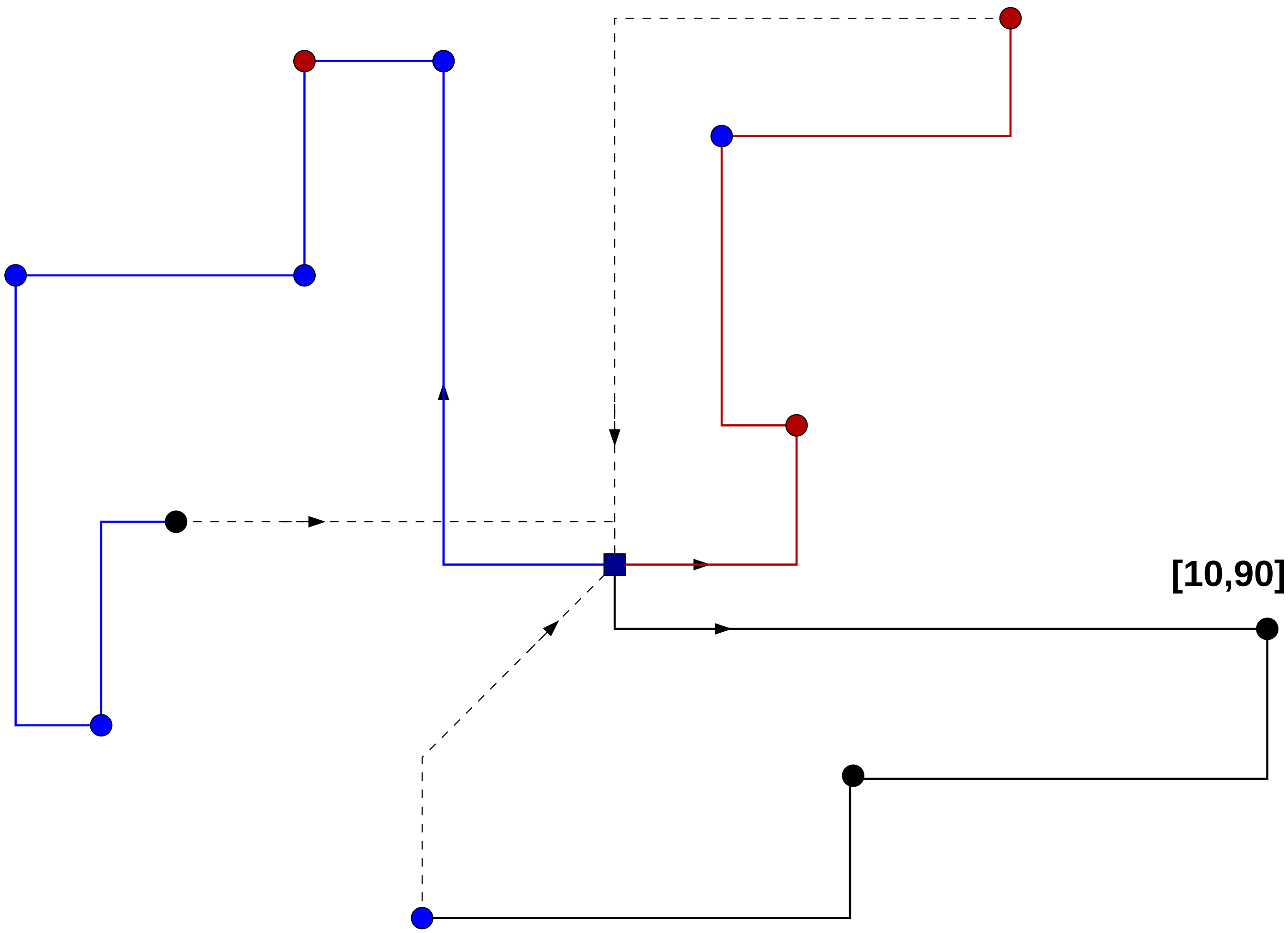
Tours of length 6633 and <u>6514</u> miles





Shortest tour: 0, 1, 2, 3, 5, 6, 9, 8, 7, 4
Shortest tour length: 6514

Vehicle Routing



Minimize distance traveled by trucks at a depot delivering to a set of customers within prescribed time windows (used in package delivery by Fedex, USPS etc.)

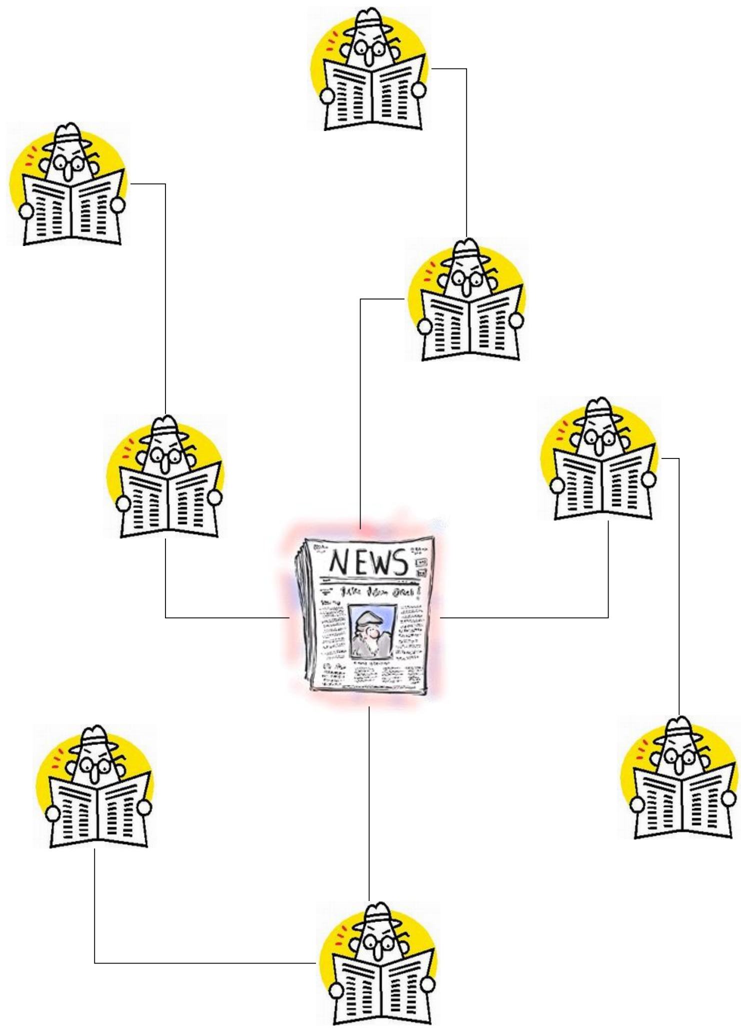
2014 survey in OR-MS Magazine lists 15+ vendors of VRP software.

Min-max vehicle routing

1 

4 

120 



1996 Whizzkids challenge

▷ 5000 Dutch Guilders prize sponsored by CMG

▷ Winners: Hemel, van Erk, Jenniskens (U. Eindhoven students)
▷ Max path length of 1183
▷ Local search techniques, 15,000 hours of computing time.

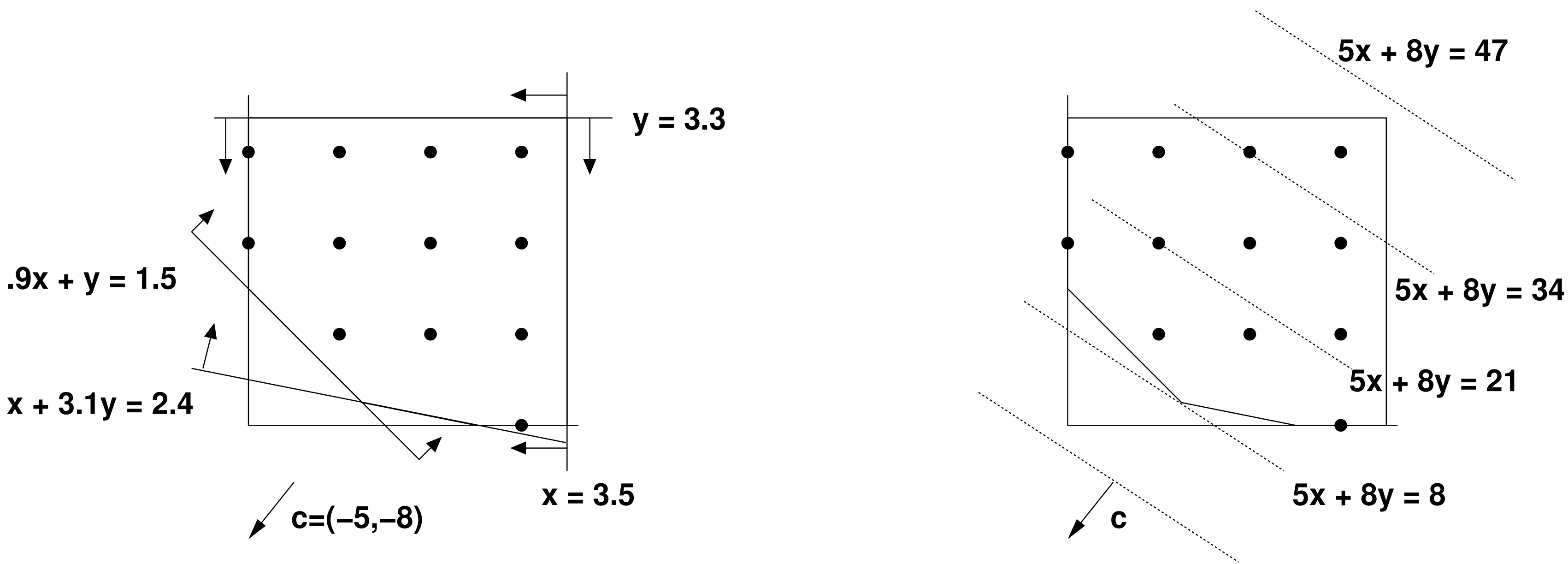
Optimal solution? Lower bound of 1160 given by Hurkens '97.

Integer programming

$\min 5x + 8y$ subject to

$.9x + y \geq 1.5, \; x + 3.1y \geq 2.4$

$0 \leq x \leq 3.5, \; 0 \leq y \leq 3.3, \; x, y$ integral

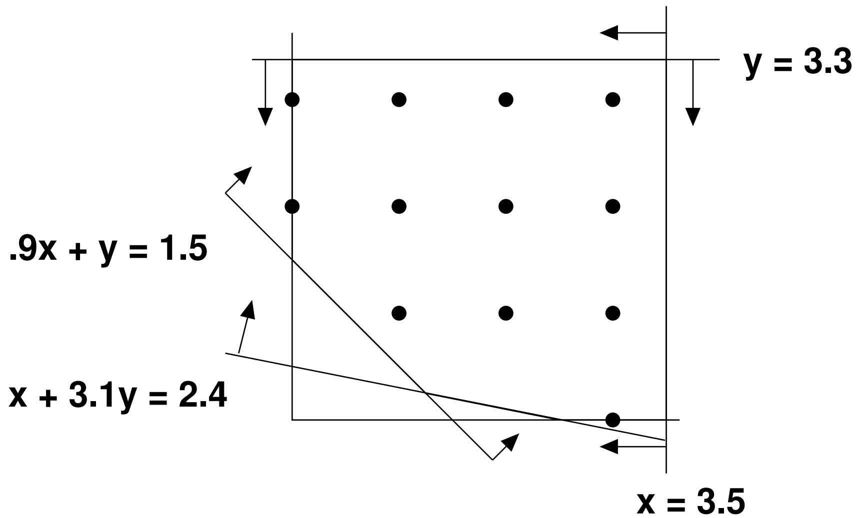


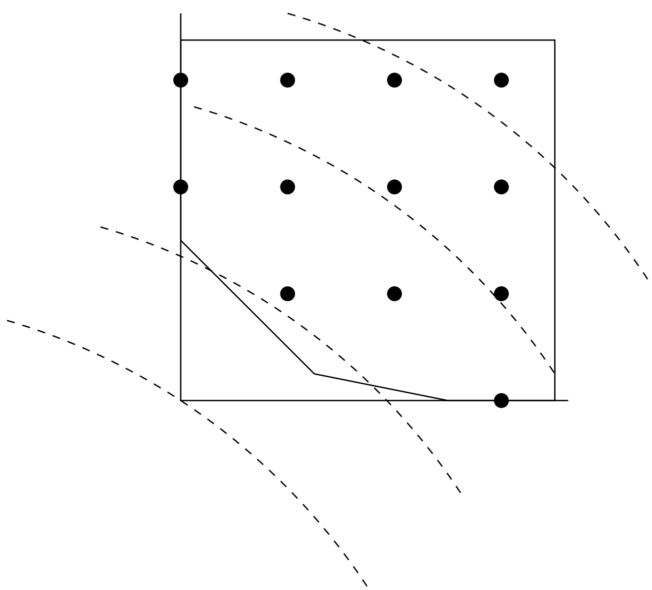
Integer Quadratic programming

$$\min 2x^2 + 5y^2 + 6xy + 3x \text{ subject to}$$

$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$

$$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3, \;\; x, y \;\; \text{integral}$$

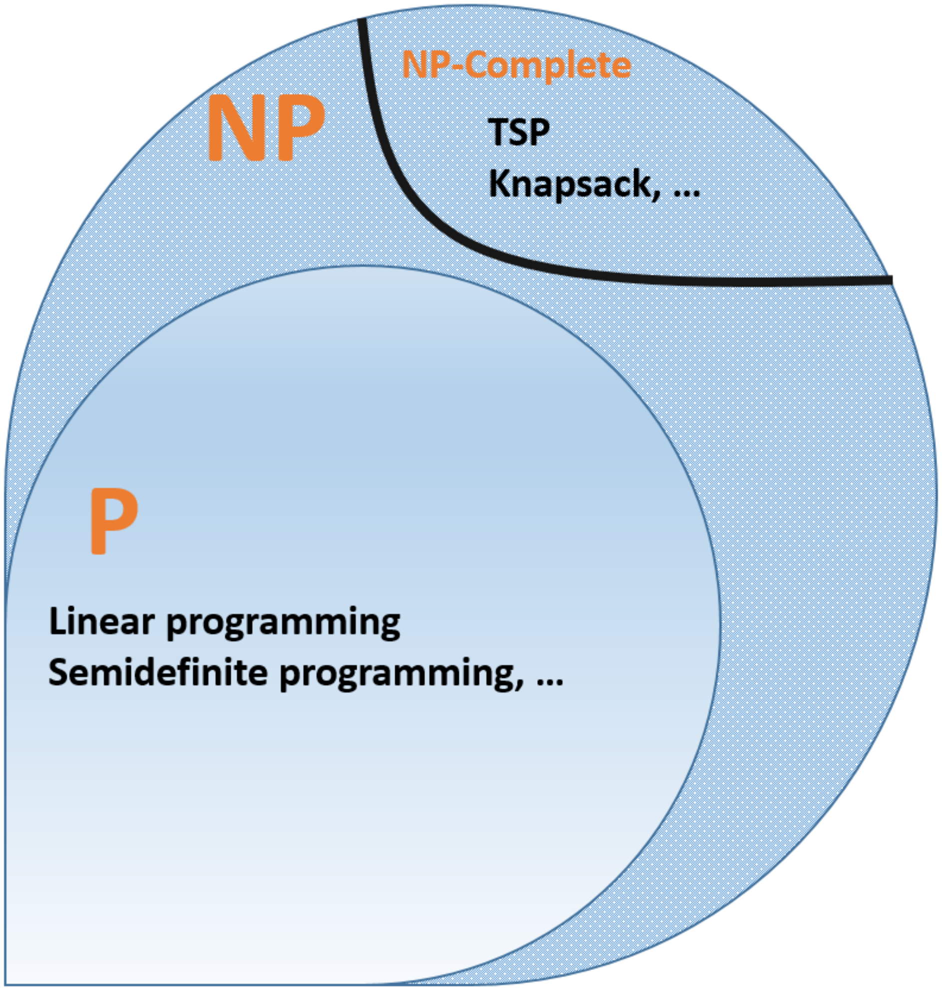




Computational Complexity

NP-completeness

The problem of determining if there exists a TSP tour of length less than k is NP-complete.



Running time growth

- ▷ Traveling salesman problem: $O(n^2 2^n)$ algorithm by Held and Karp
- ▷ Shortest path problem: $O(n^2)$ algorithm by Djikstra

function	5	10	30	64
n^2	25	100	900	4096
$n^2 \log n$	58.0	332.2	$4,416.2$	$24,576$
2^n	32	1024	$1,073,741,824$	$18,446,744,073,709,551,616$
1.1^n	1.6	2.6	17.4	445.8

Important: For real-life applications, the data/problem size are restricted.

Time taken by Pisinger's MINKNAP algorithm on knapsack instances with n items and item weights chosen uniformly at random from $1, \ldots, R$.

	uncorrelated			strongly correlated		
n/R	100	1000	10000	100	1000	10000
100	.002	.002	.002	.002	.002	.076
1000	.002	.002	.003	.019	.078	.172
10000	.004	.005	.010	.050	1.19	25.2

Formulations

0-1 Knapsack formulations

Profits p_i and weights w_i are assumed to nonnegative

integer program:

$$\begin{array}{ll} \text{Maximize} & p_1x_1 + p_2x_2 + \ldots + p_nx_n \\ \text{s.t.} & w_1x_1 + w_2x_2 + \ldots w_nx_n \leq c \\ & x_1, x_2, \ldots, x_n \in \{0, 1\}. \end{array}$$

For *unbounded knapsack* replace $\{0, 1\}$ by $\{\text{integers}\}$ above.

nonlinear integer program:

$$\begin{array}{ll} \text{Maximize} & p_1x_1 + p_2x_2 + \ldots + p_nx_n \\ \text{s.t.} & w_1x_1^2 + w_2x_2^2 + \ldots w_nx_n^2 \leq c \\ & x_1, x_2, \ldots, x_n \in \{0, 1\}. \end{array}$$

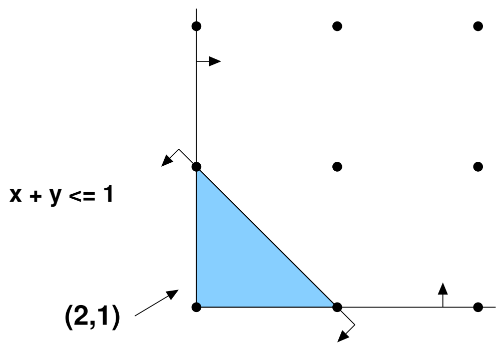
0-1 Knapsack relaxations

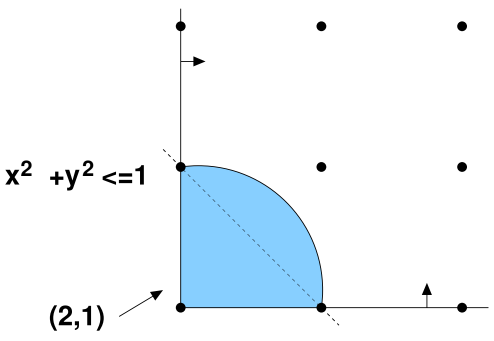
$$\begin{aligned} \text{Maximize} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \\ & x_1, x_2 \in \{0, 1\}. \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1, x_2 \in \{0, 1\}. \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1 \\ & x_1, x_2 \in [0, 1]. \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1, x_2 \in [0, 1]. \end{aligned}$$

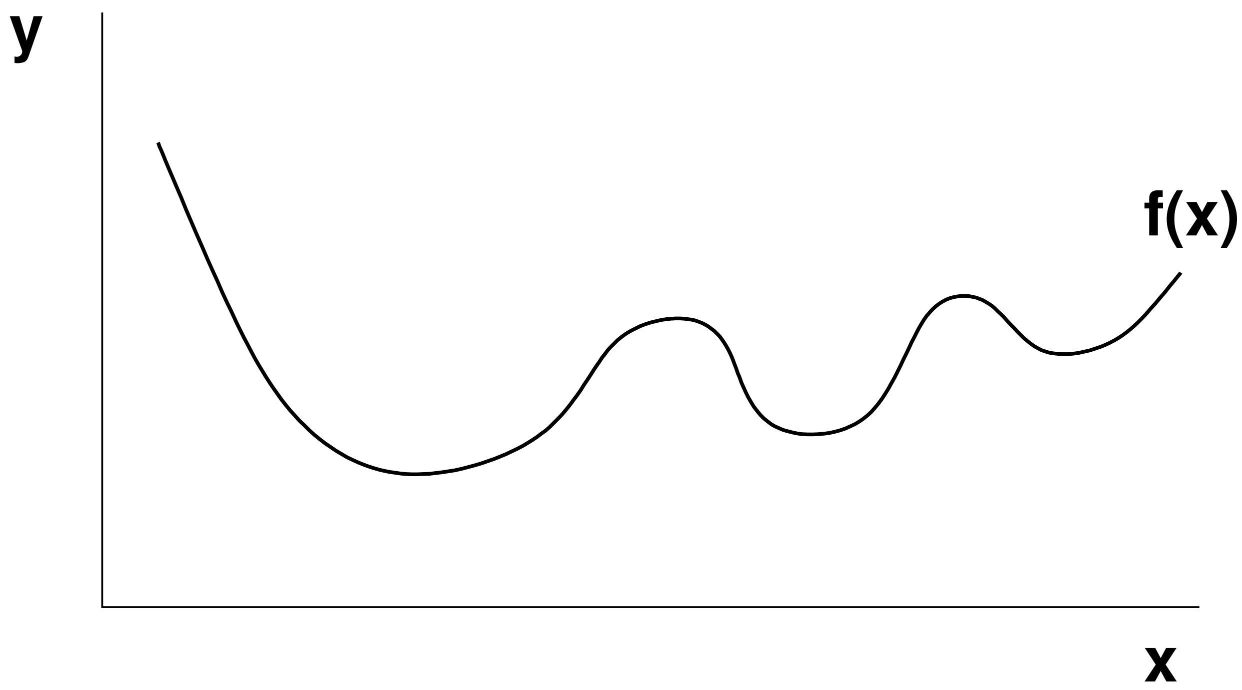




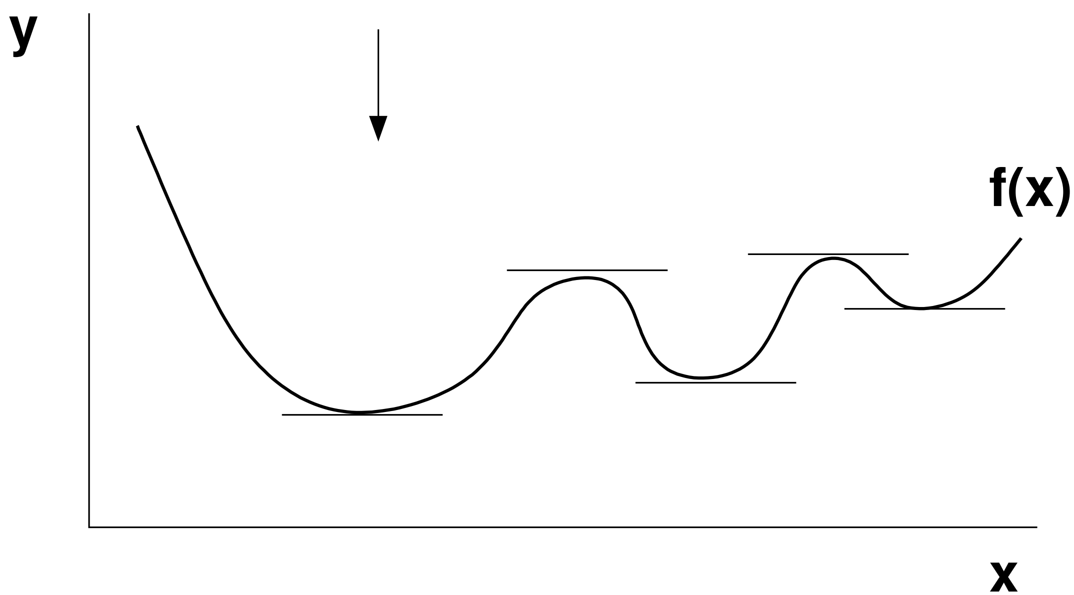
Solution techniques

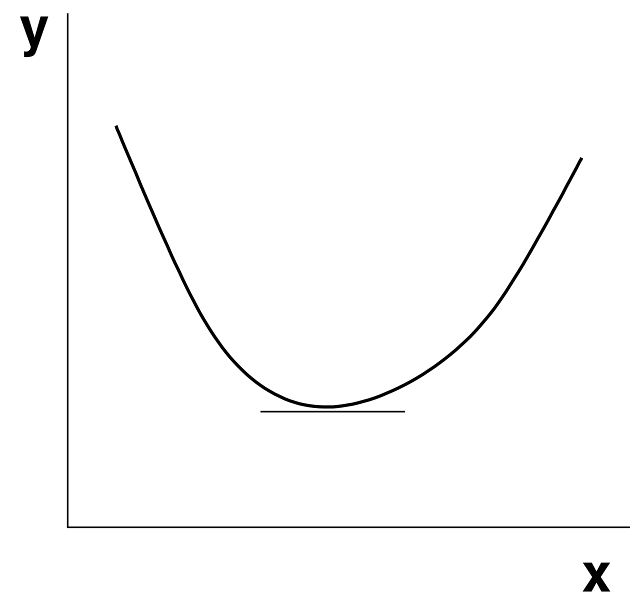
Basic optimization

Minimize $f(x)$ for x in some domain



Optimality conditions





Necessary condition for optimality of x is $f'(x) = 0$. $f''(x) > 0$ is sufficient condition for local optimality. For convex functions, first condition is sufficient.

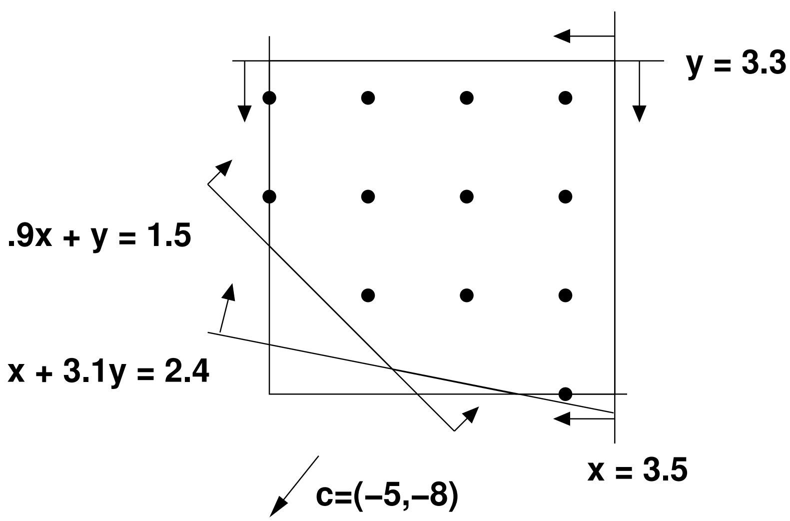
For constrained optimization, KKT conditions are necessary (Kuhn, Tucker '54, Karush '39).

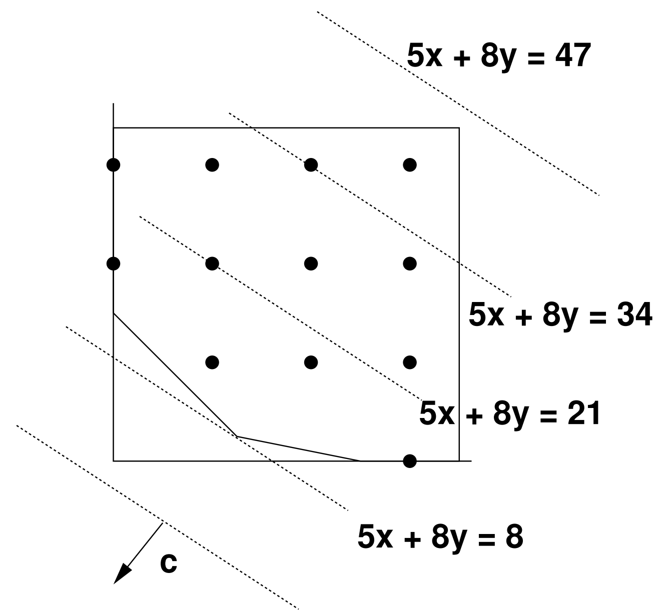
Integer programming

$\min 5x + 8y$ subject to

$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$

$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3, \;\; x, y \;$ integral



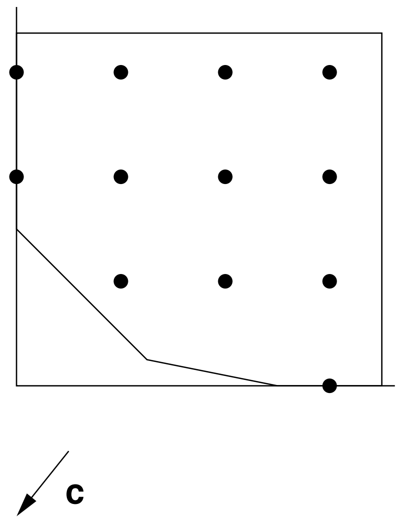


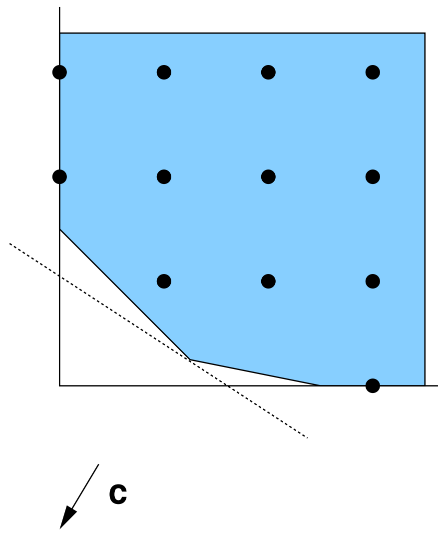
LP relaxation

$$\min 5x + 8y \text{ subject to}$$

$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$

$$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3$$





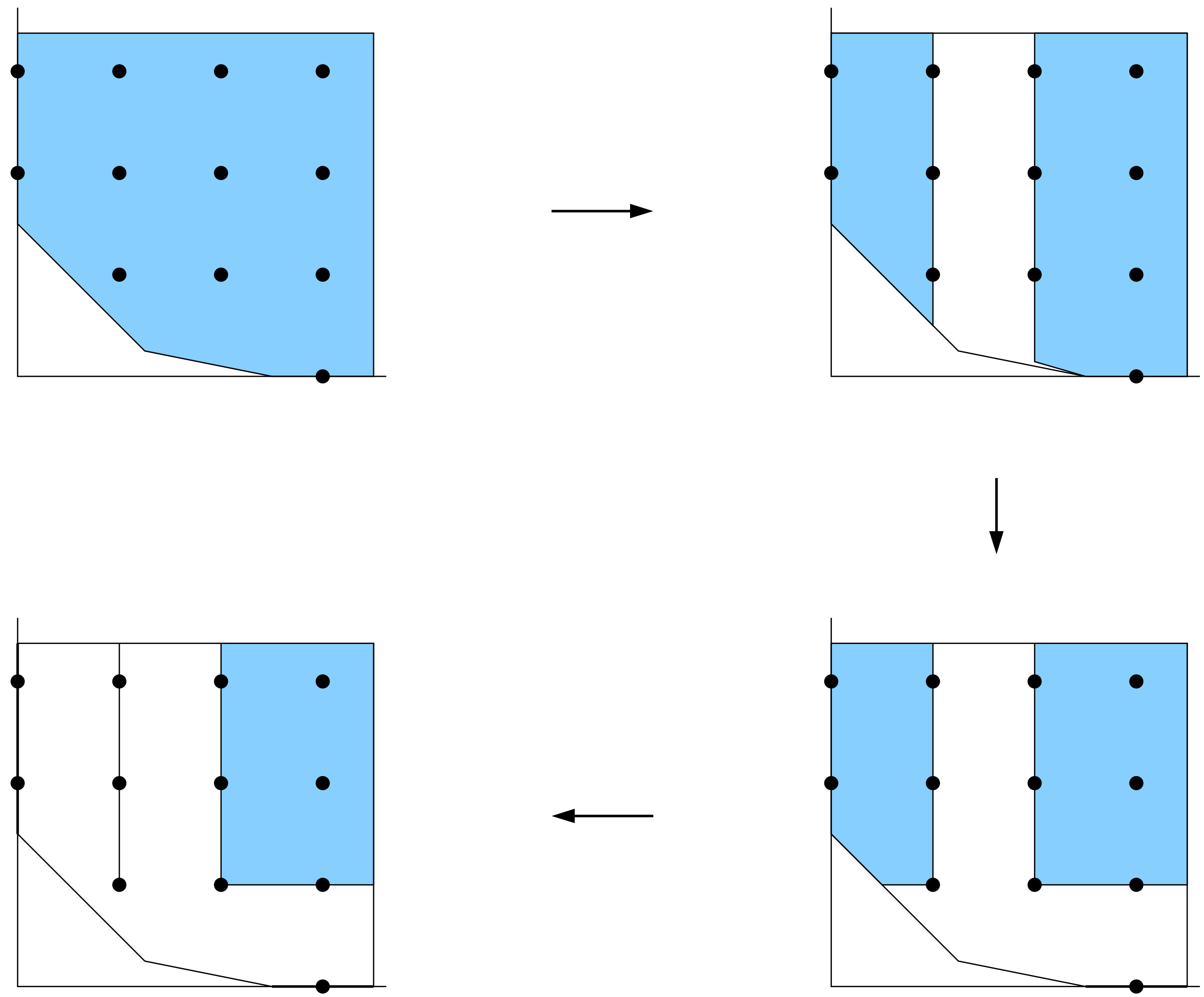
LP relaxation + branching

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5,\ \ x + 3.1y \geq 2.4$$
$$0 \leq x \leq 3.5,\ \ 0 \leq y \leq 3.3$$

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5,\ \ x + 3.1y \geq 2.4$$
$$0 \leq x \leq 1,\ \ 0 \leq y \leq 3.3$$

$$\min 5x + 8y \text{ subject to}$$
$$.9x + y \geq 1.5,\ \ x + 3.1y \geq 2.4$$
$$2 \leq x \leq 3.5,\ \ 0 \leq y \leq 3.3$$

Branch and bound



cplex-log2.txt

```
Problem 'pp08a' read.
....
Reduced MIP has 133 rows, 234 columns, and 468 nonzeros.
Reduced MIP has 64 binaries, 0 generals, 0 SOSs, and 0 indicators.
....
        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                      27080.0000                     77      ---
      0     0     2748.3452    51  27080.0000     2748.3452       77   89.85%
*     0+    0                      14300.0000     2748.3452       77   80.78%
*     0+    0                       7950.0000     2748.3452       77   65.43%
      0     2     2748.3452    51   7950.0000     2748.3452       77   65.43%
Elapsed real time =   0.03 sec. (tree size =  0.00 MB, solutions = 3)
*   100+   94                       7860.0000     2848.3452      428   63.76%
*   100+   90                       7640.0000     2848.3452      428   62.72%
   2862  2111     6556.5595    28   7640.0000     3981.3452     9387   47.89%
   6557  5339     6788.4524    21   7640.0000     4254.2976    20447   44.32%
* 10017+ 8320                       7630.0000     4369.3452    30879   42.73%
* 10017+ 8067                       7520.0000     4369.3452    30879   41.90%
* 10017+ 8047                       7510.0000     4369.3452    30879   41.82%
* 10017+ 7947                       7480.0000     4369.3452    30879   41.59%
  10017  7949     7152.1667    16   7480.0000     4369.3452    30879   41.59%
....
 467260 381944    6279.9524    23    7480.0000    5330.2500  1336479   28.74%
Elapsed real time =  76.80 sec. (tree size = 86.82 MB, solutions = 9)
 488008 398616    6870.4881    16    7480.0000    5340.1310  1393871   28.61%
 508767 415262    7018.3810    21    7480.0000    5350.3452  1451784   28.47%
 529510 431893    5359.7738    26    7480.0000    5359.7738  1509653   28.35%
 550267 448498    5819.7024    30    7480.0000    5368.3929  1567040   28.23%
 570955 465047    7091.7738    13    7480.0000    5377.4405  1624524   28.11%
....
 760995 616110    6726.4405    24    7480.0000    5445.6548  2152219   27.20%
 778020 629628    6542.1548    30    7480.0000    5451.3214  2199840   27.12%
 794094 642371    6215.4881    25    7480.0000    5456.2024  2244463   27.06%
 811975 656559       cutoff          7480.0000    5461.4405  2294026   26.99%
 829297 670288    6740.9167    28    7480.0000    5466.6786  2342402   26.92%
 846366 683716    6716.6786    22    7480.0000    5471.6786  2389544   26.85%
Elapsed real time = 143.55 sec. (tree size = 155.11 MB, solutions = 9)
```

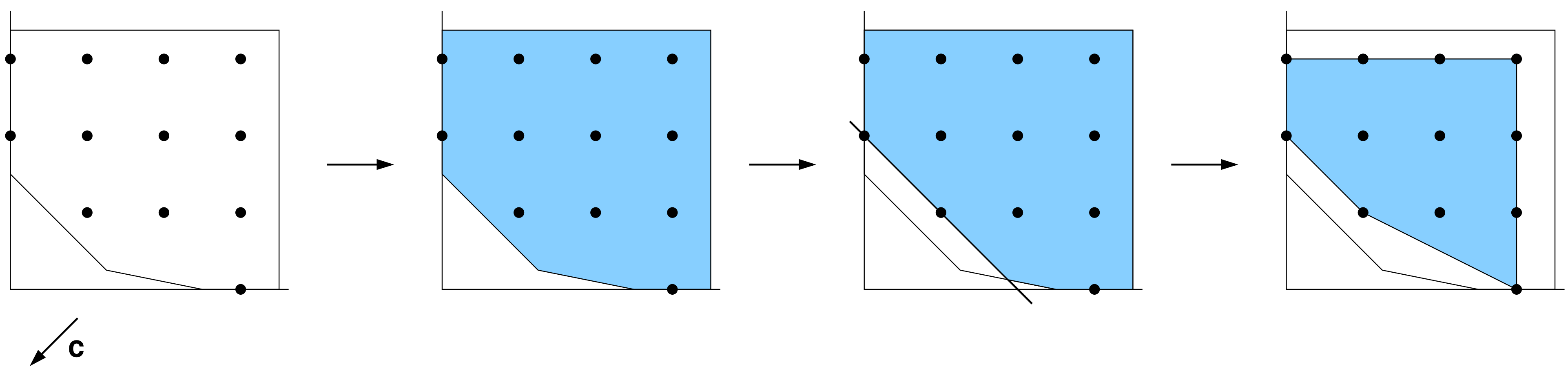
Cutting planes

cutting plane: an inequality satisfied by integral solutions of linear inequalities.

$$\min 5x + 8y \text{ subject to}$$

$$.9x + y \geq 1.5, \;\; x + 3.1y \geq 2.4$$

$$0 \leq x \leq 3.5, \;\; 0 \leq y \leq 3.3, \;\; x, y \;\; \text{integral}$$



Gomory-Chvátal cutting planes (cuts)

$$x \leq 3.5 \Rightarrow x \leq 3$$
$$y \leq 3.3 \Rightarrow y \leq 3$$

$$(.9x + y \geq 1.5) + (.1x \geq 0) \rightarrow$$
$$x + y \geq 1.5 \Rightarrow x + y \geq 2$$

$$(x + y \geq 2) \times .6 + (x + 3.1y \geq 2.4) \times .4 \rightarrow$$
$$x + 1.84y \geq 2.16 \rightarrow$$
$$x + 2y \geq 2.16 \Rightarrow x + 2y \geq 3.$$

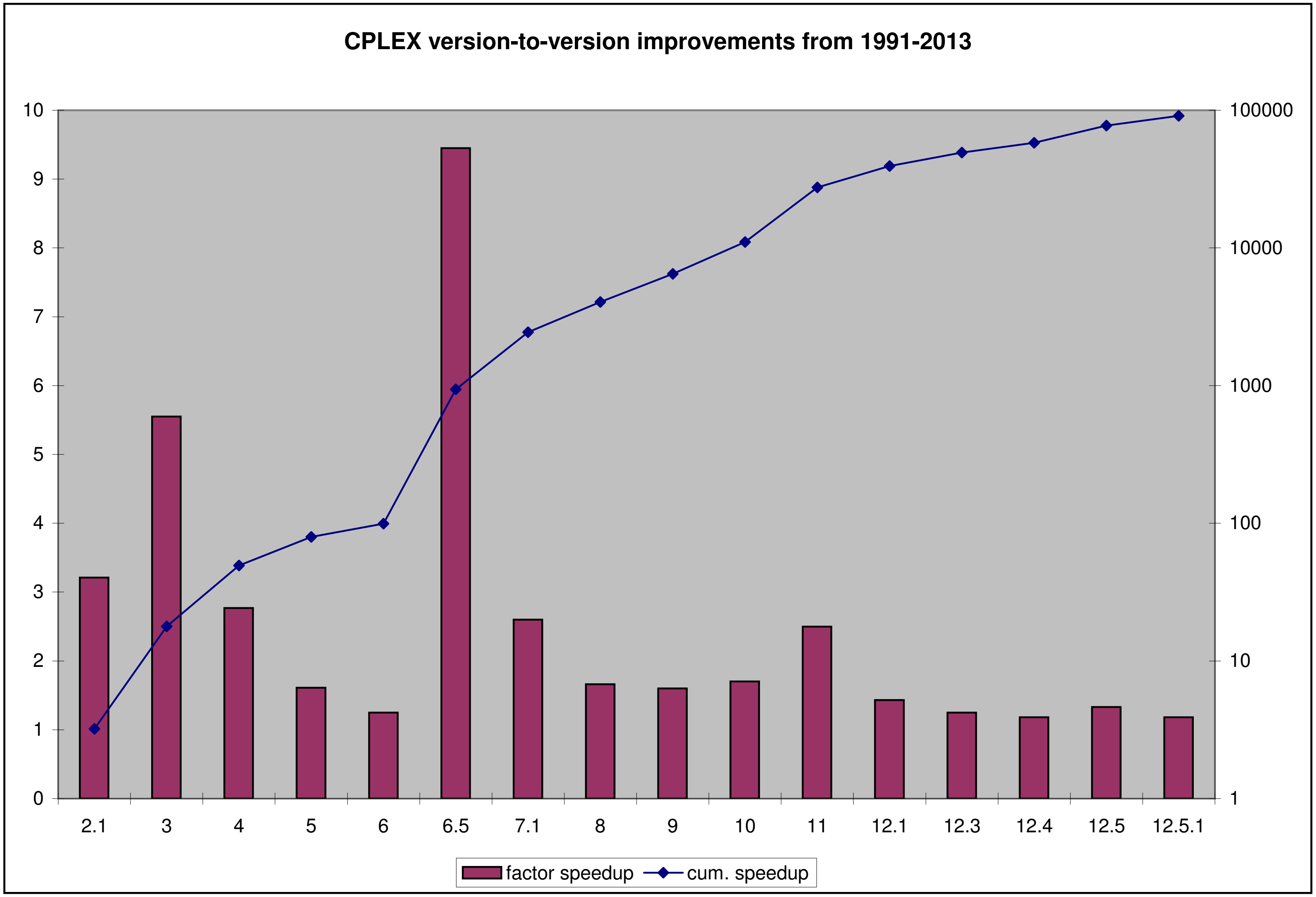
Every integer program can be solved by Gomory-Chvátal cuts (Gomory ’60), though it may take exponential time in the worst case (Pudlák ’97).

cplex-log.txt

```
Problem 'pp08a' read.
....
Reduced MIP has 133 rows, 234 columns, and 468 nonzeros.
Reduced MIP has 64 binaries, 0 generals, 0 SOSs, and 0 indicators.
....
        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Bound    ItCnt     Gap

*     0+    0                      27080.0000                     77      ---
      0     0     2748.3452    51  27080.0000     2748.3452       77   89.85%
*     0+    0                      14300.0000     2748.3452       77   80.78%
      0     0     5046.0422    48  14300.0000     Cuts: 133      153   64.71%
      0     0     6749.5837    24  14300.0000     Cuts: 130      265   52.80%
*     0+    0                      10650.0000     6749.5837      265   36.62%
      0     0     7099.1233    27  10650.0000      Cuts: 53      327   33.34%
      0     0     7171.1837    28  10650.0000      Cuts: 35      356   32.66%
*     0+    0                       7540.0000     7171.1837      356    4.89%
      0     0     7176.2716    31   7540.0000      Cuts: 19      370    4.82%
      0     0     7187.8155    33   7540.0000      Cuts: 20      388    4.67%
      0     0     7188.4198    28   7540.0000       Cuts: 4      398    4.66%
      0     0     7189.5182    30   7540.0000       Cuts: 9      409    4.65%
      0     0     7189.5877    30   7540.0000   Flowcuts: 5      413    4.65%
      0     0     7189.9535    26   7540.0000   Flowcuts: 2      420    4.64%
      0     2     7189.9535    26   7540.0000     7190.0161      420    4.64%
Elapsed real time =   0.27 sec. (tree size =  0.00 MB, solutions = 4)
*    50+   40                       7530.0000     7218.8496     1733    4.13%
*    55    44      integral     0   7520.0000     7218.8496     1783    4.00%
*    60+   45                       7490.0000     7218.8496     1892    3.62%
*    60+   38                       7420.0000     7218.8496     1892    2.71%
*   110+   53                       7400.0000     7238.6753     2712    2.18%
*   210    64      integral     0   7350.0000     7255.3139     4760    1.29%

Implied bound cuts applied:  1
Flow cuts applied:  149
Flow path cuts applied:  23
Multi commodity flow cuts applied:  5
Gomory fractional cuts applied:  34
....
Total (root+branch&cut) =    0.95 sec.
```


CPLEX version-to-version improvements from 1991-2013
10
9
8
7
6
5
4
3
2
1
0
100000
10000
1000
100
10
1
2.1
3
4
5
6
6.5
7.1
8
9
10
11
12.1
12.3
12.4
12.5
12.5.1
factor speedup
cum. speedup

Applications

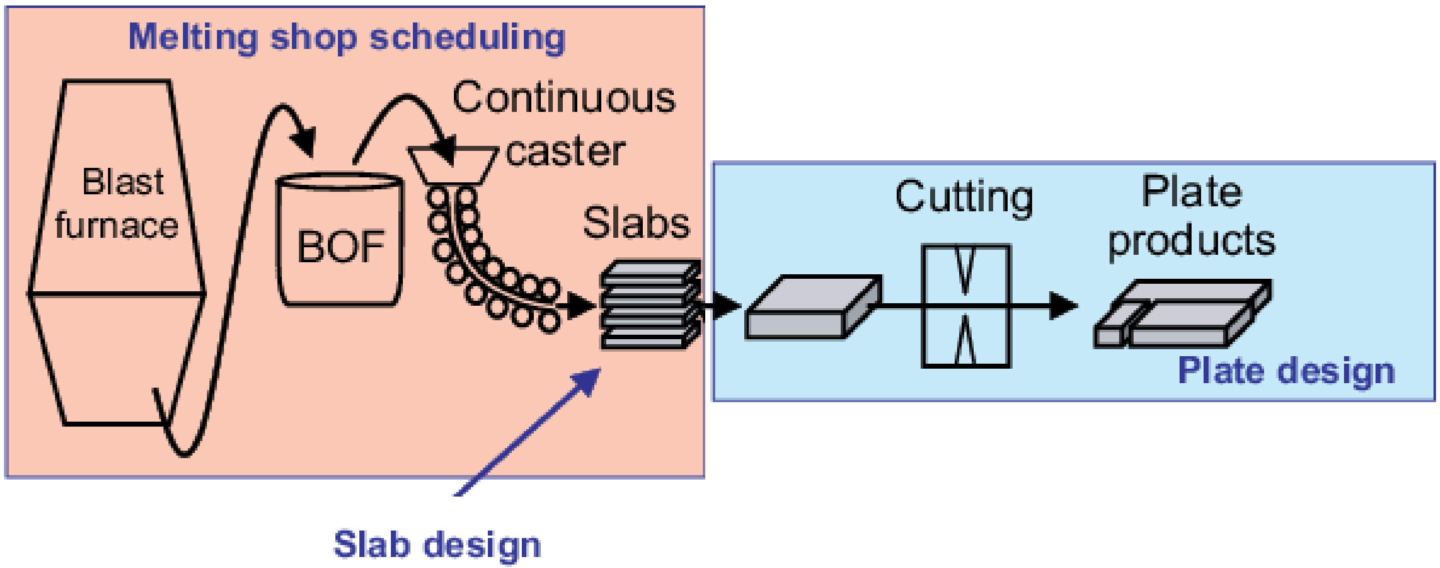
Steel industry application

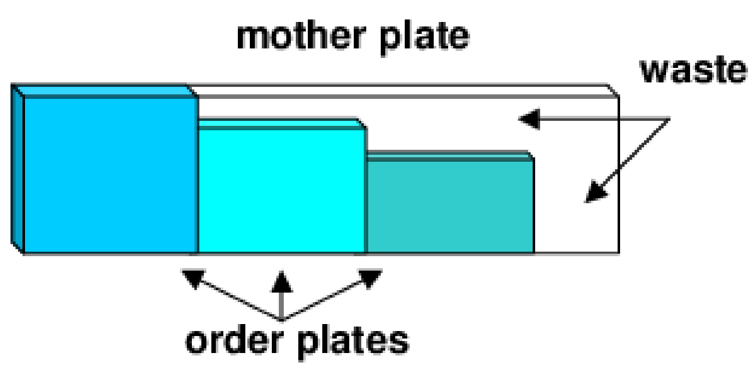
Context: Large steel plant (3 million tons of plates/year $\approx$ 10,000 tons/day) in East Asia moving from a producer-centric model to a customer-centric model

Goal: Optimization tool to generate a production design – a detailed desciption of production steps and related intermediate products

Timeline: 1.5 years
(5 man years on optimization, 25 man years on databases/GUI/analysis)
(joint work with J. Kalagnanam, C. Reddy, M. Trumbo)

Manufacturing process





Consulting Issues

◇ 2+ research man years spent defining problem (high complexity)
- Very large number of constraints including objectives masked as constraints
- 500+ pages of specifications: scope of problem not known at contract signing

◇ High level problem has non-linearities

◇ Software/data issues - 1000+ files

◇ 30 minutes of computing time allowed
- 100+ complex cutting stock problems with up to 2000 orders solved via integer programming column generation

Pipeline management

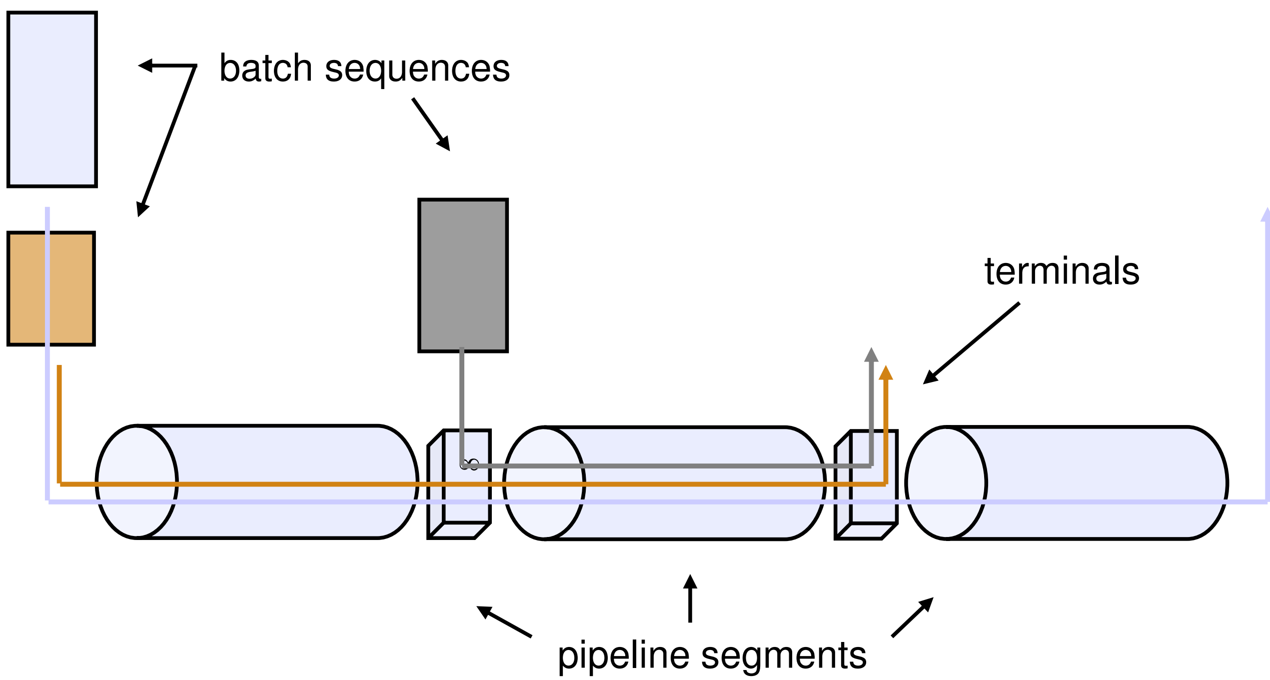
Schedule injections of batches of oil on a pipeline network while minimizing interface costs, delays, and power costs and satisfying tank constraints

(joint work with V. Austel, O. Günlük, P. Rimshnick, B. Schieber)

A pipeline network has many pipelines, each with multiple segments, each of which can run at multiple 'natural rates'.

Timeline: 2.5 years (10+ man years on optimization

Inputs to Batch Sequencing Problem



Interface cost table

	0	0.9	∞
	1.5	0	2.5
	2.7		0

Batch sequencing

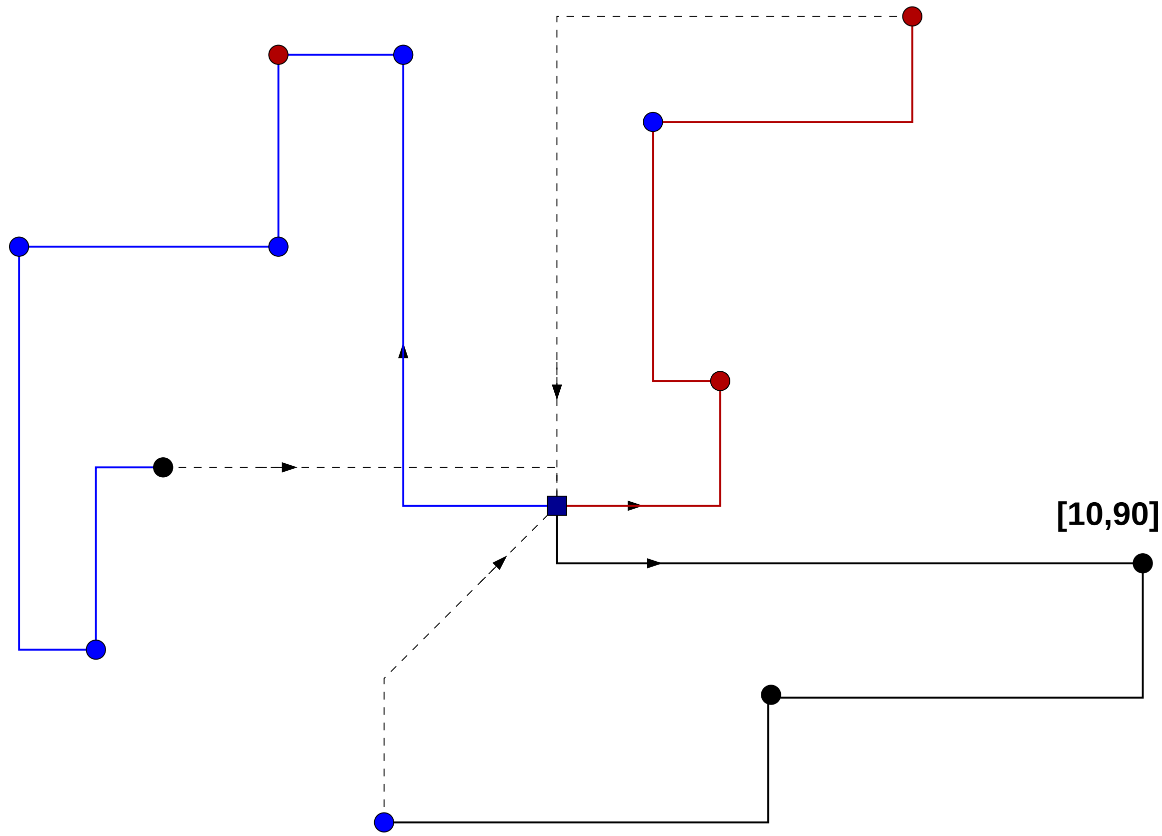
When the pipeline consists of single segment, the cost of a batch sequence depends only on interface costs of adjacent batch pairs: batch sequencing reduces to the Asymmetric TSP problem.

	0	1	2	3	4	5	6	7	8	9
0 Chicago	0									
1 Erie	449	0								
2 Chattanooga	618	787	0							
3 Kansas City	504	937	722	0						
4 Lincoln	529	1004	950	219	0					
5 Wichita	805	1132	842	195	256	0				
⋮										

Vehicle routing application

Context: Food distribution company in North America trying to improve delivery to customers within desired time windows, while minimizing travel costs.

VRPTW with driver preferences

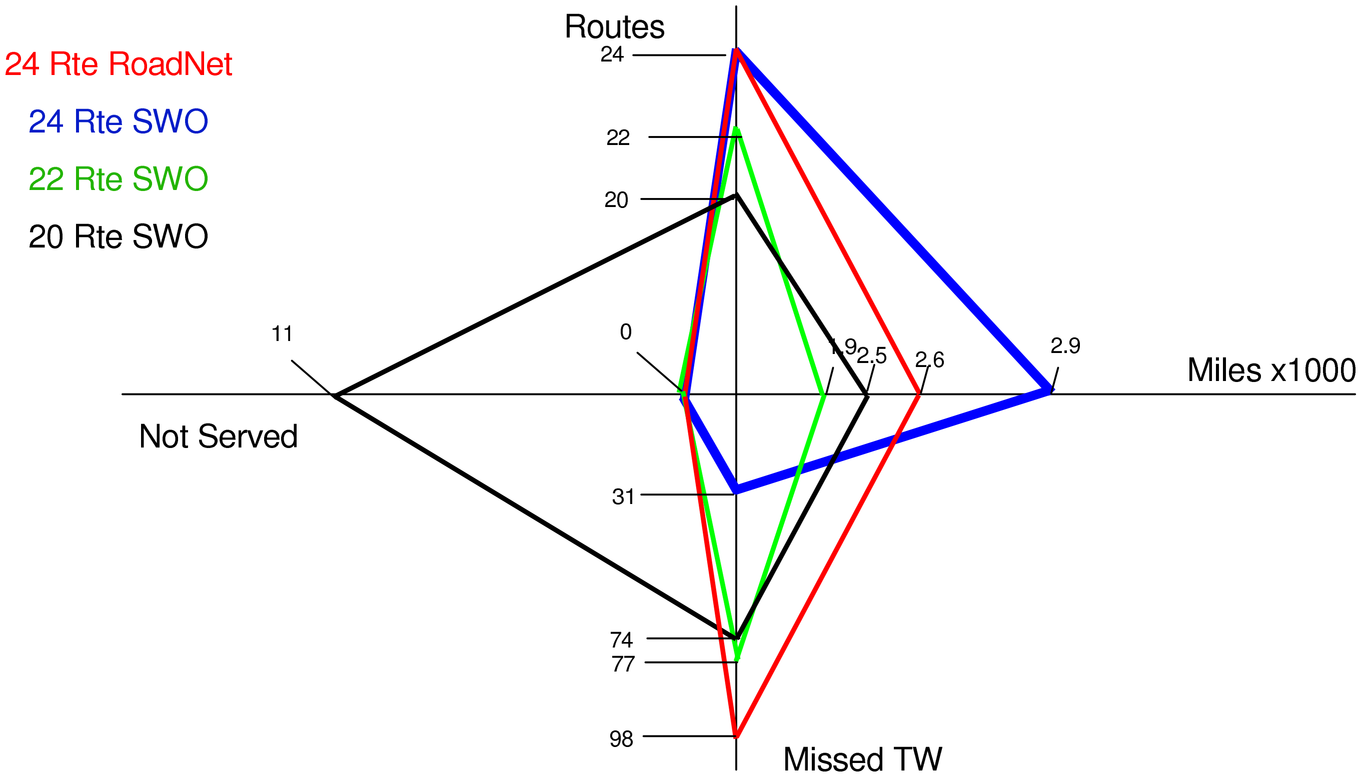


Customers have preferred drivers; penalize for delivery by non-preferred driver.

◇ 200-300 customers, 20-30 routes per shift, 3-6 shifts per day
◇ Create preference relationships between ≈ 200 drivers and 1000 customers
(joint work with O. Günlük, G. Sorkin)


Twix
SN200

Graphic Route Comparison

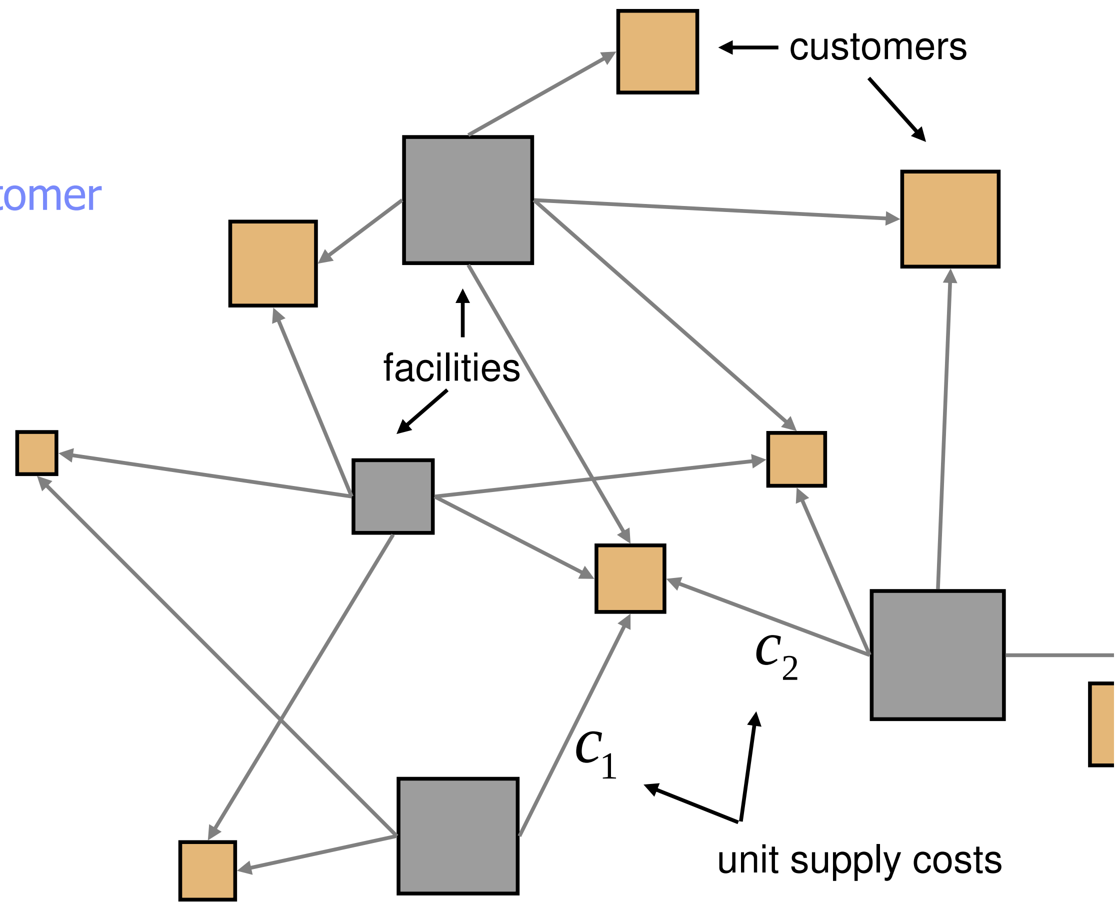


Comparison of route characteristics – Changing Input Parameters and Penalties directly impacts optimizer solution.

Facility location problem

Inputs:
  Facilities + capacities
  Cost of opening each facility
  Customers + demands
  Unit cost of supplying a customer
  from each facility

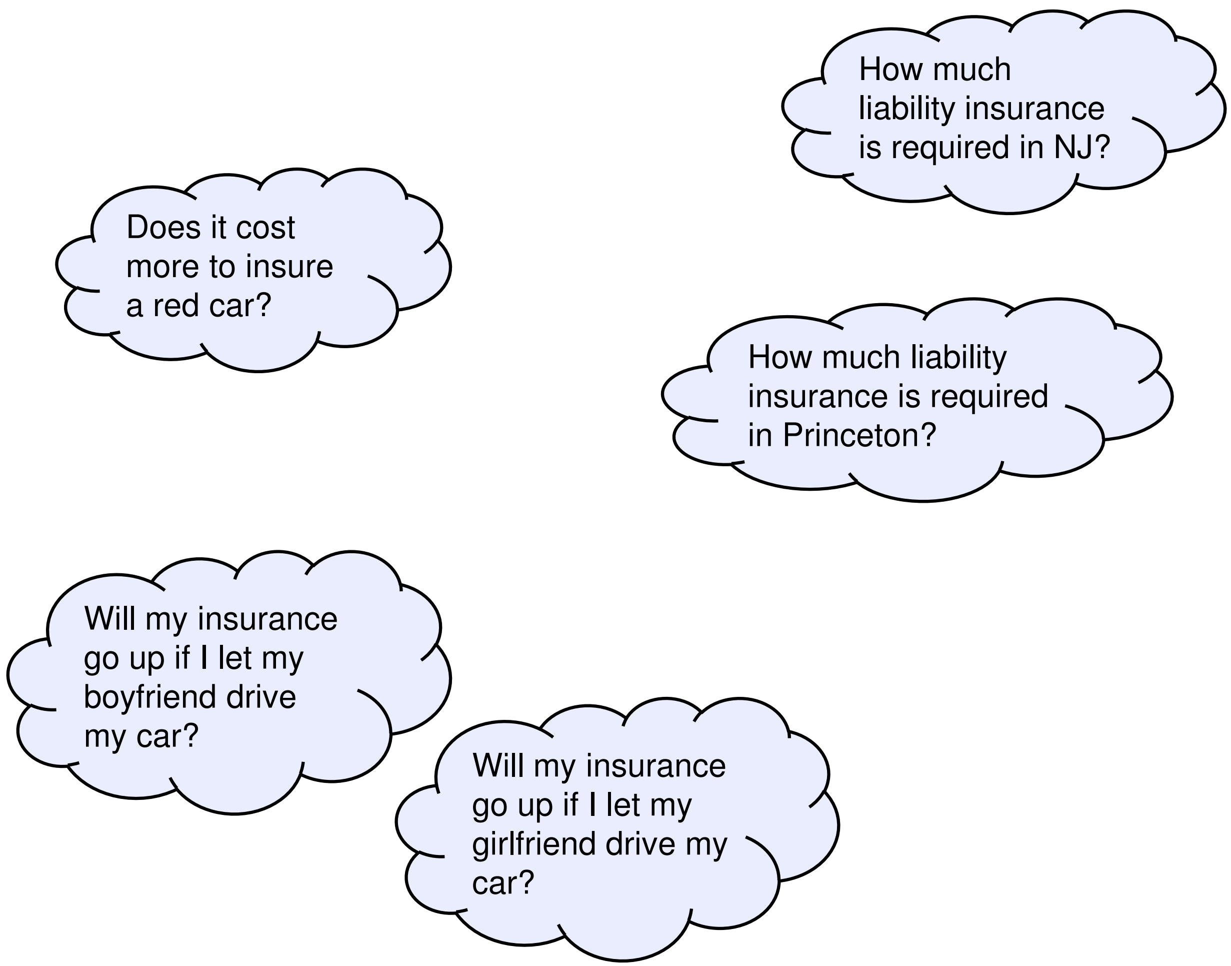
Goal:
  Minimize cost of opening
  facilities to satisfy demand



Related to Fermat-Weber problem

Machine learning application

▷ Insurance company wants to answer a long list of customer questions, but has a budget for only 500 answers (Dmitry Malioutov).



▷ The problem is an “active learning” problem: try to optimize which questions to answer.

▷ Balance “information gain” vs. “diversity” for each answered question.

Inputs:

1) Each node/question has a notion of how much additional information it will add by providing a human answer – this is the node cost.

2) The similarity of each question to other questions: there is no point in answering the same question 20 times, so it’s great to have a diverse set of questions to ask humans to answer.

DWave Quantum Computer

DWave: An adiabatic quantum computer performing "quantum annealing".

▷ A special-purpose analog machine employing "flux qubits" arranged in a Chimera graph structure solving the Ising Model Problem. DWave does not guarantee optimality.

DWave experiments

McGeoch and Wang: Experimental Evaluation of an adiabatic quantum system for combinatorial optimization, *ACM Conference on Computing Frontiers* 2013.

> In horserace terms, QA dominates on the Chimera-structure QUBO problems: at the largest problem size $n = 439$, CPLEX (best among the software solvers), returns comparable results running about 3600 times slower than the hardware. On the W2SAT problems, Blackbox, AK, and TABU

▷ DWave Two takes half a second versus half an hour for CPLEX 12.3. on quadratic unconstrained boolean optimization (QUBO) problems defined on a Chimera graph

New York Times:

MAY 16, 2013, 5:00 AM | 30 Comments

Google Buys a Quantum Computer

By QUENTIN HARDY

quantum physics. Their quantum computer, which performs complex calculations thousands of times faster than existing supercomputers, is expected to be in active use in the third quarter of this year.

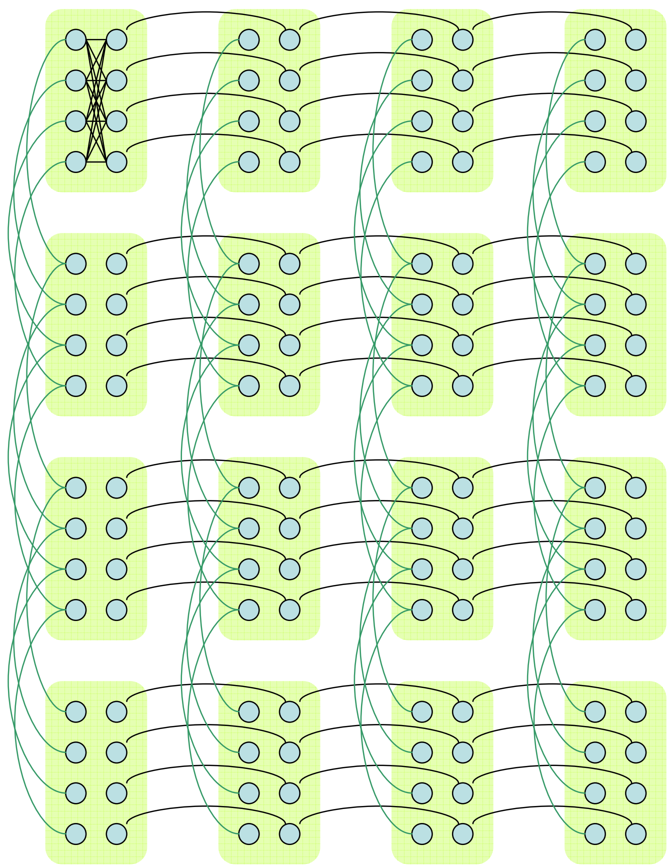
New York Times (Nov 14,2013):

This year, Google and a corporation associated with NASA acquired for study an experimental computer that appears to make use of quantum properties to deliver results sometimes 3,600 times faster than traditional supercomputers. The maker of the quantum computer, D-Wave Systems of Burnaby, British Columbia, counts Mr. Bezos as an investor.

Chimera graphs

Chimera graph C_n: $8n^2$ nodes with n^2 $K_{4,4}$ graphs arranged in a $n \times n$ grid.

C_4



QUBO problem

The Ising model problem is equivalent to the QUBO problem.

▷ The quadratic unconstrained boolean optimization problem (QUBO) problem:Given an $n \times n$ matrix Q:

$$\text{Min} \sum_{i,j} Q_{ij} x_i x_j \text{ subject to } x \in \{0, 1\}^n. \tag{1}$$

QUBO problems

McGeoch and Wang solve QUBO-miqp using the CPLEX MIQP solver

The CPLEX MIQP solver does branch-and-bound based on the QP relaxation:

$$\text{Min} \quad \sum_{i,j} Q_{ij} x_i x_j \quad \text{subject to} \quad x \in [0, 1]^n. \tag{2}$$

QUBO problems

QUBO-milp: Classical mixed-integer linear programming formulation (assume Q is upper triangular):

$$\begin{aligned}
\text{Min } \textstyle\sum_{i<j} Q_{ij} z_{ij} + \sum_{i=1}^{n} Q_{ii} x_i & & (3)\\
\text{subject to} & & \\
x \in \{0,1\}^n, & & (4)\\
z_{ij} \le x_i & \quad \forall i < j, & (5)\\
z_{ij} \le x_j & \quad \forall i < j, & (6)\\
x_i + x_j - z_{ij} \le 1 & \quad \forall i < j, & (7)\\
z_{ij} \ge 0 & \quad \forall i < j. & (8)
\end{aligned}$$

For any fixed i, j, the constraints (5)-(8) are called Fortet inequalities or McCormick inequalities and force z_{ij} to equal $x_i x_j$ when $x_i, x_j \in \{0, 1\}$.

Experiments

▷ CPLEX 12.3/QUBO-MILP takes 93.8 seconds in the worst case and not half an hour: Dash, Puget '14

▷ Simulated annealing heuristic takes 0.02 seconds on 512 node instances: Boixo, Ronnow, Isacker, Wang, Wecker, Lidar, Martinis, Troyer '13.

▷ Specialized heuristic takes 0.01 seconds on 439 node instances: Selby '13.

Conclusions

▷ Many real-life optimization problems can be modeled as instances of NP-hard problems. However, as the data and problem sizes are restricted, such problems can often be solved with customized techniques.

▷ Linear-integer programming is the most widely used optimization tool in practical applications, but some important problems (e.g., portfolio optimization) are modeled as nonlinear (quadratic) integer programs.

▷ Linear constraints are more common in combinatorial problems, whereas nonlinear constraints are more common in systems where the physics is important.