ORF 363/COS 323

Computing and Optimization in the Physical and Social Sciences

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Princeton, ORFE

Lecture 1

Fall 2021

ORF 363 classroom protocols

- Please wear your mask correctly, over your nose and mouth, and keep it on at all-ϵ times.
- Lifting the mask to take sips of a beverage is permitted.
- You can ask questions with a mask or by lifting the mask. I am happy to repeat your question if needed.
- If you don't feel well or have a runny nose, sore throat, etc., please stay home. We will work with you.
- If you test positive and need to isolate, please contact me to confirm arrangements for keeping up with the class.

- Let's take a moment to be grateful ☺ We survived covid and a tornado, and we are finally back together!

What is optimization?

- Roughly, can think of optimization as the science of making the most out of every situation.
- You've probably all done it many times recently:

- What courses to take?
  - To maximize learning.
  - To maximize GPA (?!)
  - Courses can't conflict.
  - Not before 10AM.
  - Professor rating > 4.5.

- What furniture to buy?
  - To minimize cost.
  - To maximize comfort.
  - Must fit in your room.
  - Must have 3 drawers.
  - Not too heavy.

- What path to choose for a run?
  - Minimize (2020 version) or maximize (2019/2021? version) encounter with other runners.
  - Path must be between 5 and 7 miles.
  - Chosen roads must have side walks.
  - Path should get you back home.

- **Common theme:**
  - You make decisions and choose one of many alternatives.
  - You hope to maximize or minimize something (you have an objective).
  - You cannot make arbitrary decisions. Life puts constraints on you.

How is this class different from your every-day optimization?

- We'll be learning techniques for dealing with problems that have
  - Thousands (if not millions) of variables
  - Thousands (if not millions) of constraints
- These problems appear every day in the industry, in science, in engineering
- Hopeless to make decisions in your head and with rules of thumb
- Need mathematical techniques that translate to algorithms
  - Algorithms then get implemented on a computer to solve your optimization problem
- We typically model a physical or social scenario with a precise mathematical description
- In this mathematical model, we care about finding *the best solution*
- Whenever we can't find the best solution, we would like to know how far off our proposed solution is

Examples of optimization problems

In finance

- In what proportions to invest in 500 stocks?
  - To maximize return.
  - To minimize risk.
  - No more than 1/5 of your money in any one stock.
  - Transactions costs < $70.
  - Return rate > 2%.

In control engineering

- How to drive an autonomous vehicle from A to B?
  - To minimize fuel consumption.
  - To minimize travel time.
  - Distance to closest obstacle > 2 meters.
  - Speed < 40 miles/hr.
  - Path needs to be smooth (no sudden changes in direction).

In machine learning

- How to assign likelihoods to transactions being fraudulent?
  - To minimize probability of a false negative.
  - To penalize overfitting on training set.
  - Probability of false positive < .15.
  - Misclassification error on training set < 5%.

Computing and Optimization

- This class will give you a broad introduction to
  "optimization from a computational viewpoint."

- Optimization and computing are very close areas of applied mathematics:
  - For a host of major problems in computer science, the best algorithms currently come from the theory of optimization.
  - Conversely, foundational work by computer scientists has led to a shift of focus in optimization theory from "mathematical analysis" to "computational mathematics."

- Several basic topics in scientific computing (that we'll cover in this course) are either special cases or fundamental ingredients of more elaborate optimization algorithms:
  - Least squares, root finding, solving linear systems, solving linear inequalities, approximation and fitting, matrix factorizations, conjugate gradients,…

Agenda for today

- Meet your teaching staff
- Get your hands dirty with algorithms
    - Game 1
    - Game 2
- Modelling with a mathematical program
    - Fermat’s last theorem!
- Course logistics and expectations

Meet your teaching staff (1/2)

- **Amir Ali Ahmadi** (Amir Ali, or Amirali, is my first name)

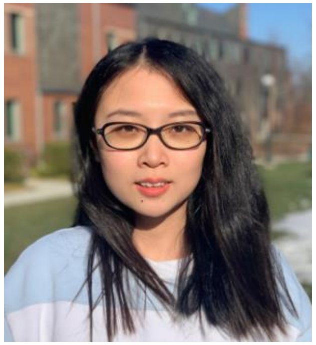
http://aaa.princeton.edu/ aaa@p...

- I am a Professor at ORFE, and affiliated faculty at COS, ECE, MAE, PACM, CSML
- I came to Princeton from MIT, EECS, after a fellowship at IBM Research

- **Cemil Dibek**
- cdibek@p...
- Will defend his PhD thesis in October!
- Will hopefully teach ORF 363 in Spring 2022 and Fall 2022
- Will give a limited number of lectures this semester (math review lectures next week)

Joint office hours: **Tuesdays, 3:15-6:15pm, Sherrerd 122**
(Typically, AAA 3:15-4:45pm, Cemil 4:45-6:15pm)

Meet your teaching staff (2/2)



- **Maxine Yu (AI)** – ORFE grad student
- Office hours: **Mon 5-7pm, Sherrerd 122**
- mengxiny@p...

- **Abraar Chaudhry (AI)** – ORFE grad student
- Office hours: **Mon 7-9pm, Sherrerd 122**
- azc@p...

- **Igor Silin (AI)** – ORFE grad student
- Office hours: **Wed 5-7pm, Sherrerd 122**
- isilin@p...

- **Cole Becker (UCA)** – ECE Senior
- Office hours: **Wed 7-9pm, Sherrerd 122**
- colebecker@p...

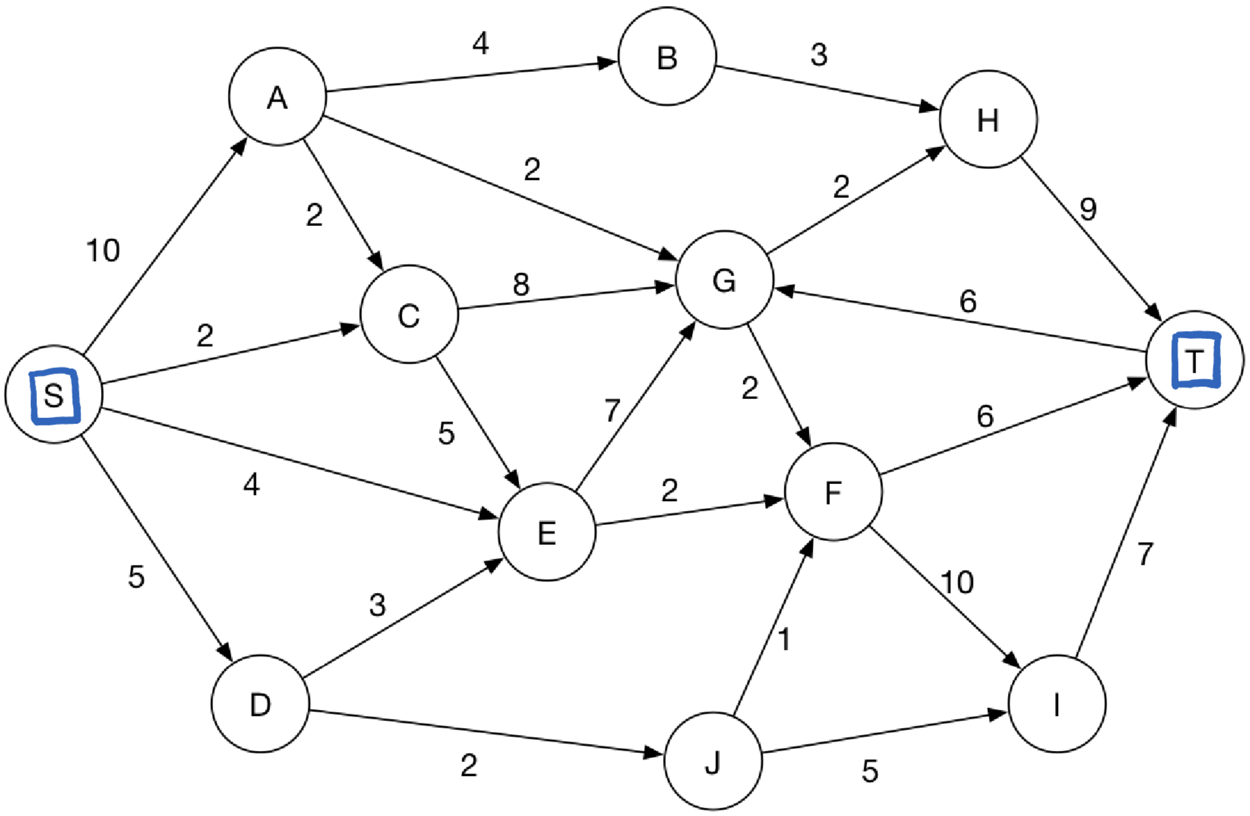
Course website

Course website:
http://aaa.princeton.edu/orf363

- Will have all lectures, problem sets, exams (a copy will also be posted on Canvas)
- Solutions will be posted only on Canvas
- Please set up notification settings on Canvas so you receive our announcements via email
- Please sign up to **Ed Discussion** via Canvas (and use it frequently!)
- Use Ed Discussion instead of email (unless your question is personal)

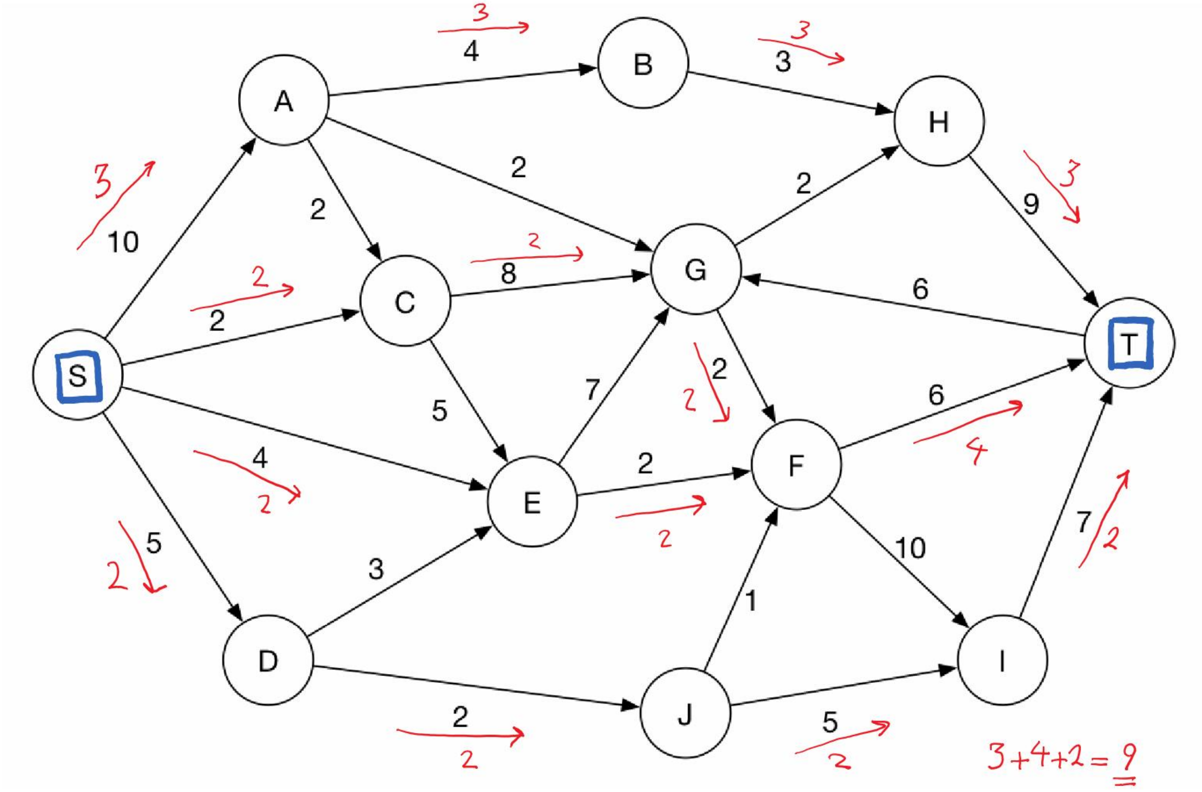
Let’s get to the games!

Let's ship some oil together!



- **Rules of the game:**
  - Cannot exceed capacity on the edges.
  - For each node, except for S ant T, flow in = flow out (i.e., no storage).
  - **Goal:** ship as much oil as you can from S to T.

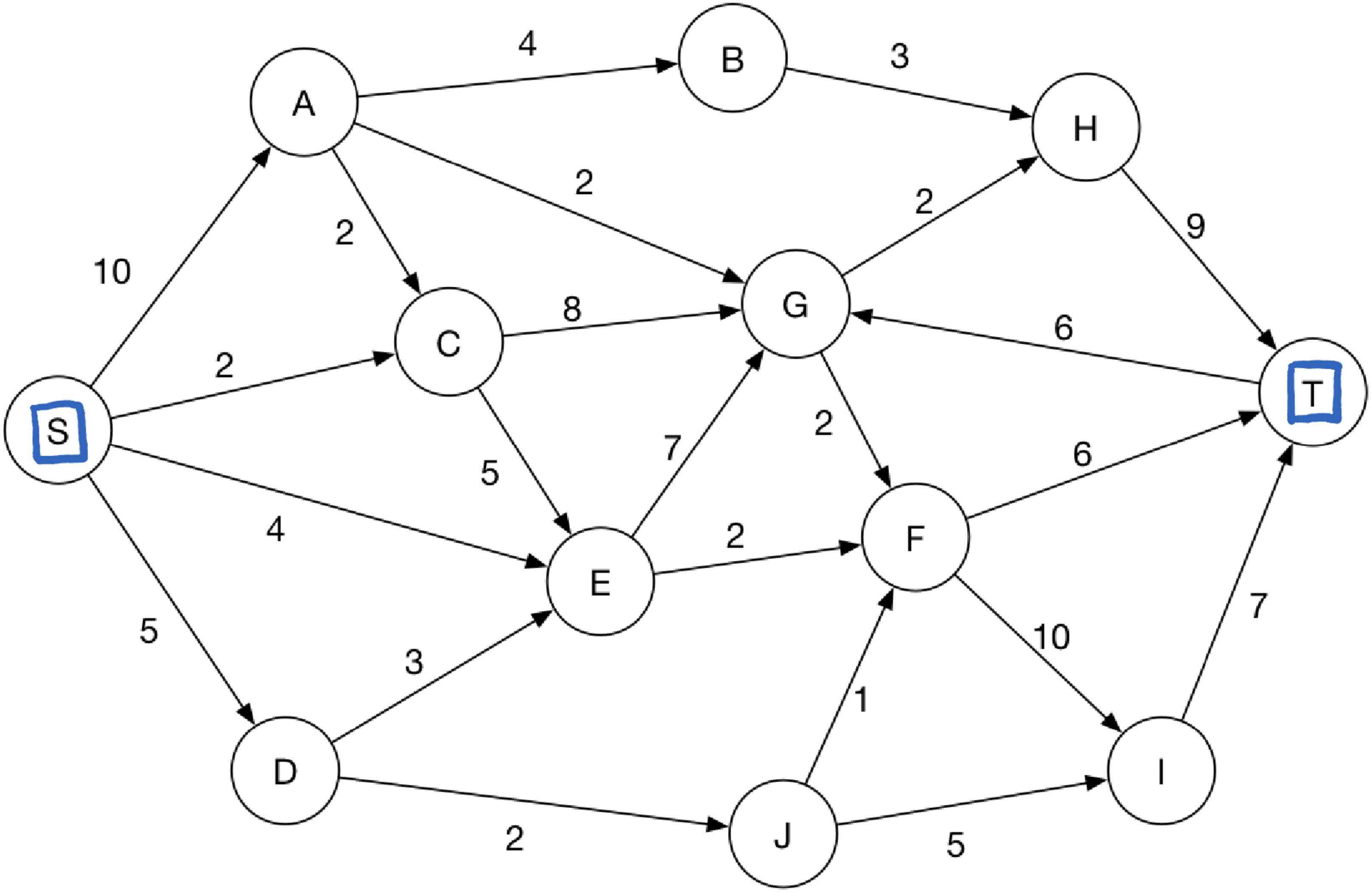
- Let me start things off for you. Here is a flow with value 9:



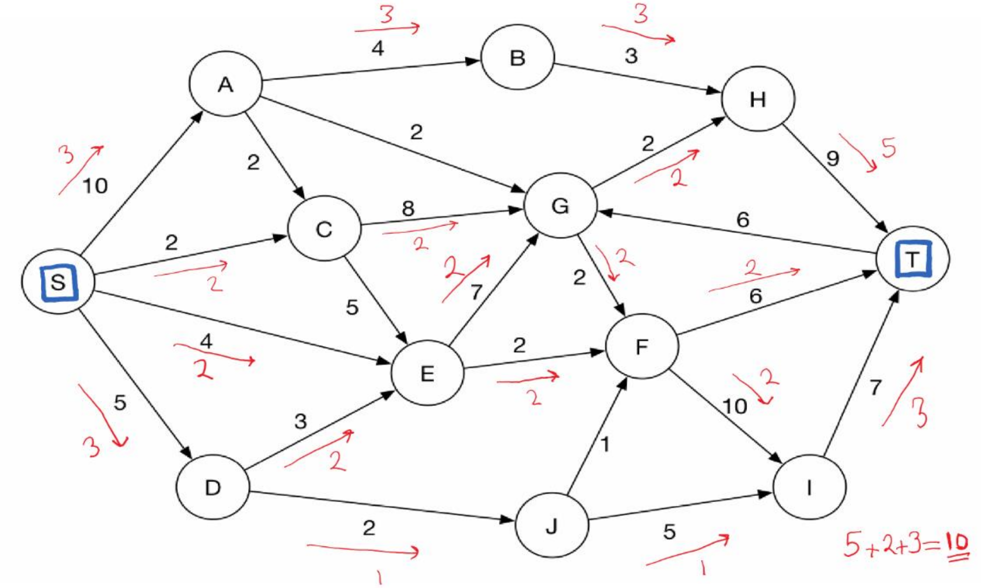
- Can you do better? How much better?

- You all get a copy of this graph.
  - You have 6 minutes!

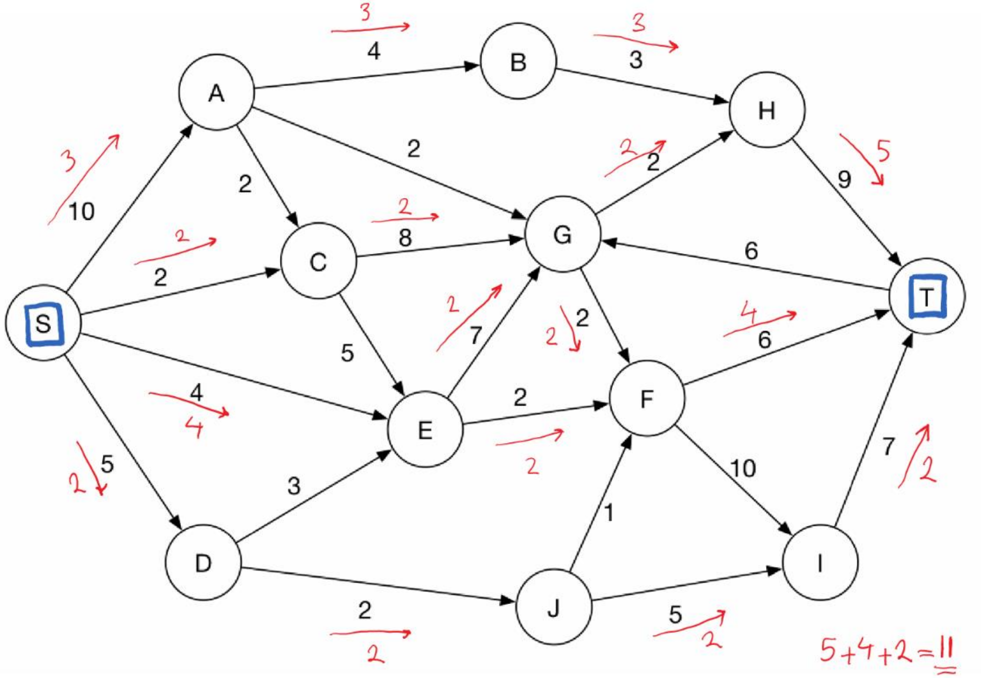
You tell me, I draw…



A couple of good attempts



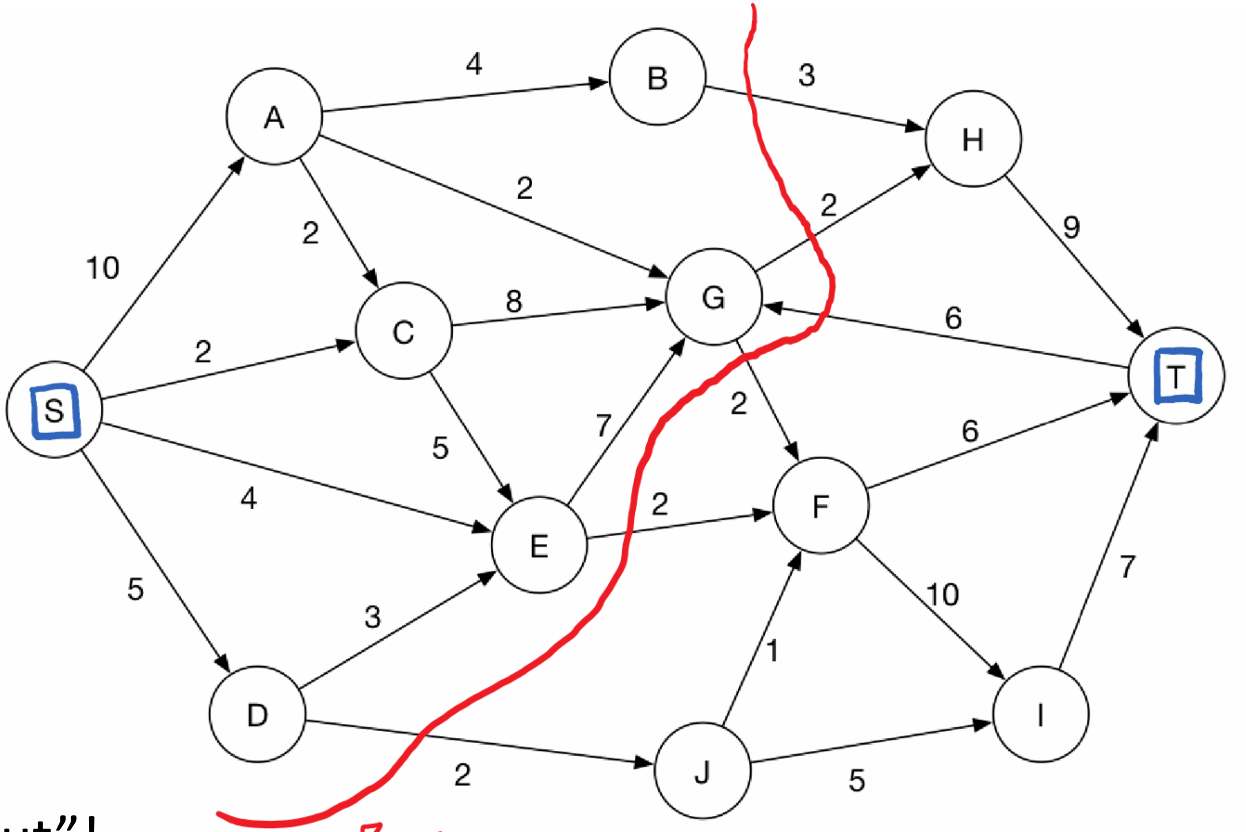
- Flow of value 10
- Can you do better?



- Flow of value 11
- Can you do better?
- How can you prove that it’s impossible to do better?

11 is the best possible!

- Proof by magic:



$3+2+2+2+2=\underline{\underline{11}}$

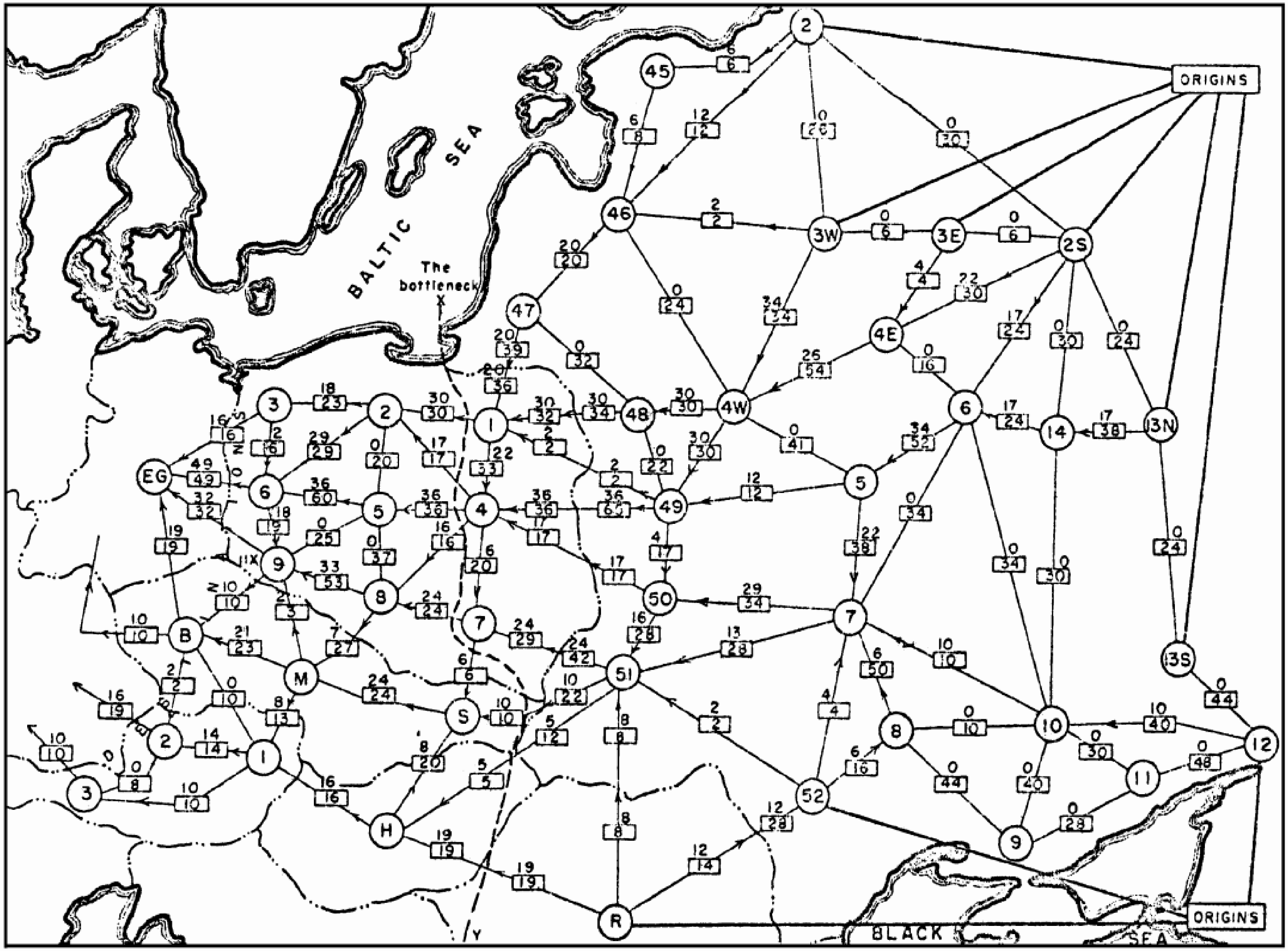
- The rabbit is the red "cut"!

- Any flow from S to T must cross the red curve.

- So it can have value at most 11.

- And here is the magic: such a proof is *always* possible!

Let’s try a more realistic graph

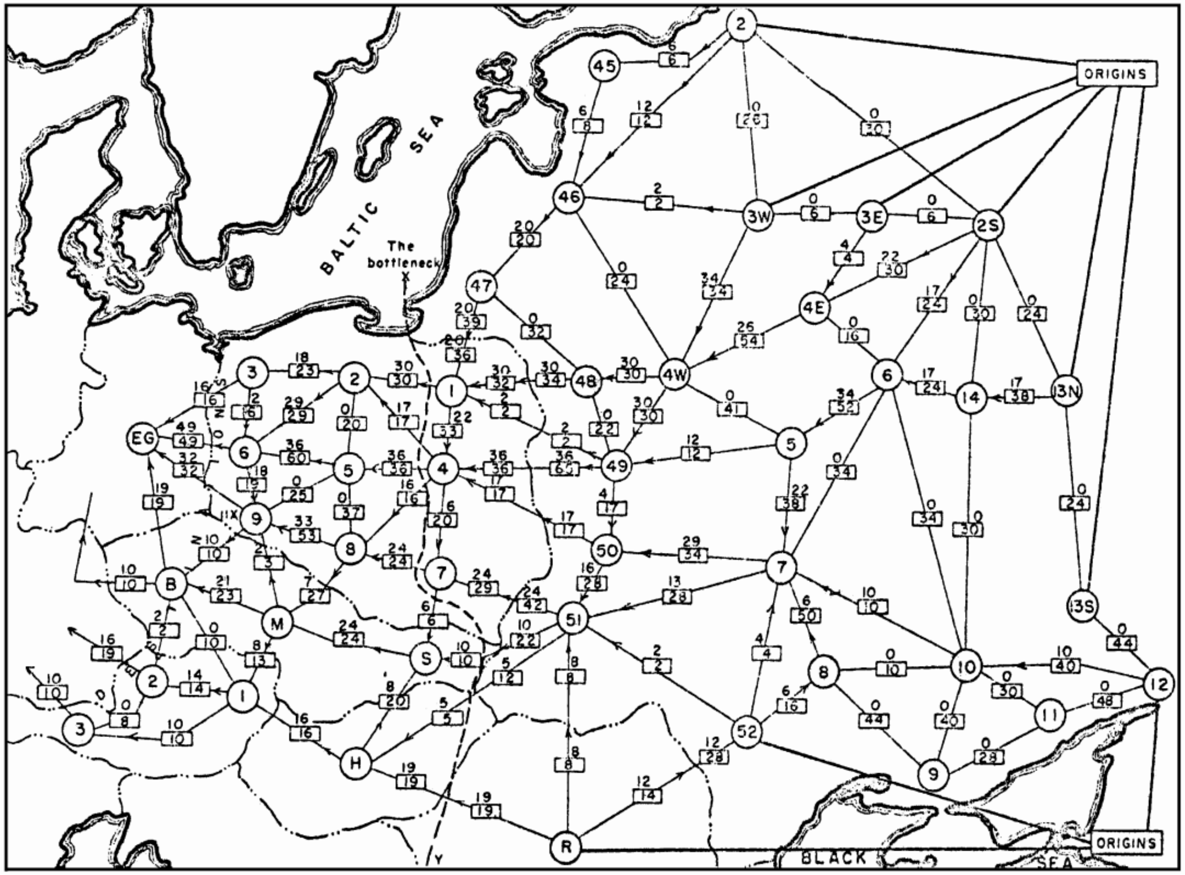


- You have 6 minutes! ;)

- How long do you think an optimization solver would take (on my laptop) to find the best solution here?

- How many lines of code do you think you have to write for it?

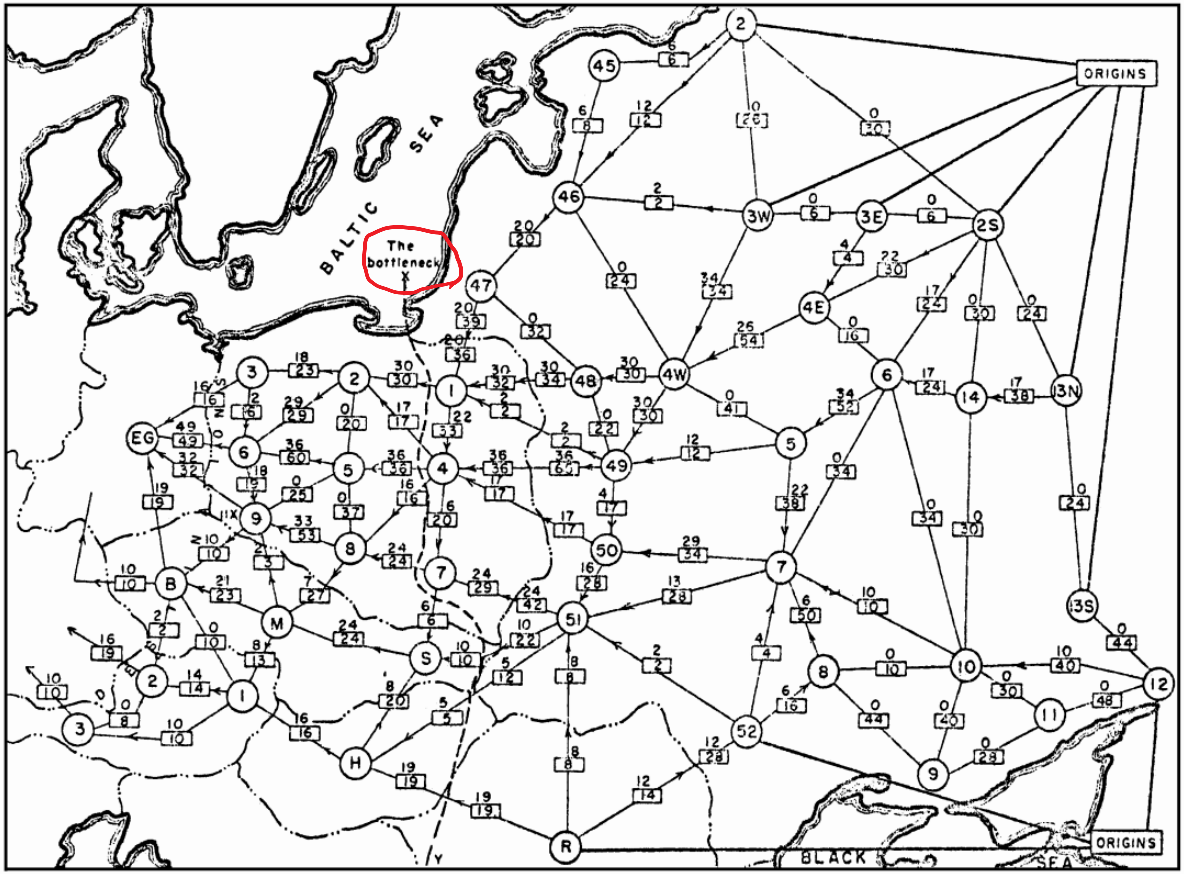
- How would someone who hasn't seen optimization approach this?



  - Trial and error?
  - Push a little flow here, a little there…
  - Do you think they are likely to find the best solution?

A bit of history behind this map

- From a secret report by Harris and Ross (1955) written for the Air Force.
- Railway network of the Western Soviet Union going to Eastern Europe.
- Declassified in 1999.
- Look at the min-cut on the map (called the “bottleneck”)!
- There are 44 vertices, 105 edges, and the max flow is 163K.



- Harris and Ross gave a heuristic which happened to solve the problem optimally in this case.
- Later that year (1955), the famous Ford-Fulkerson algorithm came out of the RAND corporation. The algorithm always finds the best solution (for rational edge costs).

More on this history: [Sch05]

Let’s look at a second problem

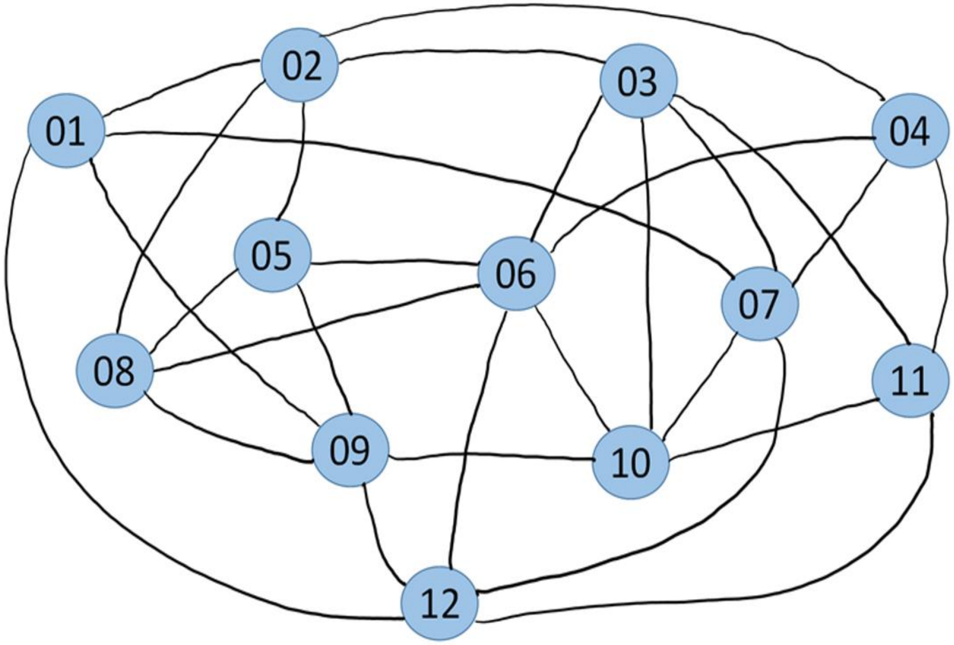
…and tell me which one you thought was easier

Two finals in one day? No thanks.

- The department chair at ORFE would like to schedule the final exams for 12 graduate courses offered this semester.
- He wants to have as many exams as possible on the same day, so everyone gets done quickly and goes on vacation.
- There is just one constraint:

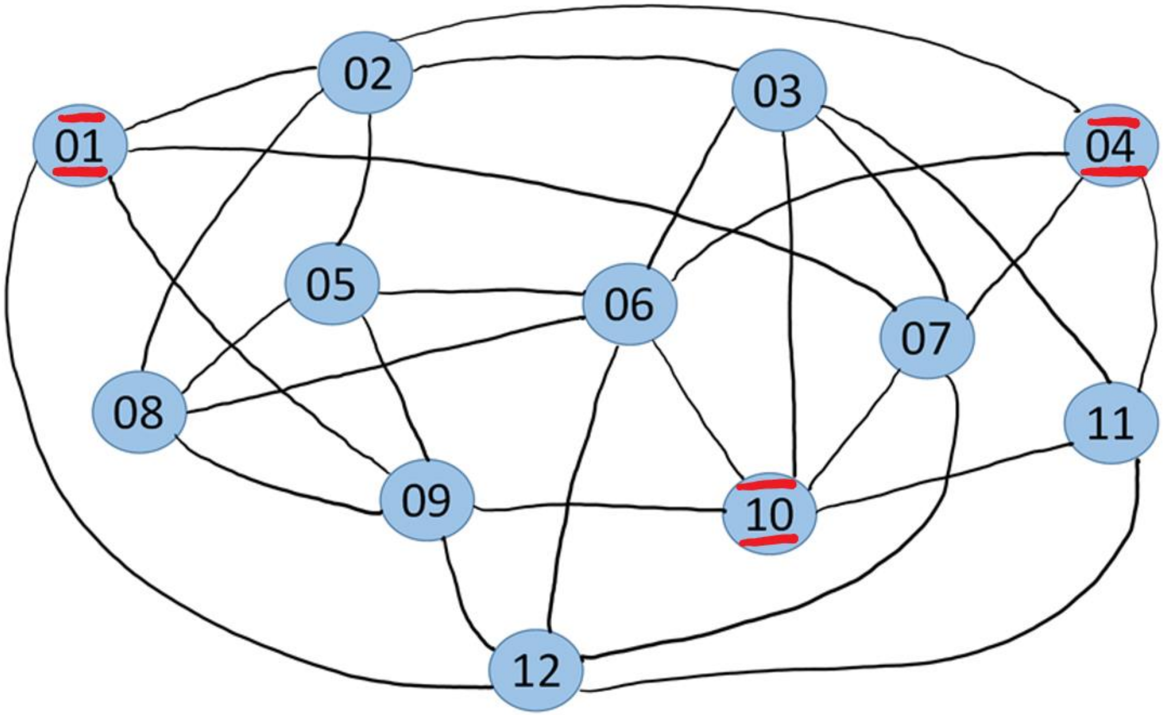
No student should have >1 exam on that day.

- The nodes of this graph are the 12 courses.
- There is an edge between two nodes if and only if there is at least one student who is taking both courses.
- If we want to schedule as many exams as possible on the same day, what are we looking for in this graph?



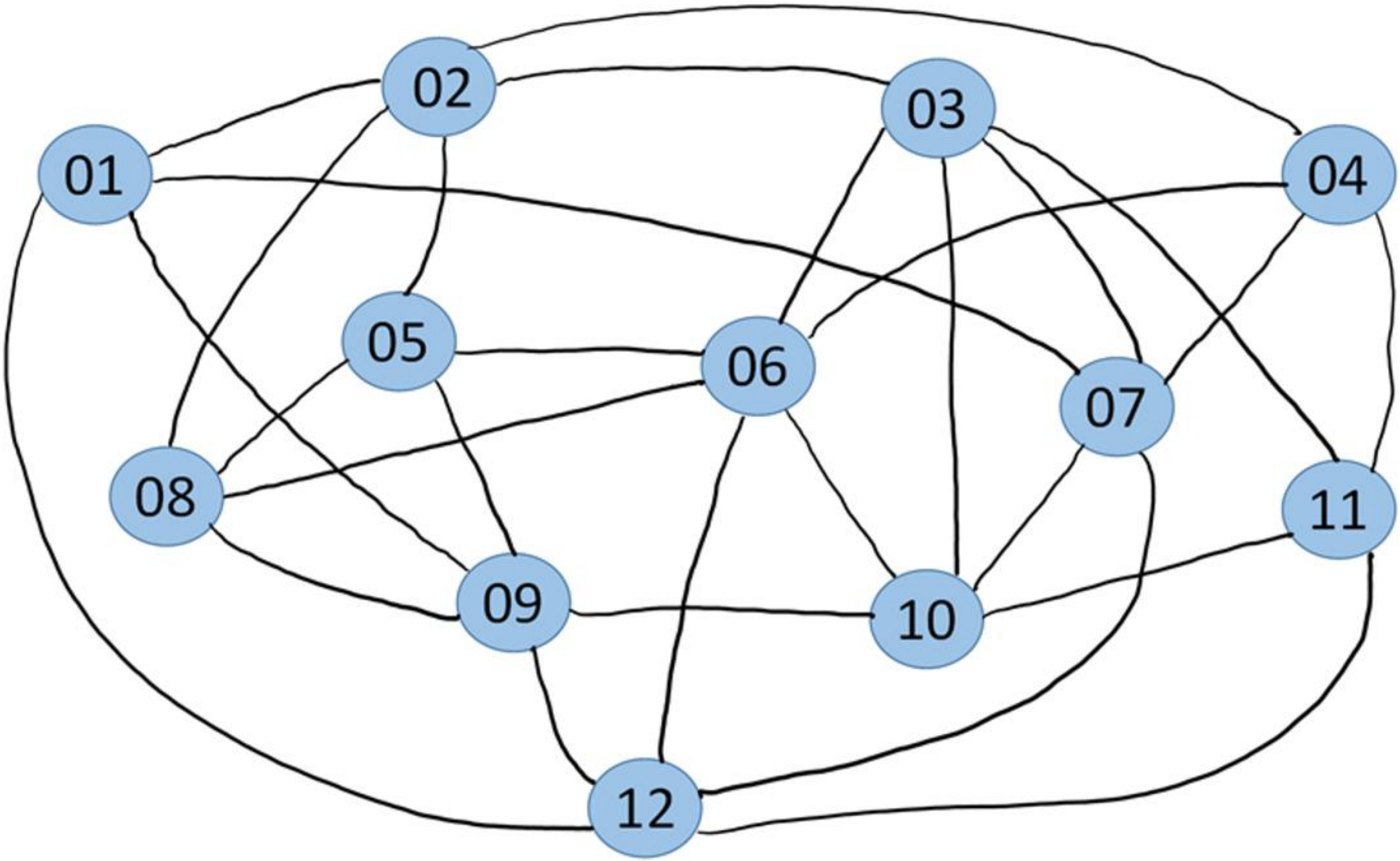
- The largest collection of nodes such that no two nodes share an edge.

- Let me start things off for you. Here is 3 concurrent final exams:

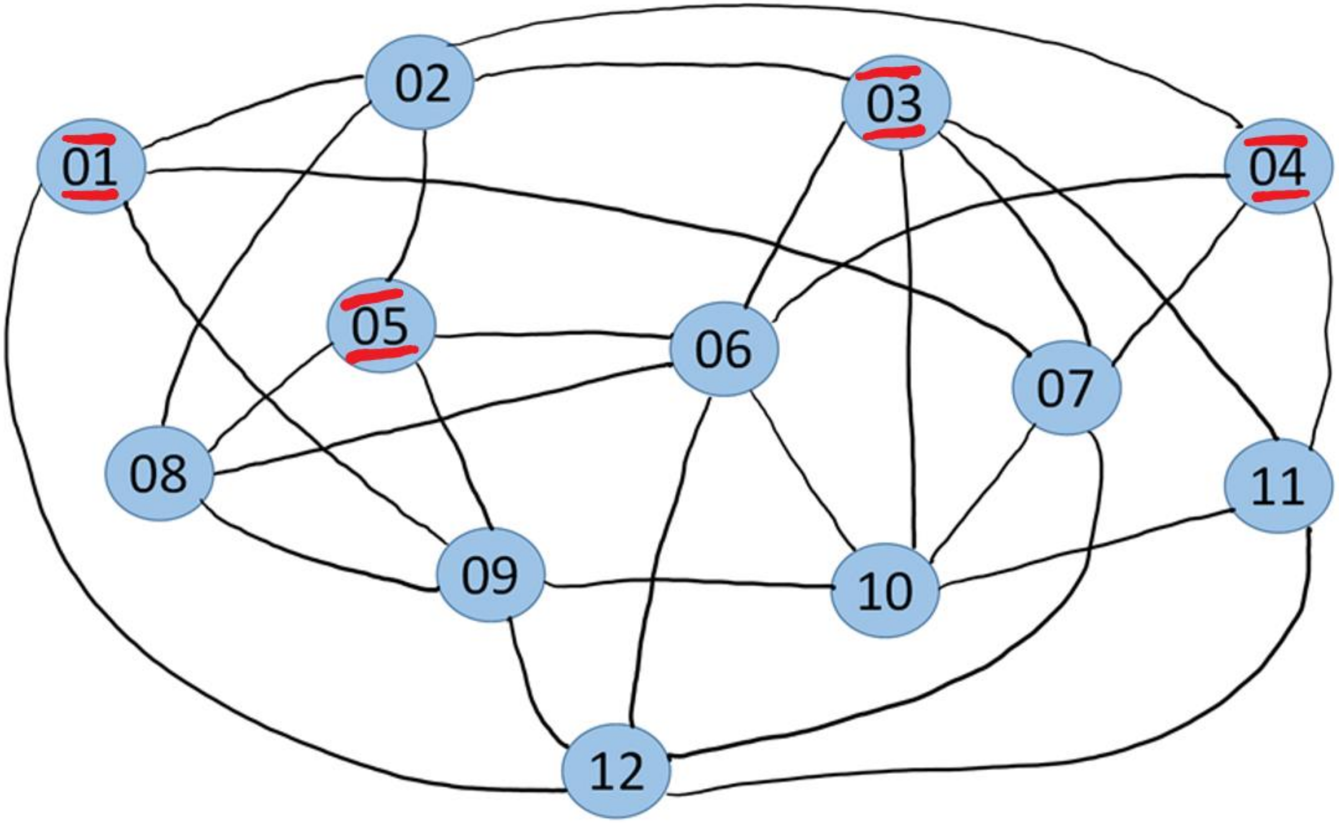


- Can you do better?
- How much better?
- You all get a copy of this graph.
  - You have 6 minutes!

You tell me, I draw…



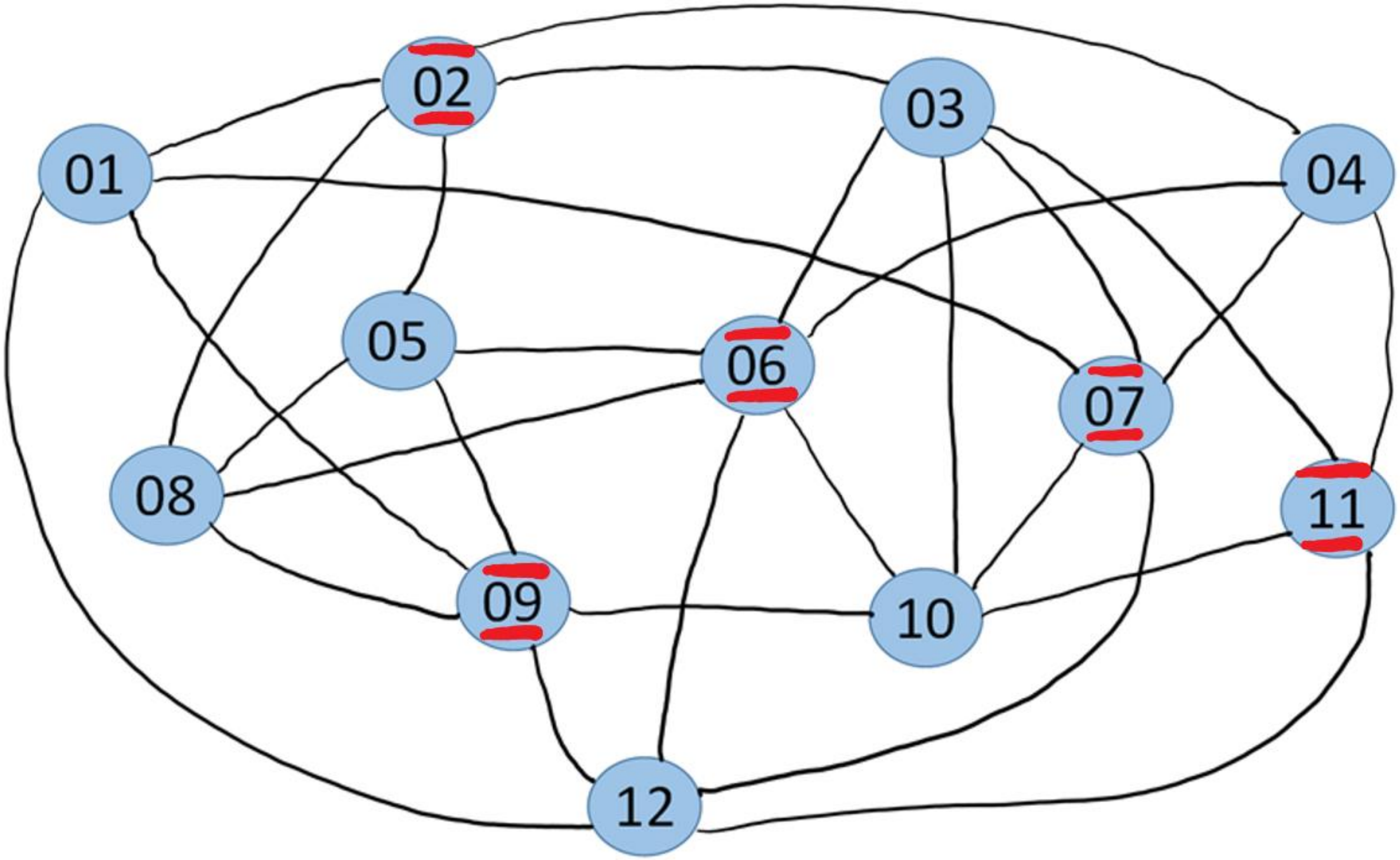
A couple of good attempts



4 exams

- Can you do better?

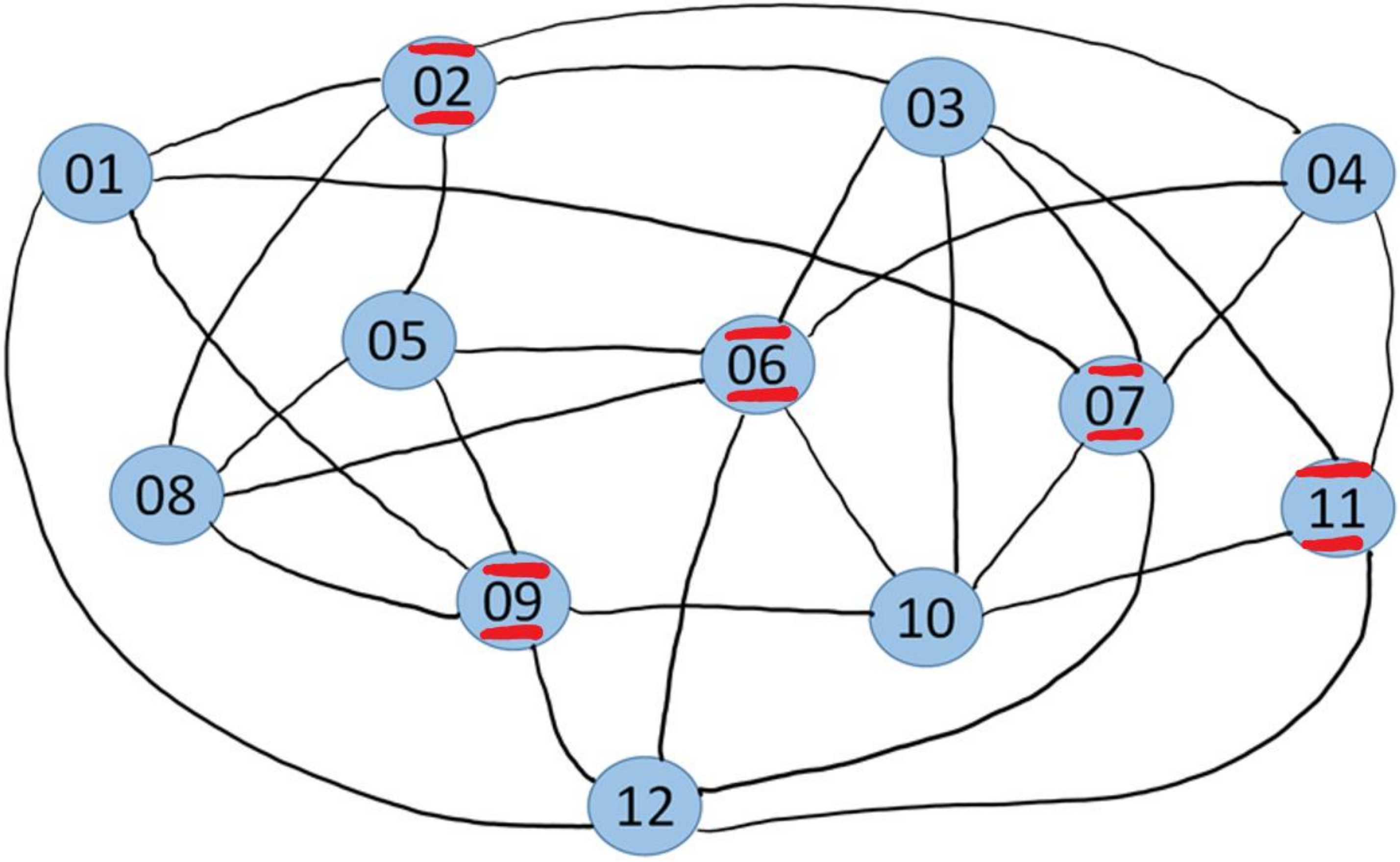
A couple of good attempts



5 exams

- Can you do better?

A couple of good attempts



5 exams

- Tired of trying?
- Is this the best possible?

5 is the best possible!

- Proof by magic?

- Unfortunately not ☹
- No magician in the world has pulled out such a rabbit to this day! (By this we mean a trick that would work on *all* graphs.)

- Of course there is always a proof:
  - Try all possible subsets of 6 nodes.
  - There are 924 of them.
  - Observe that none of them work.
- But this is no magic. It impresses nobody. We want a “short” proof. (We will formalize what this means.) Like the one in our max-flow example.
- Let’s appreciate this further…

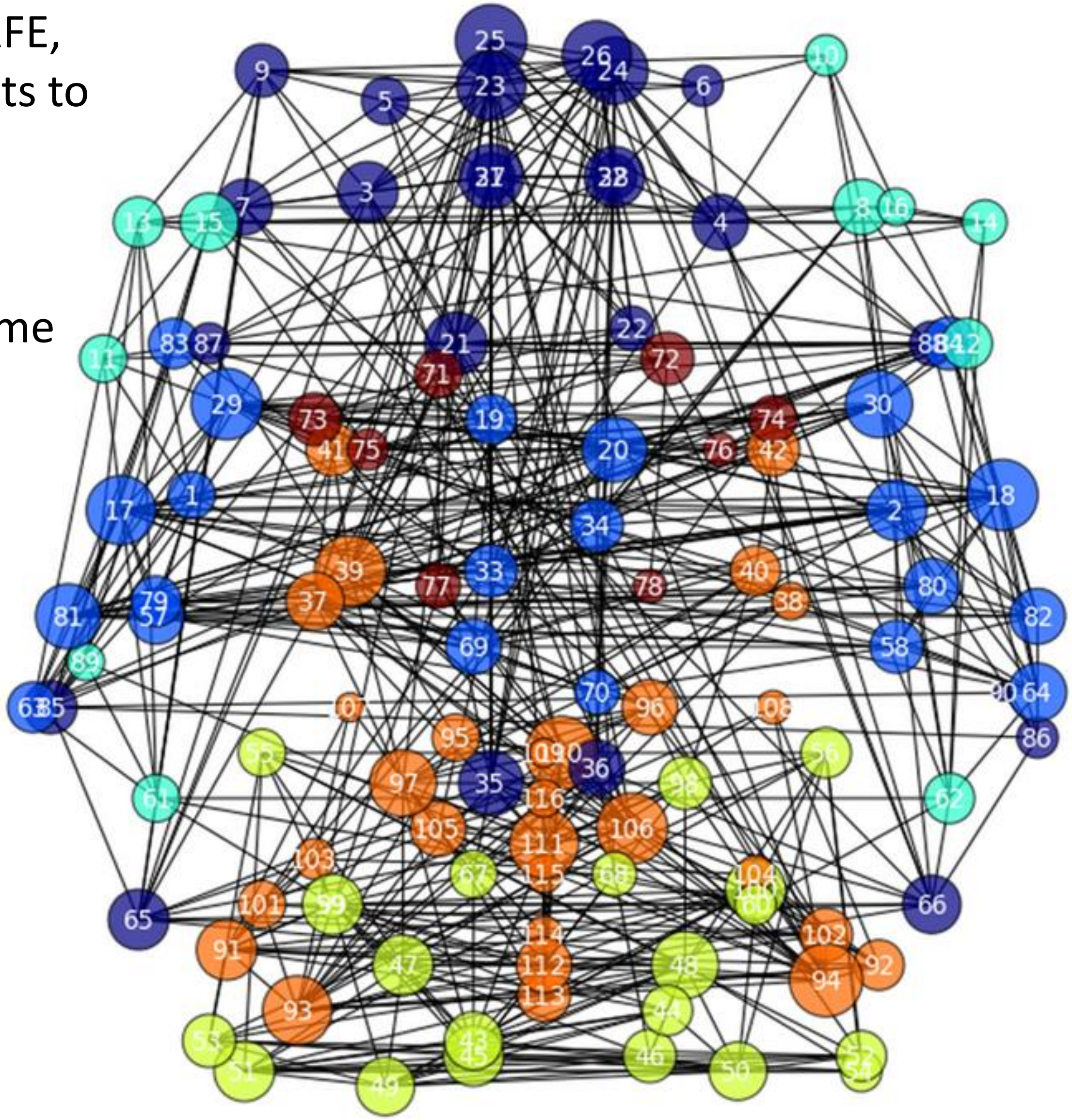
Let's try another graph

- Encouraged by the success of ORFE, now the Dean of Engineering wants to the same for 115 SEAS courses.

- How many final exams on the same day are possible? Can you do 17?
  - You have 6 minutes! ;)

- Want to try out all possibilities for 17 exams?

- There are over 80000000000000000000 of them!

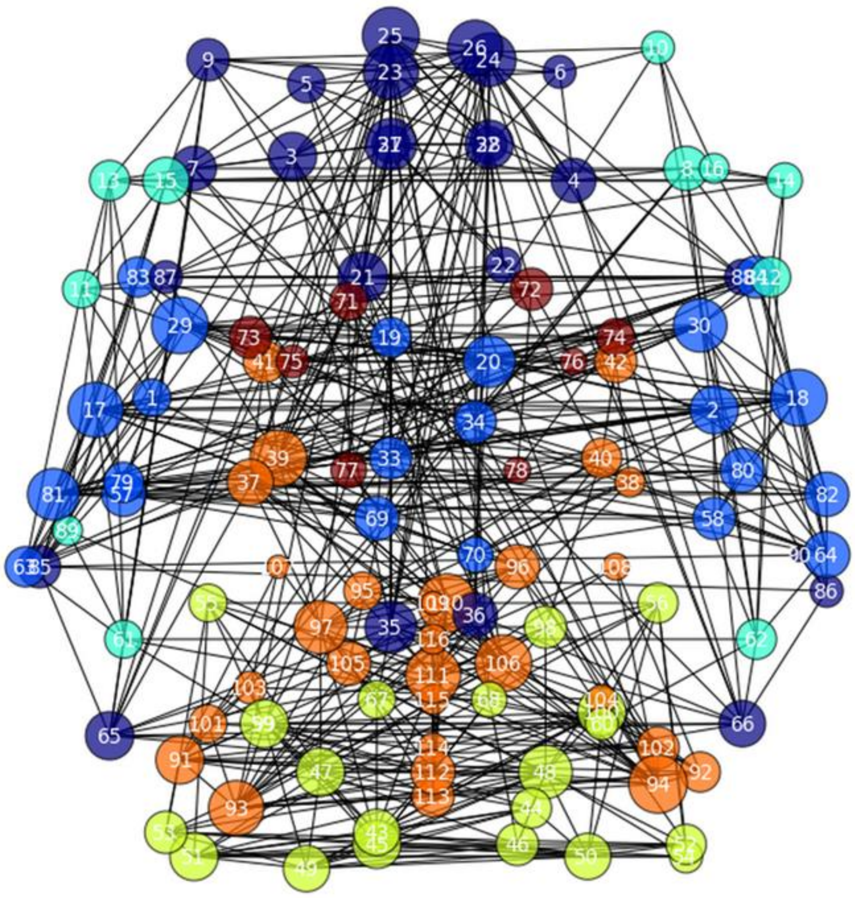


But there is some good news

- Even though finding the best solution always may be too much to hope for, techniques from optimization (and in particular from the area of *convex optimization*) often allow us to find high-quality solutions with performance guarantees.

- For example, an optimization algorithm may quickly find 16 concurrent exams for you.

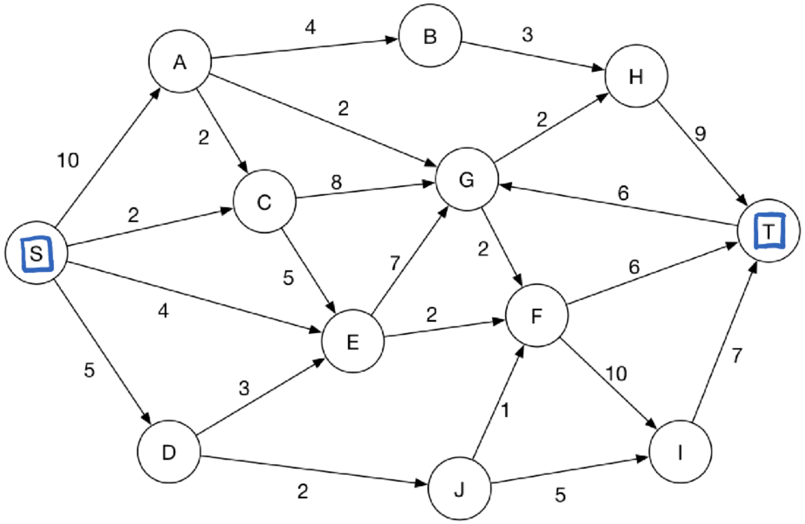
- You really want to know if 17 is impossible. Instead, another optimization algorithm (or sometimes the same one) tells you that 19 is impossible.

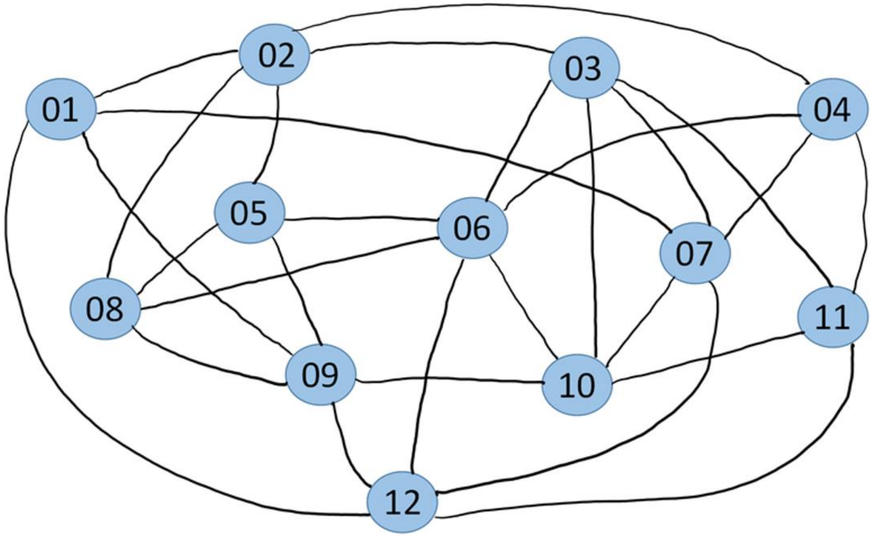


- This is very useful information! You know you got 16, and no one can do better than 19.

  - **We sill see a lot of convex optimization in this class!**

Which of the two problems was harder for you?

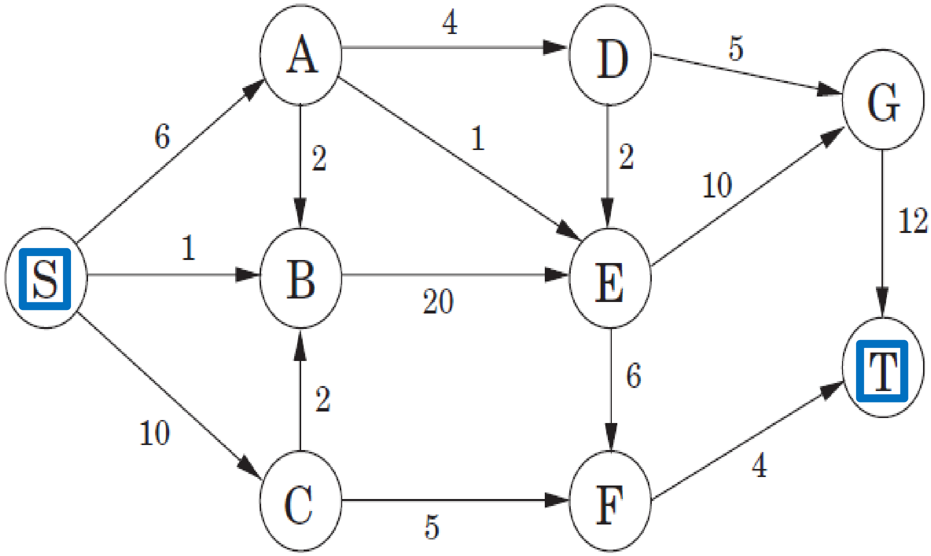




- Not always obvious. A lot of research in optimization and computer science goes into distinguishing the “tractable” problems from the “intractable” ones.
- The two brain teasers actually just gave you a taste of the P vs. NP problem. (If you have not heard about this, that’s OK. You will soon.)
- The first problem we can solve efficiently (in “polynomial time”).
- The second problem: no one knows. If you do, you literally get $1M!
  - More importantly, your algorithm immediately translates to an efficient algorithm for thousands of other problems no one knows how to solve.

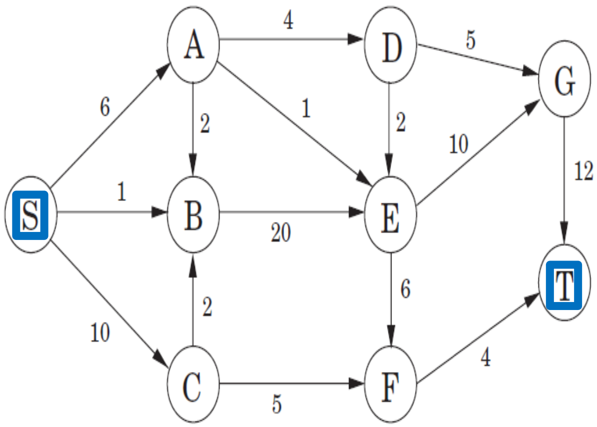
Modelling problems as a mathematical program

Let’s revisit our first game



- What were your decision variables?
- What were your constraints?
- What was your objective function?

- **Rules of the game:**
  - Cannot exceed capacity on the edges.
  - For each node, except for S ant T, flow in = flow out (i.e., no storage).
  - **Goal:** ship as much oil as you can from S to T.



$x_{SA}, x_{AD}, x_{BE}, \ldots, x_{GT}$ ← Decision variables

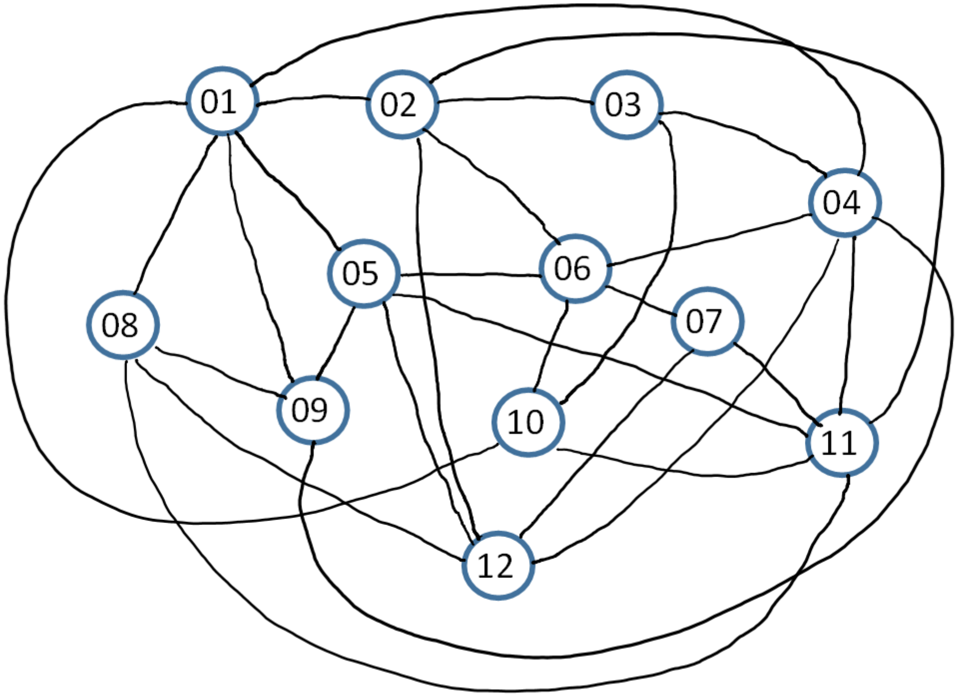
max. $x_{SA} + x_{SB} + x_{SC}$ ← Objective function

s.t.

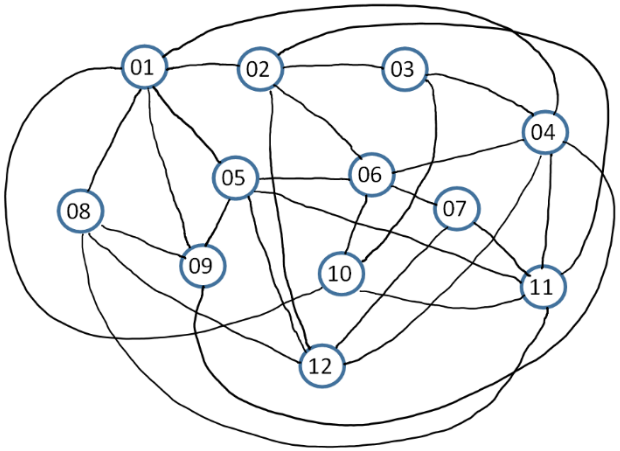
Constraints

- $x_{SA}, x_{AD}, x_{BE}, \ldots, x_{GT} \geq 0$
- $x_{SA} \leq 6, x_{AB} \leq 2, x_{EG} \leq 10, \ldots, x_{GT} \leq 12$
- $$\begin{cases} x_{SA} = x_{AD} + x_{AB} + x_{AE} \\ x_{SC} = x_{CB} + x_{CF} \\ \vdots \\ x_{CF} + x_{EF} = x_{FT}. \end{cases}$$

Let’s revisit our second game



- What were your decision variables?
- What were your constraints?
- What was your objective function?



$x_1, x_2, \ldots, x_{12}$ ← Decision variables

max. $x_1 + x_2 + \cdots + x_{12}$ ← Objective function

s.t.

- $x_i\,(1-x_i)=0, \quad i=1,\ldots,12$
- $$\left[\begin{array}{l} x_1 + x_2 \leq 1 \\ x_1 + x_8 \leq 1 \\ x_4 + x_6 \leq 1 \\ \vdots \\ x_{12} + x_8 \leq 1 \end{array}\right.$$ (one per edge)

← Constraints

Why one hard and one easy? How can you tell?

$x_{SA}, x_{AD}, x_{BE}, \dots, x_{GT}$

max. $x_{SA} + x_{SB} + x_{SC}$

s.t.

- $x_{SA}, x_{AD}, x_{BE}, \dots, x_{GT} \geq 0$
- $x_{SA} \leq 6, x_{AB} \leq 2, x_{EG} \leq 10, \dots, x_{GT} \leq 12$
- $$\left[\begin{array}{l} x_{SA} = x_{AD} + x_{AB} + x_{AE} \\ x_{SC} = x_{CB} + x_{CF} \\ \vdots \\ x_{CF} + x_{EF} = x_{FT}. \end{array}\right.$$

$x_1, x_2, \dots, x_{12}$

max. $x_1 + x_2 + \dots + x_{12}$

s.t.

- $x_i(1 - x_i) = 0, \quad i = 1, \dots, 12$
- $$\left[\begin{array}{l} x_1 + x_2 \leq 1 \\ x_1 + x_8 \leq 1 \\ x_4 + x_6 \leq 1 \\ \vdots \\ x_{12} + x_8 \leq 1 \end{array}\right.$$ (one per edge)

- **Caution:** just because we can write something as a mathematical program, it doesn't mean we can solve it.

Fermat's Last Theorem

- Can you give me three positive integers x, y, z such that

$$x^2 + y^2 = z^2?$$

- Sure:

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)

And there are infinitely many more…

- How about $x^3 + y^3 = z^3$?

- How about $x^4 + y^4 = z^4$?

- How about $x^5 + y^5 = z^5$?

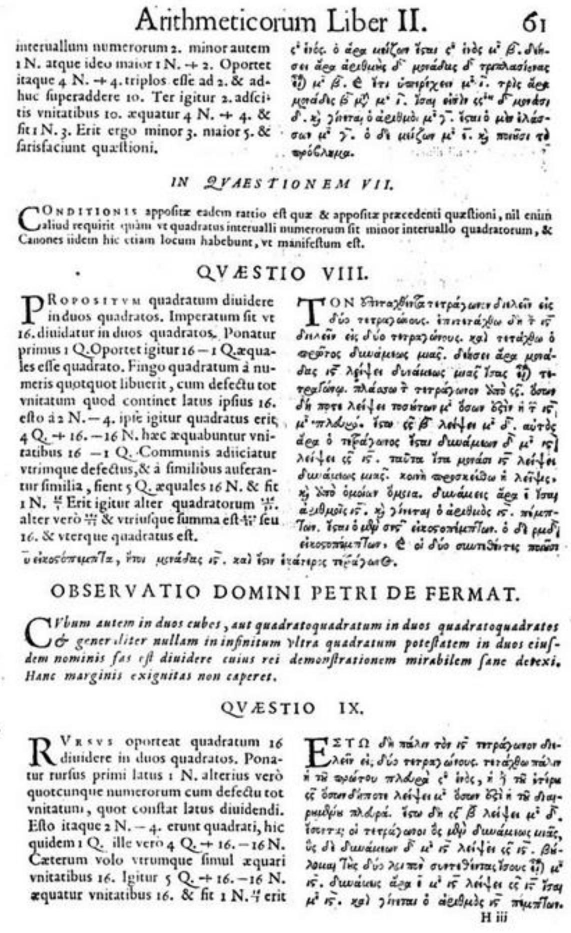
Fermat's Last Theorem

- Fermat's conjecture (1637):

For $n \geq 3$, the equation $x^n + y^n = z^n$ has no solution over positive integers.

- Proved in 1994 (357 years later!) by Andrew Wiles.

(Was on the faculty in our math department until a few years ago.)



Arithmeticorum Liber II. 61

interuallum numerorum 2. minor autem 1 N. atque ideo maior 1 N. + 2. Oportet itaque 4 N. + 4. triplos eſſe ad 2. & adhuc ſuperaddere 10. Ter igitur 2. adſcitis vnitatibus 10. æquatur 4 N. + 4. & fit 1 N. 3. Erit ergo minor 3. maior 5. & ſatisfaciunt quæſtioni.

ςς̅ ἑνός. ὁ ἄρα μείζων ἔσται ςς̅ ἑνὸς μ̊ β̅. δεήσει ἄρα ἀριθμοὺς δ̅ μονάδας δ̅ τριπλασίονας εἶναι μ̊ β̅. καὶ ἔτι ὑπερέχειν μ̊ ι̅. τρὶς ἄρα μονάδες β̅ μετὰ μ̊ ι̅. ἴσαι εἰσὶν ςς̅ δ̅ μονάσι δ̅. καὶ γίνεται ὁ ἀριθμὸς μ̊ γ̅. ἔσται ὁ μὲν ἐλάσσων μ̊ γ̅. ὁ δὲ μείζων μ̊ ε̅. καὶ ποιοῦσι τὸ πρόβλημα.

IN QVAESTIONEM VII.

CONDITIONIS appoſitæ eadem ratio eſt quæ & appoſitæ præcedenti quæſtioni, nil enim aliud requirit quàm vt quadratus interualli numerorum ſit minor interuallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifeſtum eſt.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum ſit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales eſſe quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipſius 16. eſto à 2 N. − 4. ipſe igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à ſimilibus auferantur ſimilia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$ & vtriuſque ſumma eſt $\frac{400}{25}$ ſeu 16. & vterque quadratus eſt.

ΤΟΝ ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους. ἐπιτετάχθω δὴ τὸν ις̅ διελεῖν εἰς δύο τετραγώνους. καὶ τετάχθω ὁ πρῶτος δυνάμεως μιᾶς. δεήσει ἄρα μονάδας ις̅ λείψει δυνάμεως μιᾶς ἴσας εἶναι τετραγώνῳ. πλάσσω τὸν τετράγωνον ἀπὸ ςς̅. ὅσων δή ποτε λείψει τοσούτων μ̊ ὅσων ἐστὶν ἡ τῶν ις̅ μ̊ πλευρά. ἔστω ςς̅ β̅ λείψει μ̊ δ̅. αὐτὸς ἄρα ὁ τετράγωνος ἔσται δυνάμεων δ̅ μ̊ ις̅ λείψει ςς̅ ις̅. ταῦτα ἴσα μονάσι ις̅ λείψει δυνάμεως μιᾶς. κοινὴ προσκείσθω ἡ λεῖψις, καὶ ἀπὸ ὁμοίων ὅμοια. δυνάμεις ἄρα ε̅ ἴσαι ἀριθμοῖς ις̅. καὶ γίνεται ὁ ἀριθμὸς ις̅ πέμπτων. ἔσται ὁ μὲν σνς̅ εἰκοστοπέμπτων. ὁ δὲ ρμδ̅ εἰκοστοπέμπτων, καὶ οἱ δύο συντεθέντες ποιοῦσι υ̅ εἰκοστόπεμπτα, ἤτοι μονάδας ις̅. καὶ ἔστιν ἑκάτερος τετράγωνος.

OBSERVATIO DOMINI PETRI DE FERMAT.

CVbum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum poteſtatem in duos eiuſdem nominis fas eſt diuidere cuius rei demonſtrationem mirabilem ſane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rurſus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot conſtat latus diuidendi. Eſto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque ſimul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit

ΕΣΤΩ δὴ πάλιν τὸν ις̅ τετράγωνον διελεῖν εἰς δύο τετραγώνους. τετάχθω πάλιν ἡ τοῦ πρώτου πλευρὰ ςς̅ ἑνός, ἡ δὲ τοῦ ἑτέρου ςς̅ ὁσωνδήποτε λείψει μ̊ ὅσων ἐστὶν ἡ τοῦ διαιρουμένου πλευρά. ἔστω δὴ ςς̅ β̅ λείψει μ̊ δ̅. ἔσονται οἱ τετράγωνοι ὃς μὲν δυνάμεως μιᾶς, ὃς δὲ δυνάμεων δ̅ μ̊ ις̅ λείψει ςς̅ ις̅. βούλομαι τοὺς δύο λοιπὸν συντεθέντας ἴσους εἶναι μ̊ ις̅. δυνάμεις ἄρα ε̅ μ̊ ις̅ λείψει ςς̅ ις̅ ἴσαι μ̊ ις̅. καὶ γίνεται ὁ ἀριθμὸς ις̅ πέμπτων.

H iii

Fermat's Last Theorem

- Fermat's conjecture (1637):

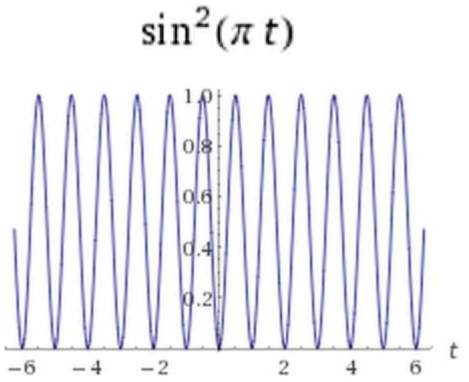
For $n \geq 3$, the equation $x^n + y^n = z^n$ has no solution over positive integers.

- Consider the following optimization problem (mathematical program):

$$\min_{x,y,z,n} \quad (x^n + y^n - z^n)^2$$

$$\text{s.t.} \quad x \geq 1,\ y \geq 1,\ z \geq 1,\ n \geq 3,$$

$$\sin^2 \pi n + \sin^2 \pi x + \sin^2 \pi y + \sin^2 \pi z = 0.$$



- Innocent-looking optimization problem: 4 variables, 5 constraints.

- If you could show the optimal value is non-zero, you would prove Fermat's conjecture!

Course objectives

- The skills I hope you acquire:

- Ability to view your own field through the lens of optimization and computation
  - To help you, we'll draw applications from operations research, statistics, finance, machine learning, engineering, …
- Learn about several topics in scientific computing
- More mathematical maturity and ability for rigorous reasoning
  - There will be some proofs in lecture. Easier ones on homework.
- Enhance your coding abilities (nothing too fancy, simple MATLAB)
  - There will be a MATLAB component on every homework and on the take-home final.
  - You are free to use any other programming language instead (e.g., Python)
- Ability to recognize hard and easy optimization problems
- Ability to use optimization software
  - Understand the algorithms behind the software for some easier subclass of problems.

Software you need to download

- Right away:

MATLAB

Available for free to Princeton students

You are free to use any other programming language instead (e.g., Python). The solutions that we provide will be in MATLAB.

- In the next week or two (will appear on HW#2 or #3):

CVX

http://cvxr.com/cvx/

Course logistics

- See syllabus.
- Course website:

http://aaa.princeton.edu/orf363

- For those interested:
  - Princeton Optimization Seminar
  - http://orfe.princeton.edu/events/optimization-seminar

- Image credits and references:

- [DPV08] S. Dasgupta, C. Papadimitriou, and U. Vazirani. Algorithms. McGraw Hill, 2008.
- [Sch05] A. Schrijver. On the history of combinatorial optimization (till 1960). In “Handbook of Discrete Optimization”, Elsevier, 2005. http://homepages.cwi.nl/~lex/files/histco.pdf