ORF 363/COS 323	Problem set 3	Fall 2023, Princeton University
Instructor: A.A. Ahmadi		
TAs: Beneventano, Chaudhry, Hua, Li, Lok		
Due on Tuesday, October 10, 2023, at 1:30pm EST, on Gradescope		

For all problems that involve coding, please include your code. Our instructions are given with MATLAB in mind (recommended language), but you are free to use a different language.

Problem 1: Radiation treatment planning[1]

In radiation treatment, radiation is delivered to a patient, with the goal of killing or damaging the cells in a tumor, while carrying out minimal damage to other tissue. The radiation is delivered in beams, each of which has a known pattern; the level of each beam can be adjusted. (In most cases multiple beams are delivered at the same time, in one 'shot', with the treatment organized as a sequence of 'shots'.) We let b_j denote the level of beam j, for $j = 1, \dots, n$. These must satisfy $0 \leq b_j \leq B^{\max}$, where $B^{\max}$ is the maximum possible beam level. The exposure area is divided into m voxels, labeled $i = 1, \dots, m$. The dose d_i delivered to voxel i is linear in the beam levels, i.e., $d_i = \sum_{j=1}^{n} A_{ij} b_j$. Here $A \in \mathbb{R}_+^{m \times n}$ is a (known) matrix that characterizes the beam patterns. We now describe a simple radiation treatment planning problem.

A (known) subset of the voxels, $\mathcal{T} \subset \{1, \dots, m\}$, corresponds to the tumor or target region. We require that a minimum radiation dose D^{target} be administered to each tumor voxel, i.e., $d_i \geq D^{\text{target}}$ for $i \in \mathcal{T}$. For all other voxels, we would like to have $d_i \leq D^{\text{other}}$, where D^{other} is a desired maximum dose for non-target voxels. This is generally not feasible, so instead we settle for minimizing the penalty

$$E = \sum_{i \notin \mathcal{T}} (d_i - D^{\text{other}})_+,$$

where $(\cdot)_+$ denotes the nonnegative part of its argument (i.e., $(z)_+ = \max\{0, z\}$). We can interpret E as the total nontarget excess dose.

1. Show that the treatment planning problem is convex. The optimization variable is $b \in \mathbb{R}^n$; the problem data are $B^{\max}$, A, $\mathcal{T}$, D^{target}, and D^{other}.

2. Solve the problem instance with data generated by the file `treatment_planning_data.m`. If you are using Python, you can download `Atumor.csv` and `Aother.csv`, then use

[1] Courtesy of S. Boyd and L. Vandenberghe.

`treatment_planning_data.py` to load them. Here we have split the matrix A into `Atumor`, which contains the rows corresponding to the target voxels, and `Aother`, which contains the rows corresponding to other voxels. Plot the dose histogram for the target voxels, and also for the other voxels. (You can use the MATLAB function `hist` to plot histograms.) Make a brief comment on what you see. *Remark:* The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.

Problem 2: Hands off my paintings

You are the newly appointed head of Security at the Princeton Art Museum and your task is to protect five particularly valuable paintings from theft at nighttime. These paintings are all put on the same wall with the following positions (see figure):

Painting i	x_1^i	y_1^i	x_2^i	y_2^i
1	40	335	160	237
2	70	235	172	85
3	190	385	310	259
4	320	365	440	219
5	280	185	400	90

Here, the origin is taken to be at the bottom left corner of the wall, (x_1^i, y_1^i) are the coordinates of the top left corner of painting i, and (x_2^i, y_2^i) are the coordinates of its bottom right one.

Figure 1: The museum wall

Because you are lazy and don't want to stand in front of the paintings all night, you install a detector that monitors them on the wall and triggers an alarm as soon as one of them moves. The detector's coverage area is a disk and the amount of energy it consumes is proportional to the area of the disk. (Recall that a disk of center $u \in \mathbb{R}^2$ and radius $r > 0$ is given by $B(u, r) = \{z \in \mathbb{R}^2 \mid ||z - u||_2 \leq r\}$.)

1. Use CVX to find a disk of minimum area that covers the *entirety* of all paintings. Justify your approach. Report the center and the radius of this disk. Also justify why the problem that you are formulating is a convex optimization problem.

2. Plot the circle corresponding to the boundary of the optimal disk on top of the paintings. You can use the function `Circledraw` that is provided (MATLAB) or the function `Circle` in matplotlib (Python) and the following code to load the image of the wall and draw a circle centered at (x, y) with radius r.

   MATLAB code:

```
wall = imread('wall_with_paintings.png');
imshow(wall)
hold on
Circledraw(x, y, r, 'red')
```

   Python code:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Circle
from matplotlib.image import imread

wall = imread('wall_with_paintings.png')
wall_flipped = np.flipud(wall)
plt.imshow(wall_flipped, origin='lower')
circle = Circle((x, y), r, color='red', fill=False)
plt.gca().add_patch(circle)
plt.show()
```

3. Name the artists who painted the paintings that touch the boundary of the optimal disk. (We are sure our ORF 363 students know their painters, but just in case, the following names may help: Matisse, Da Vinci, Monet, Picasso, Van Gogh, and Google.)

Problem 3: Minimizers of convex problems

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\Omega \subseteq \mathbb{R}^n$ be a convex set. Show that the set of minimizers of f over Ω is convex (i.e., a convex set).

Problem 4: True or False? (Provide a proof or a counterexample.)

(a) A quadratic function $f(x) = x^T Q x + b^T x + c$ is convex if and only if it is quasiconvex.

(b) A convex homogeneous polynomial $p(x)$ of degree $d \geq 2$ is nonnegative. (Recall that a polynomial is homogeneous of degree d if all of its monomials have degree exactly d, and that $p(x)$ is nonnegative if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.)