ORF 363/COS 323	Problem set 7	Fall 2023, Princeton University
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Due on Thursday, November 30, 2023, at 1:30pm EST, on Gradescope		

For all problems that involve coding, please include your code.

Problem 1: Nearest correlation matrix

You are the CEO of HoneyMoney Technologies LLC, a new hedge fund firm in NYC whose proprietary optimization algorithms has Wall Street raving. Your main competitor, RenaissancE Technologies[1], has sent in a spy, disguised as a summer intern, to interfere with your investments. The spy has gotten his hands on your *correlation matrix* C of n important stocks[2], to which he has added some random noise, leaving you with a matrix $\hat{C}$. We remark that to be a valid correlation matrix, a matrix must be symmetric, positive semidefinite, and have all diagonal entries equal to one. The spy has been careful enough to make sure that the resulting matrix $\hat{C}$ is symmetric and has ones on the diagonal, but he hasn't noticed that his change has made $\hat{C}$ not positive semidefinite.

(a) Suppose we have

$$\hat{C} = \begin{pmatrix} 1.00 & -0.76 & 0.07 & -0.96 \\ -0.76 & 1.00 & 0.18 & 0.07 \\ 0.07 & 0.18 & 1.00 & 0.41 \\ -0.96 & 0.07 & 0.41 & 1.00 \end{pmatrix}.$$

Using CVX or CVXPY, recover the original matrix by finding the nearest correlation matrix to $\hat{C}$ in Frobenius norm (i.e., the correlation matrix C that minimizes $||C - \hat{C}||_F$). Give your optimal solution.

(b) Show that for any symmetric matrix $\hat{C}$, the problem of finding the closest correlation matrix to $\hat{C}$ in Frobenius norm has a unique solution. (You can use the fact that an optimal solution to this problem exists without proof.)

[1]Not to be confused with Renaissance Technologies that would never do such a thing.

[2]If you are curious, the correlation matrix is an $n \times n$ symmetric matrix used frequently in investment banking. Its (i, j)-th entry is a number between -1 and 1, with numbers close to 1 meaning that stocks i and j are likely to move up together, close to -1 meaning that the two stocks are likely to move in opposite directions, and close to zero meaning that they are likely uncorrelated. The problem of finding the closest correlation matrix to a given matrix is an important problem in financial engineering; see e.g. this article.

Problem 2: Existence of rational solutions to LPs and SDPs

An important consideration for the development of solvers for optimization is to determine whether a solution can be returned over the same number field as the input. In this problem, we examine this question for LPs and SDPs.

Classify the following two statements as "true" or "false". You must provide a proof or a counterexample, and your arguments should only use what you have seen in this class.

(a) Consider a linear program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b, \end{aligned}$$

where A is an $m \times n$ matrix. Suppose that the feasible set is nonempty and bounded. If the entries of A, b, c are rational[3], then there exists at least one optimal solution x^* whose entries are all rational. (Hint: You may want to use the fact that the conjugate gradient algorithm accomplishes a certain task in a finite number of iterations.)

(b) Consider a semidefinite program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \end{aligned}$$

where $A_0, \ldots, A_n$ are symmetric $m \times m$ matrices. Suppose that the feasible set is nonempty and bounded. If the entries of $A_0, \ldots, A_n$ and c are rational, then there exists at least one optimal solution x^* whose entries are all rational.

[3]Recall that a real number is rational if it is the ratio of two integers (e.g., $\frac{7}{9}$).