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| ORF 363/COS 323 | PROBLEM SET 3 | Fall 2024, Princeton University |
| Instructor: A.A. Ahmadi | | |
| TAs: Aydin, Budway, Hua, Tziampazis | | |
| Due on Tuesday, October 8, 2024, at 1:30pm EST, on Gradescope | | |

For all problems that involve coding, please include your code.

**Problem 1: Radiation treatment planning**[1]

In radiation treatment, radiation is delivered to a patient, with the goal of killing or damaging the cells in a tumor, while carrying out minimal damage to other tissue. The radiation is delivered in beams, each of which has a known pattern; the level of each beam can be adjusted. (In most cases multiple beams are delivered at the same time, in one 'shot', with the treatment organized as a sequence of 'shots'.) We let $b_j$ denote the level of beam $j$, for $j = 1, \ldots, n$. These must satisfy $0 \leq b_j \leq B^{\max}$, where $B^{\max}$ is the maximum possible beam level. The exposure area is divided into $m$ voxels, labeled $i = 1, \ldots, m$. The dose $d_i$ delivered to voxel $i$ is linear in the beam levels, i.e., $d_i = \sum_{j=1}^{n} A_{ij} b_j$. Here $A \in \mathbb{R}_+^{m \times n}$ is a (known) matrix that characterizes the beam patterns. We now describe a simple radiation treatment planning problem.

A (known) subset of the voxels, $\mathcal{T} \subset \{1, \ldots, m\}$, corresponds to the tumor or target region. We require that a minimum radiation dose $D^{\text{target}}$ be administered to each tumor voxel, i.e., $d_i \geq D^{\text{target}}$ for $i \in \mathcal{T}$. For all other voxels, we would like to have $d_i \leq D^{\text{other}}$, where $D^{\text{other}}$ is a desired maximum dose for non-target voxels. This is generally not feasible, so instead we settle for minimizing the penalty

$$E = \sum_{i \notin \mathcal{T}} (d_i - D^{\text{other}})_+,$$

where $(\cdot)_+$ denotes the nonnegative part of its argument (i.e., $(z)_+ = \max\{0, z\}$). We can interpret $E$ as the total nontarget excess dose.

1. Show that the treatment planning problem is convex. The optimization variable is $b \in \mathbb{R}^n$; the problem data are $B^{\max}$, $A$, $\mathcal{T}$, $D^{\text{target}}$, and $D^{\text{other}}$.

2. Solve the problem instance with data generated by the file `treatment_planning_data.m`. If you are using Python, you can download `Atumor.csv` and `Aother.csv`, then use `treatment_planning_data.py` to load them. Here we have split the matrix $A$ into

[1] Courtesy of S. Boyd and L. Vandenberghe.

`Atumor`, which contains the rows corresponding to the target voxels, and `Aother`, which contains the rows corresponding to other voxels. Plot the dose histogram for the target voxels, and also for the other voxels. You can use the function `hist` in both MATLAB and Python (Matplotlib package) to plot histograms. Make a brief comment on what you see. *Remark:* The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.


**Problem 2: Would your GPA be higher at Yale?**

An article appeared recently in the New York Times with the title "*Nearly Everyone Gets A's at Yale. Does That Cheapen the Grade?*"[2] After reading the article, you may wonder whether it is easier to get an A at Yale than at Princeton and, if so, how one could adjust GPAs to account for course difficulty. In this problem, we approach this question using an optimization-based idea proposed by Professor Vanderbei and his collaborators.

We begin by asking four familiar Princeton/Yale students to take some courses at their own institution and a few similar ones at the other institution. The letter grades of these students are summarized in Table 1. Their GPAs are calculated using the grade points in Table 2.

We assume that the grade point $g_{ij}$ that student $i$ receives in course $j$ should nearly be equal to $a_i + b_j$, where $a_i$ is the "*aptitude*" of student $i$ and $b_j$ is the "*inflatedness*" of course $j$. In

[2] `https://www.nytimes.com/2023/12/05/nyregion/yale-grade-inflation.html`

| | Princeton ORF 363 | Princeton ENG 351 | Princeton ORF 309 | Yale CPSC 365 | Yale ENGL 305 | Yale STAT 241 | GPA | Aptitude |
|---|---|---|---|---|---|---|---|---|
| M. Obama | A- | A | B+ | | A+ | | 3.825 | ? |
| J. Bezos | A+ | A- | B- | A | | | 3.675 | ? |
| M. Streep | B- | | | A | A | A | 3.675 | ? |
| R. DeSantis | | | B- | A+ | A+ | A+ | 3.9 | ? |
| Inflatedness | ? | ? | ? | ? | ? | ? | | |

Table 1: Performance of four students in six courses

| Letter Grade | A+ | A | A- | B+ | B | B- |
|---|---|---|---|---|---|---|
| Grade Point | 4.3 | 4.0 | 3.7 | 3.3 | 3.0 | 2.7 |

Table 2: Converting letter grades to grade points

our example, $i \in \{1, \ldots, 4\}$, and $j \in \{1, \ldots, 6\}$. We normalize the inflatedness scores with the constraint $\sum_{j=1}^{6} b_j = 0$ (negative inflatedness scores correspond to more difficult courses). This leads us to the following optimization problem which simultaneously computes student aptitudes and course inflatedness scores:

$$\begin{array}{ll} \min\limits_{a \in \mathbb{R}^4, b \in \mathbb{R}^6} & \sum_{(i,j) \in \mathcal{G}} (g_{ij} - a_i - b_j)^2 \\ \text{s.t.} & \sum\limits_{j=1}^{6} b_j = 0. \end{array} \tag{1}$$

Here, the index set $\mathcal{G}$ denotes the student-course pairs for which a grade is available.

(a) Is problem (1) a convex optimization problem? Why or why not?

(b) Use Tables 1 and 2 to solve problem (1) via `cvx` or `cvxpy`. Fill in the question marks in Table 1 with your optimal solution.

(c) How do the four students rank based on their aptitude (which can be thought of as an "adjusted GPA")? Compare this to the GPA-based ranking. Which courses have the lowest/highest inflatedness score?

**Problem 3: Minimizers of convex problems**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\Omega \subseteq \mathbb{R}^n$ be a convex set. Show that the set of minimizers of $f$ over $\Omega$ is convex (i.e., a convex set).

**Problem 4: True or False?** (Provide a proof or a counterexample.)

(a) A quadratic function $f(x) = x^T Qx + b^T x + c$ is convex if and only if it is quasiconvex.

(b) A convex homogeneous polynomial $p(x)$ of degree $d \geq 2$ is nonnegative. (Recall that a polynomial is homogeneous of degree $d$ if all of its monomials have degree exactly $d$, and that $p(x)$ is nonnegative if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.)