*Any typos should be emailed to a_a_a@princeton.edu.*

In this lecture, we see semidefinite programming (SDP) relaxations for nonconvex quadratically constrained quadratic programming (QCQP). There is a well-known special case which has an exact SDP formulation – this is known as the S-lemma and will be the subject of Section 1. After that, we present the relaxations in more generality.

# 1 The S-lemma

The goal in this section is to solve a QCQP with a single constraint

$$\begin{aligned} \min_{x}.\ & q_b(x) \\ \text{s.t. } & q_a(x) \geq 0, \end{aligned} \tag{1}$$

where $q_a, q_b : \mathbb{R}^n \to \mathbb{R}$ are quadratic functions; i.e.,

$$\begin{aligned} q_a(x) &= x^T Q_a x + u_a^T x + c_a, \\ q_b(x) &= x^T Q_b x + u_b^T x + c_b. \end{aligned}$$

- This problem is a convex optimization problem if $Q_b \succeq 0$ and $Q_a \preceq 0$.
- In this lecture, however, we will be making no convexity assumptions. Nevertheless, we show that this nonconvex problem can be solved efficiently. (This, by the way, goes to show that equating "tractibility" and "convexity" is not automatic!)
- Problem (1) appears in many different areas, such as stability problems in dynamics and control, the "trust region problem" in nonlinear programming [3], and robust second order cone programming [1].

What is key to solving problem (1) is the following celebrated result known as the S-lemma [4].

**Theorem 1** (S-lemma (Yakubovich '71 [5]))**.** *Suppose* $\exists \bar{x} \in \mathbb{R}^n$ *s.t.* $q_a(\bar{x}) > 0$. *If*

$$\forall x,\ [q_a(x) \geq 0 \Rightarrow q_b(x) \geq 0],$$

*then*

$$\exists \lambda \geq 0,\ s.t.\ q_b(x) \geq \lambda q_a(x)\ \forall x.$$

You can think of the second inequality as a *certificate* for the first implication.

## 1.1 Using the S-lemma to reformulate the problem as an SDP

The S-lemma is useful for solving problem (1) as it will allow us to rewrite it as an SDP! Note that

$$\begin{bmatrix} \min_x . \;\; q_b(x) \\ \text{s.t.} \;\; q_a(x) \geq 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \max \gamma \\ \text{s.t.} \;\; q_b(x) \geq \gamma \text{ whenever } q_a(x) \geq 0 \end{bmatrix}$$

The latter problem can be rewritten as

$$\begin{bmatrix} \max \gamma \\ \text{s.t.} \;\; [q_a(x) \geq 0] \Rightarrow q_b(x) - \gamma \geq 0 \end{bmatrix} \underset{S-lemma}{\Leftrightarrow} \begin{bmatrix} \max_{\gamma,\lambda} \gamma \\ q_b(x) - \gamma \geq \lambda q_a(x) \; \forall x \\ \lambda \geq 0 \end{bmatrix}$$

Replacing $q_a$ and $q_b$ by their expressions, we get

$$\begin{aligned} &\max_{\gamma,\lambda} . \;\; \gamma \\ &x^T Q_b x + u_b^T x + c_b - \gamma \geq \lambda(x^T Q_a x + u_a^T x + c_a) \; \forall x \\ &\lambda \geq 0. \end{aligned}$$

This can be rewritten equivalently in matrix form

$$\begin{aligned} &\max_{\gamma,\lambda} . \;\; \gamma \\ &\begin{pmatrix} x \\ \hline 1 \end{pmatrix}^T \left(\begin{array}{c|c} Q_b - \lambda Q_a & \frac{1}{2}(u_b - \lambda u_a) \\ \hline \frac{1}{2}(u_b^T - \lambda u_a^T) & c_b - \gamma - \lambda c_a \end{array}\right) \begin{pmatrix} x \\ \hline 1 \end{pmatrix} \geq 0. \end{aligned}$$

Finally, this problem is equivalent to

$$\begin{aligned} &\max_{\gamma,\lambda} . \;\; \gamma \\ &\text{s.t.} \;\; M \succeq 0, \\ &\lambda \geq 0, \end{aligned} \tag{2}$$

where

$$M = \left(\begin{array}{c|c} Q_b - \lambda Q_a & \frac{1}{2}(u_b - \lambda u_a) \\ \hline \frac{1}{2}(u_b^T - \lambda u_a^T) & c_b - \gamma - \lambda c_a \end{array}\right).$$

This last equivalence has to be justified: ($\Leftarrow$) is trivially true. For ($\Rightarrow$), we need to show that $y^T M y \geq 0$ for all $y \in \mathbb{R}^{n+1}$. By contradiction, suppose $\exists y$ such that $y^T M y < 0$. Then,

- if the $(n+1)^{th}$ coordinate of $y$ is nonzero, then rescale $y$ by this coordinate and we obtain a contradiction.
- if the $(n+1)^{th}$ coordinate of $y$ is zero, then by continuity of $y \to y^T M y$, there exists $\bar{y} \neq 0$ such that $\bar{y}^T M \bar{y} < 0$ and the $(n+1)^{th}$ coordinate is nonzero; this brings us back to the previous case.

Notice that formulation (2) of problem (1) is an SDP and can be solved efficiently.

## 1.2 Regularity assumption in the S-lemma

The reqgularity assumption of existence of a point $\bar{x} \in \mathbb{R}^n$ s.t. $q_a(\bar{x}) > 0$ in the statement of the S-lemma is indeed needed as the following example demonstrates.

Example: Let

$$q_a(x) = -x^2$$
$$q_b(x) = -x(x-1) = -x^2 + x.$$

Then $\forall x,\ q_a(x) \geq 0 \Rightarrow q_b(x) \geq 0$. But we cannot have $q_b(x) \geq \lambda q_a(x)$ for some $\lambda \geq 0$. Suppose that we did; this would mean that

$$-x^2 + x \geq -\lambda x^2,\ \forall x,$$

which is impossible as this inequality would be violated around zero since the linear term there dominates the quadratic term.

The regularity assumption above is in fact easy to check. Rewrite $q_a$ in matrix form:

$$q_a(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}^T \begin{pmatrix} Q_a & u_a/2 \\ u_a^T/2 & c_a \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Then the regularity condition is equivalent to the matrix

$$M_a := \begin{pmatrix} Q_a & u_a/2 \\ u_a^T/2 & c_a \end{pmatrix}$$

having at least one positive eigenvalue.

Proof: ($\Leftarrow$) If there exists a positive eigenvalue, let $\begin{pmatrix} \omega \\ t \end{pmatrix}$ be its corresponding eigenvector. Then, if $t \neq 0$, take $\bar{x} = \frac{1}{t}\omega$. If $t = 0$, then by continuity, there exists $\epsilon > 0$ small enough such that

$$\begin{pmatrix} \omega \\ \epsilon \end{pmatrix}^T M_a \begin{pmatrix} \omega \\ \epsilon \end{pmatrix} > 0,$$

and the previous argument can be repeated. ($\Rightarrow$) If $M_a$ does not have a positive eigenvalue, then $M_a \preceq 0$ and

$$\begin{pmatrix} x \\ 1 \end{pmatrix}^T M_a \begin{pmatrix} x \\ 1 \end{pmatrix} \leq 0$$

for all $x$.

## 1.3 Theorems similar to the S-lemma

The S-lemma is a theorem about strong alternatives – it tells you that exactly one of the following conditions can be true (under the regularity assumption):

(1) $\{q_a(x) \geq 0, q_b(x) < 0\}$ is feasible

(2) $\exists \lambda \geq 0$ s.t. $q_b(x) \geq \lambda q_a(x), \ \forall x$.

Recall from Lecture 5 that the Farkas lemmas had a similar flavor for linear inequalities:

$$\{Ax = b, \ x \geq 0\} \text{ is infeasible } \Leftrightarrow \exists y, \text{ s.t. } A^T y \leq 0, b^T y > 0.$$

There is in fact a version of the Farkas lemma, called the homogeneous Farkas lemma, which is even more analogous to the S-lemma (in the linear case):

$$\begin{bmatrix} a_0^T x < 0 \\ a_i^T x \geq 0, \ i = 1, \ldots, m \end{bmatrix} \text{ is infeasible } \Leftrightarrow \exists \lambda_i \geq 0, i = 1, \ldots, m, \text{ s.t. } a_0 = \sum_{i=1}^{m} \lambda_i a_i.$$

Note that all these theorems give "certificates" of infeasibility for a set of inequalities. Later in the course, we will see the concept of "sum of squares (SOS) optimization" which is a generalization of the same idea to arbitrary systems of polynomial equations and inequalities.

### 1.4 Proof of the S-lemma

Our proof follows [1] with some details filled in.

First, we will prove the S-lemma in the homogeneous case.

**Theorem 2** (The homogeneous S-lemma)**.** *Consider the quadratic optimization problem:*

$$\begin{aligned} \min_{x}.\ & x^T Bx \\ s.t.\ & x^T Ax \geq 0. \end{aligned} \tag{3}$$

*Suppose* $\exists \bar{x}$ *s.t.* $\bar{x}^T A\bar{x} > 0$ *and suppose that* $\forall x$, $x^T Ax \geq 0$ *implies* $x^T Bx \geq 0$. *Then,* $\exists \lambda \geq 0$, *s.t.* $B \succeq \lambda A$.

The proof will crucially use the following lemma which is interesting in its own right.

**Lemma 1.** *Given two symmetric matrices* $P$ *and* $Q$*, if* $Tr(P) \geq 0$*, and* $Tr(Q) < 0$*, then* $\exists e \in \mathbb{R}^n$ *s.t.* $e^T Pe \geq 0$*, but* $e^T Qe < 0$.

*Proof.* Let us write $Q = U^T \Lambda U$ (where $\Lambda$ is diagonal and $U$ is orthonormal). Observe that

$$\text{Tr}(Q) = \text{Tr}(U^T \Lambda U) = \text{Tr}(UU^T \Lambda) = \text{Tr}(\Lambda) =: \theta < 0.$$

Let $\eta \in \mathbb{R}^n$ be a random vector, whose entries are iid and $\pm 1$ with equal probability. Let us multiply $P$ and $Q$ on both sides by $U^T \eta$:

- $(U^T \eta)^T Q(U^T \eta) = \eta^T U^T QU\eta = \eta^T \Lambda \eta = \text{Tr}(\Lambda) = \theta < 0,\ \forall \eta,$
- $(U^T \eta)^T P(U^T \eta) = \eta^T (UPU^T)\eta.$
  Let's compute the expectation of this latter expression. For a general matrix $G \in S^{n\times n}$,

  $$E[\eta^T G\eta] = E[\sum_{ij} G_{ij}\eta_i\eta_j] = \text{Tr}(G).$$

  So $E[\eta^T (UPU^T)\eta] = \text{Tr}(UPU^T) = \text{Tr}(UU^T P) = \text{Tr}(P) \geq 0.$

This means that $\exists \bar{\eta} \in \{-1, 1\}^n$ s.t. $(U^T \bar{\eta})^T P(U^T \bar{\eta}) \geq 0$ as otherwise the expectation would not be nonnegative. We can then take $e = U^T \bar{\eta}$. □

*Proof.* (of the homogeneous S-lemma, i.e., Theorem 2) Observe that under the assumptions of the theorem, the optimal value of (3) is always zero (why?). Consider a new optimization problem[1]:

$$
\begin{aligned}
&\min_{x}.\ x^T B x \\
&\text{s.t. } x^T A x \geq 0 \\
&x^T x = n.
\end{aligned}
\tag{4}
$$

Note that strict feasibility of (3) implies strict feasibility of (4). Indeed, let $x$ be strictly feasible for (3), then $x \neq 0$ ($x = 0$ is not strictly feasible). So one can rescale $x$ to $\tilde{x}$ so that $\tilde{x}^T \tilde{x} = n$, but we still have $\tilde{x}^T A \tilde{x} > 0$. Also observe that under the assumption $x^T A x \geq 0 \Rightarrow x^T B x \geq 0$, the optimal value of problem (4) must be nonnegative.

Taking $X = xx^T$, notice that the previous problem is equivalent to

$$
\begin{aligned}
&\min_{X \in S^{n \times n}}.\ \mathrm{Tr}(BX) \\
&\text{s.t. } \mathrm{Tr}(AX) \geq 0 \\
&\mathrm{Tr}(X) = n \\
&X \succeq 0 \\
&\mathrm{rank}(X) = 1.
\end{aligned}
\tag{5}
$$

We can obtain an SDP relaxation for problem (5) simply by dropping the rank constraint:

$$
\begin{aligned}
&\min_{X \in S^{n \times n}}.\ \mathrm{Tr}(BX) \\
&\text{s.t. } \mathrm{Tr}(AX) \geq 0 \\
&\mathrm{Tr}(X) = n \\
&X \succeq 0.
\end{aligned}
\tag{6}
$$

The dual of this SDP reads

$$
\begin{aligned}
\max_{\mu, \lambda}.\ \ & n\mu \\
& \lambda A + \mu I \preceq B \\
& \lambda \geq 0.
\end{aligned}
\tag{7}
$$

If we argue that

[1] The reason for adding the new constraint $x^T x = n$ will become clear shortly.

(i) there is no duality gap between (6) and (7),

(ii) the optimal value of (6) is $\geq 0$,

then we would be done as the dual program tells us that $\exists \lambda \geq 0,\ \mu \geq 0$ s.t. $B - \lambda A \succeq \mu I \succeq 0 \Rightarrow \exists \lambda \geq 0$ s.t. $B \succeq \lambda A$. Let us argue these two claims separately.

(i) To show that there is no duality gap, we show that both problems are strictly feasible. To see this for the primal, take

$$\hat{X} = \frac{\bar{x}\bar{x}^T + \alpha I}{\text{Tr}(\bar{x}\bar{x}^T + \alpha I)} n,$$

where $\alpha > 0$ is small and $\bar{x}$ is strictly feasible for (4). Notice that such an $X$ is strictly feasible. For the dual, it is easy to see that if we fix $\lambda > 0$ then we can pick $\mu$ negative enough such that $B - \lambda A - \mu I \succ 0$.
Note that if we had not added the constraint $x^T x = n$ to (3), then we would not have the dual variable $\mu$ in 7 which helped us argue that the dual is strictly feasible.

(ii) The optimal value of (6) is nonnegative.
Observe that the feasible set of (6) is compact because positive semidefiniteness of $X$ implies that $||X||_2 \leq \text{Tr}(X) \leq n$. This means that $X^*$ is achieved. Since $X^* \succeq 0$, it has a Cholesky decomposition $X^* = DD^T$. We have

$$\begin{aligned} \text{Tr}(AX^*) &= \text{Tr}(ADD^T) = \text{Tr}(D^T AD) \geq 0, \\ \text{Tr}(BX^*) &= \text{Tr}(D^T BD) =: n\theta^*, \end{aligned}$$

where $\theta^*$ is by definition the optimal value of (7) (and (6)) divided by $n$. Suppose for the sake of contradiction that we had $\theta^* < 0$. Then by Lemma 1 (taking $P = D^T AD$ and $Q = D^T BD$) $\exists e \in \mathbb{R}^n$ s.t.

$$\begin{aligned} e^T D^T ADe \geq 0 &\Rightarrow (De)^T Q(De) \geq 0 \\ e^T D^T BDe < 0 &\Rightarrow (De)^T B(De) < 0. \end{aligned}$$

This contradicts the hypothesis of S-lemma (i.e., $x^T Ax \geq 0 \Rightarrow x^T Bx \geq 0$). So $\theta^* \geq 0$ and the optimal value of (6) is nonnegative.

This concludes the proof of the homogeneous S-lemma. $\square$

Now, let's prove the general case. As a reminder, we have two quadratic functions

$$q_a(x) = x^T Q_a x + u_a^T x + c_a,$$
$$q_b(x) = x^T Q_b x + u_b^T x + c_b.$$

We are supposing that $\exists \bar{x} \in \mathbb{R}^n$ s.t. $q_a(\bar{x}) > 0$ (our regularity assumption). We want to show that if

$$\forall x,\ q_a(x) \geq 0 \Rightarrow q_b(x) \geq 0,$$

then

$$\exists \lambda \in \mathbb{R} \geq 0 \text{ s.t. } q_b(x) \geq \lambda q_a(x)\ \forall x.$$

*Proof.* Let us homogenize the polynomials with a new variable $t \in \mathbb{R}$:

$$\tilde{q}_a(x,t) = x^T Q_a x + u_a^T x t + c_a t^2$$
$$\tilde{q}_b(x,t) = x^T Q_b x + u_b^T x t + c_b t^2.$$

Observe that the regularity assumption is satisfied on $\tilde{q}_a$: if $\exists \bar{x}$ s.t. $q_a(\bar{x}) > 0$, then take the point $(\bar{x}, 1)$ and observe that $\tilde{q}_a(\bar{x}, 1) > 0$.

<u>Claim:</u> For all $x, t$, $\tilde{q}_a(x,t) \geq 0 \Rightarrow \tilde{q}_b(x,t) \geq 0$.

<u>Proof:</u> Suppose $\exists x, t$ such that $\tilde{q}_a(x,t) \geq 0$ but $\tilde{q}_b(x,t) < 0$.

(1) If $t \neq 0$, then evaluation at $(\frac{x}{t}, 1)$ gives a contradiction as it implies the same inequalities for $q_a$ and $q_b$.

(2) If $t = 0$, and $\tilde{q}_a(x,t) > 0$, then by continuity, get a nonzero $t$ and repeat the previous step.

(3) If $t = 0$ and $\tilde{q}_a(x,t) = 0$. This means that $x^T Q_a x = 0$ and $x^T Q_b x < 0$. Then change $t$ slightly to make it nonzero while keeping $\tilde{q}_b < 0$. After that, change $x$ to $\gamma x$ for $|\gamma|$ large enough so that

- In $\tilde{q}_a$, $u_a^T(\gamma x)t$ becomes positive and dominates the constant term, while $(\gamma x)^T Q(\gamma x)$ clearly stays at zero.
- In $\tilde{q}_b$, $(\gamma x)^T Q_b(\gamma x)$ becomes large and negative and dominates the other terms.

We can then repeat step (1).

With this claim established, we can apply the homogeneous S-lemma. This tells us that $\exists \lambda \geq 0$ such that

$$\tilde{q_b}(x,t) \geq \lambda \tilde{q_a}(x,t) \; \forall x, t.$$

Set $t = 1$ and we get that $\exists \lambda \geq 0$ such that

$$q_b(x) \geq \lambda q_a(x) \; \forall x.$$

□

# 2 Lower bounds for nonconvex QCQP

## 2.1 Generalization of the S-lemma

For more than two quadratics, there is no direct analogue of the S-lemma, but we can still get lower bounds on a general QCQP by applying the same concept. Consider a general QCQP:

$$\begin{aligned} \text{min.} \; & x^T H_0 x + 2c_0^T x + d_0 \\ \text{s.t.} \; & x^T H_i x + 2c_i^T x + d_i \leq 0, \; i \in I, \\ & x^T H_j x + 2c_j^T x + d_j = 0, \; j \in J. \end{aligned} \tag{8}$$

The optimal value of the following SDP gives a lower bound on the optimal value of our QCQP (why)?

$$\begin{aligned} & \max_{\gamma, \lambda \in \mathbb{R}^{|I|}, \eta \in \mathbb{R}^{|J|}} \gamma \\ & \text{s.t.} \; M \succeq 0, \\ & \lambda \geq 0, \end{aligned} \tag{9}$$

where

$$M = \left( \begin{array}{c|c} H_0 + \sum_{i \in I} \lambda_i H_i - \sum_{j \in J} \eta_j H_j & c_0 + \sum_{i \in I} \lambda_i c_i - \sum_{j \in J} \eta_j c_j \\ \hline (c_0 + \sum_{i \in I} \lambda_i c_i - \sum_{j \in J} \eta_j c_j)^T & d_0 + \sum_{i \in I} \lambda_i d_i - \sum_{j \in J} \eta_j d_j - \gamma \end{array} \right).$$

The S-lemma says that when $|J| = 1$ and $|I| = 0$, the lower bound returned by the SDP is guaranteed to be exact. In other cases, solving the SDP is still valuable as it provides a lower bound (which may or may not be exact).

### 2.2 Rank relaxation for nonconvex QCQP

Consider once again the general QCQP:

$$\begin{aligned} \text{min. } & x^T H_0 x + 2c_0^T x + d_0 \\ \text{s.t. } & x^T H_i x + 2c_i^T x + d_i \leq 0, \; i \in I, \\ & x^T H_j x + 2c_j^T x + d_j = 0, \; j \in J. \end{aligned} \tag{10}$$

Introducing a new variable $X \in S^{n \times n}$, this problem is equivalent to

$$\begin{aligned} \min_{x,X}. \; & \mathrm{Tr}(H_0 X) + 2c_0^T x + d_0 \\ \text{s.t. } & \mathrm{Tr}(H_i X) + 2c_i^T x + d_i \leq 0, \; i \in I, \\ & \mathrm{Tr}(H_j X) + 2c_j^T x + d_j = 0, \; j \in J, \\ & X = xx^T. \end{aligned}$$

We now relax the constraint $X = xx^T$ into the convex constraint $X \succeq xx^T$ which can equivalently be written by taking the Schur complement as

$$\left(\begin{array}{c|c} X & x \\ \hline x^T & 1 \end{array}\right) \succeq 0.$$

Hence, (10) can be relaxed to an SDP

$$\begin{aligned} \min_{x,X}. \; & \mathrm{Tr}(H_0 X) + 2c_0^T x + d_0 \\ \text{s.t. } & \mathrm{Tr}(H_i X) + 2c_i^T x + d_i \leq 0, \; i \in I, \\ & \mathrm{Tr}(H_j X) + 2c_j^T x + d_j = 0, \; j \in J, \\ & \left(\begin{array}{c|c} X & x \\ \hline x^T & 1 \end{array}\right) \succeq 0, \end{aligned}$$

whose optimal value provides a lower bound on the optimal value of (10). The relaxations seen in these two subsections are two dual approaches to produce SDP-based lower bounds on QCQP.

# Notes

Further reading for this lecture can include Chapter 4 of [1] on semidefinite programming, and Chapter 10 of [2].

## References


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