Computational Complexity in Numerical Optimization

ORF 523

Lecture 13

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When can we solve an optimization problem efficiently?

- **Arguably, this is the main question an optimizer should be concerned with. At least if his/her view on optimization is computational.**

A quote from our first lecture:

- The right question in optimization is not
  - Which problems are optimization problems? (The answer would be everything.)
- The right question is
  - *Which optimization problems can we solve?*

Is it about the number of decision variables and constraints?

- **No!**

- Fermat's conjecture (1637):

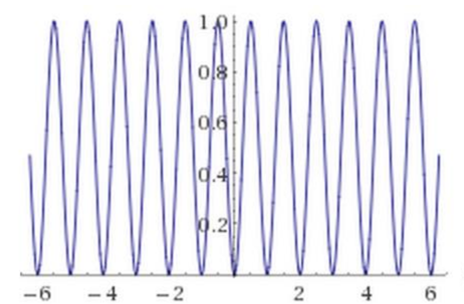
For $n \geq 3$, the equation $x^n + y^n = z^n$ does not have a solution over positive integers.

- Consider the following optimization problem:

$$\min_{x,y,z,n} \quad (x^n + y^n - z^n)^2$$

$$\text{s.t.} \quad x \geq 1,\ y \geq 1,\ z \geq 1,\ n \geq 3,$$

$$\sin^2 \pi n + \sin^2 \pi x + \sin^2 \pi y + \sin^2 \pi z = 0.$$



- Innocent-looking optimization problem: 4 variables, 5 constraints.

- If you could show the optimal value is non-zero, you would prove Fermat's conjecture!

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Is it about being linear versus nonlinear?

- **No!**

- We can solve many nonlinear optimization problems efficiently:
  - QP
  - Convex QCQP
  - SOCP
  - SDP
  - …

Is it about being convex versus nonconvex?

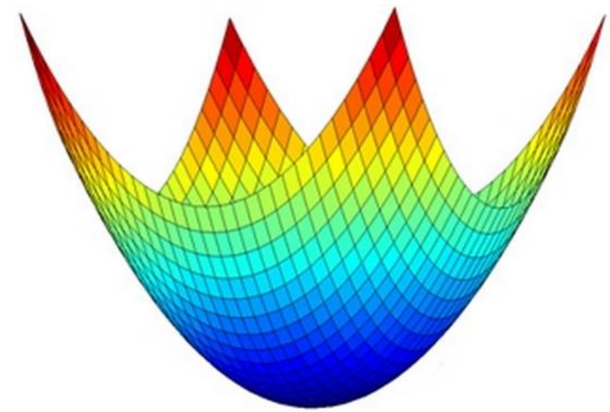
- **Hmmm…many would say**

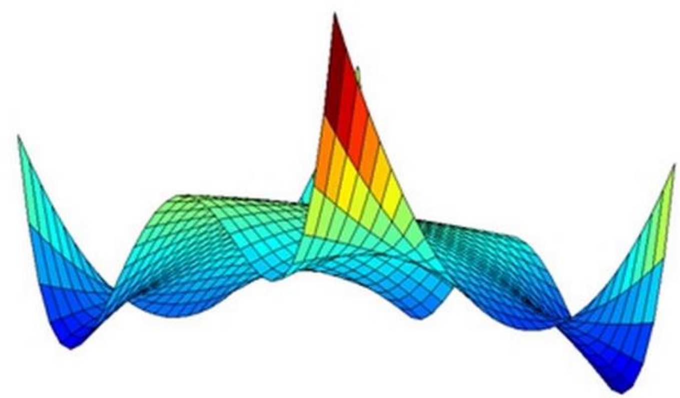
(if I could only get 10 cents for every time somebody gave a talk and justified giving a heuristic by saying that his problem was nonconvex….)

Famous quote-

Rockafellar, ’93:

“In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.”





But is it really?

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Is it about being convex versus nonconvex?

- We already showed that you can write any optimization problem as a convex problem:
  - Write the problem in epigraph form to get a linear objective
  - Replace the constraint set with its convex hull
- So at least we know it's not just about the geometric property of convexity; somehow the (algebraic) description of the problem matters
- There are many convex sets that we know we cannot efficiently optimize over
  - Or we cannot even test membership to
  - We'll see some examples (set of copositive matrices, positive polynomials, etc.).

Is it about being convex versus nonconvex?

- Even more troublesome, there are non-convex problems that are easy.
- Can you name four of them that we've seen already?

①

$$\min_x \; x^T B x$$
$$\text{s.t.} \quad x^T x = 1$$

(λ_{min})

②

$$\min_x \; x^T Q_1 x + b_1^T x + c_1$$
$$\text{s.t.} \quad x^T Q_2 x + b_2^T x + c_2 \leq 0$$

(The S-lemma)

Is it about being convex versus nonconvex?

- Even more troublesome, there are non-convex problems that are easy.
- Can you name four of them that we've seen already?

③

$$\min_{B} \|A - B\|_F$$

$$\text{rank}(B) \leq k$$

(SVD)

④

State feedback stabilization

Given $A, B, \overset{?}{\exists} K$ s.t.

$$\rho(A + BK) < 1\ ?$$

(LMI tricks)

Is it about being convex versus nonconvex?

- I admit the question is tricky
- For some of these non-convex problems, one can come up with an equivalent convex formulation
- But how can we tell when this can be done?
- We saw, e.g., that when you tweak the problem a little bit, the situation can change
  - Recall, e.g., that for output feedback stabilization we had no convex formulation
  - Or for generalization of the S-lemma to QCQP with more constraints…
- Can we have techniques for showing that (an efficiently solvable) convex formulation is impossible?

Is it about being convex versus nonconvex?

- **My view on this question:**
- Convexity is a rule of thumb.
- It's a very useful rule of thumb.
  - Often it characterizes the complexity of the problem correctly.
  - But there are exceptions.
- Incidentally, it may not even be easy to check convexity unless you are in pre-specified situations (recall the CVX rules for example).
  - Maybe good enough for many applications.
- To truly and rigorously speak about complexity of a problem, we need to go beyond this rule of thumb.
- Computational complexity theory is an essential tool for optimizers.

Why computational complexity?

- **What is computational complexity theory?**

It's a branch of mathematics that provides a formal framework for studying how efficiently one can solve problems on a computer.

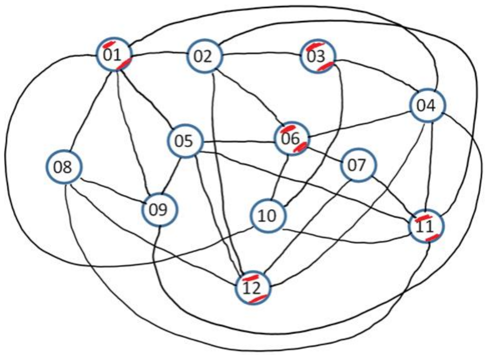
- This is absolutely crucial to optimization and many other computational sciences.
  - In optimization, we are constantly looking for algorithms to solve various problems as fast as possible. So it is of immediate interest to understand the fundamental limitations of efficient algorithms.

- To start, how can we formalize what it means for a problem to be "easy" or "hard"?
  - Let's begin by understanding what it means to have a "problem"!

Optimization problems/Decision problems/Search problems

- Let's introduce these concepts using an example we know well: stable set (aka independent set) of a graph.

- Recall that a stable set in a graph G is a subset of the nodes with no edges among them.



Optimization problem:

- Given a graph G, find its largest stable set.

Decision problem:

- Given a graph G and an integer b, decide if there exists a stable set of size $\geq b$?

(answer to a decision question is just YES or NO)

Search problem:

- Given a graph G and an integer b, find a stable set of size $\geq b$ or declare that none exists.

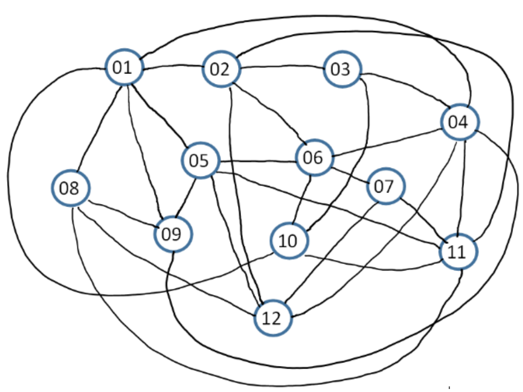
- It turns out that all three problems are equivalent, in the sense that if you could solve one efficiently, you could also solve the other two (why?). See Ex. 8.1,8.2 of [DPV].

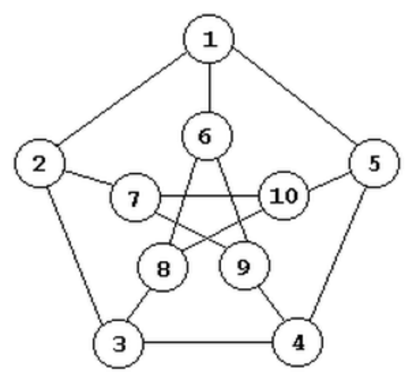
- We will focus on decision problems, since it's a bit cleaner to develop the theory there, and since it can only make our negative results stronger.

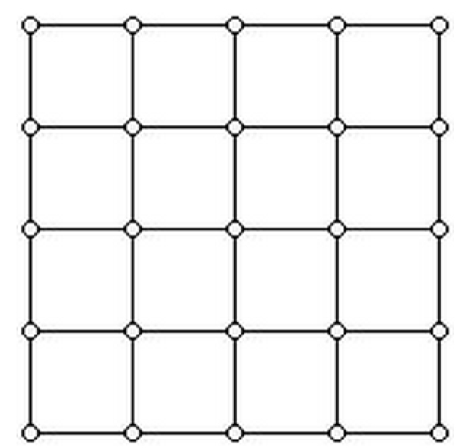
A “problem” versus a “problem instance”

- A (decision) problem is a general description of a problem to be answered with yes or no.
- Every decision problem has a *finite input* that needs to be specified for us to choose a yes/no answer.
- Each such input defines an **instance** of the problem.
- A decision problem has an infinite number of instances.
(Why doesn’t it make sense to study problems with a finite number of instances?)

- Different instances of the STABLE SET problem:
(It is common to use capital letters for the name of a decision problem.)







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Examples of decision problems

- **LINEQ**
- **Input:** An $m \times n$ matrix A and an $m \times 1$ vector b, both with rational entries.
- **Question:** Is there a solution to the linear system $Ax = b$?

An instance of LINEQ:

$$2x_1 + 7x_2 = 6$$
$$\tfrac{1}{2}x_1 - x_2 = -\tfrac{1}{3}$$

$$A = \begin{pmatrix} 2 & 7 \\ \tfrac{1}{2} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ -\tfrac{1}{3} \end{pmatrix}$$

- **ZOLINEQ**
- **Input:** An $m \times n$ matrix A and an $m \times 1$ vector b, both with rational entries.
- **Question:** Is there a 0/1 solution x to the linear system $Ax = b$?

An instance of ZOLINEQ:

$$2x_1 + 7x_2 = 6$$
$$\tfrac{1}{2}x_1 - x_2 = -\tfrac{1}{3}$$

$$A = \begin{pmatrix} 2 & 7 \\ \tfrac{1}{2} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ -\tfrac{1}{3} \end{pmatrix}$$

- **Remark.** Input is rational so we can represent it with a finite number of bits. This is the so-called "*bit model of computation*", aka the *"Turing model."*

Examples of decision problems

▪LP

▪**Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational
a rational number k

▪**Question:** Is the optimal value of the LP (in standard form) $\leq k$?

(This is equivalent to testing LP feasibility.)

An instance of LP: $A = \begin{pmatrix} 4 & 7 & 1/2 \\ 3 & -1 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $k = 5$.

▪IP

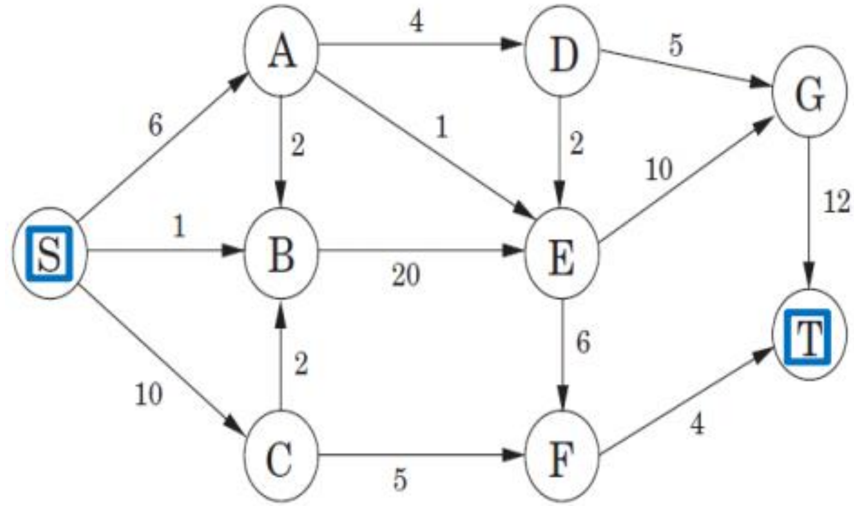
▪**Input:** same as above

▪**Question:** Is there an *integer* feasible solution to the LP with objective value $\leq k$?

Examples of decision problems

Let’s look at a problem we have seen…

An instance of MAXFLOW:



Can you formulate the decision problem?

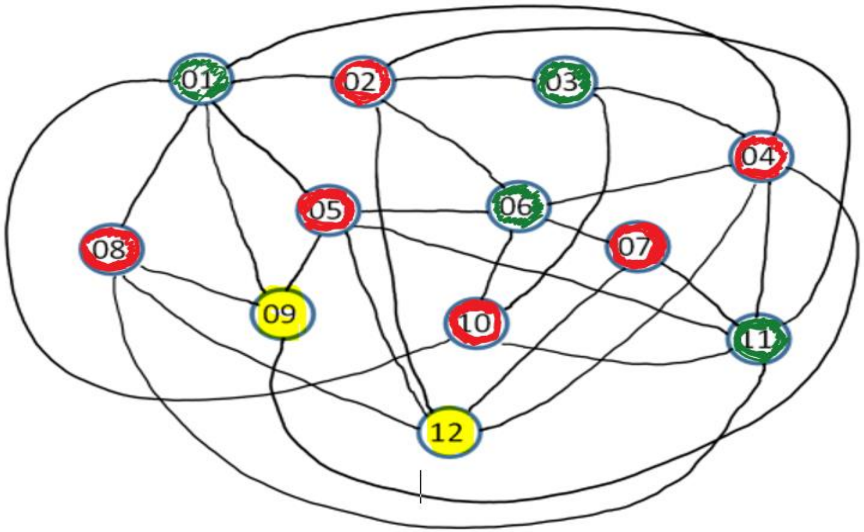
- **MAXFLOW**
- **Input:** A directed graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a designated node S, a designated node T, a rational number k.
- **Question:** Is there a flow of value $\geq k$ from S to T that respetcs the edge cost constraints and the conservation of flow constraints?

Examples of decision problems

- A graph is said to be k-colorable if there is a way to color its nodes with k colors such that no two adjacent nodes get the same color.

- For example, the following graph is 3-colorable.

- Graph coloring has important applications in job scheduling.



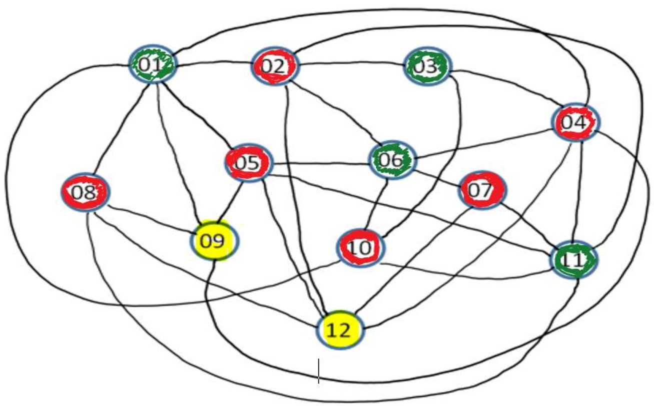
- **COLORING**
- **Input:** An undirected graph G and a positive integer k.
- **Question:** Is the graph k-colorable?

- We want to understand how fast can all these problems be solved?

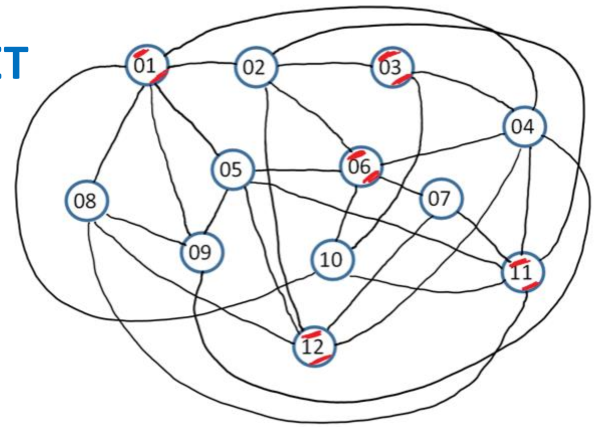
Size of an instance

- To talk about the running time of an algorithm, we need to have a notion of the "size of the input".
- Of course, an algorithm is allowed to take longer on larger instances.

- **COLORING**



- **STABLE SET**



- Reasonable candidates for input size:
  - Number of nodes n
  - Number of nodes + number of edges (number of edges can at most be n(n-1)/2)
  - Number of bits required to store the adjacency matrix of the graph

A =

0	1	0	1	1	0	0	1	1	1	0	0
1	0	1	0	0	1	0	0	0	0	1	1
0	1	0	1	0	0	0	0	0	1	0	0
1	0	1	0	0	1	0	0	1	0	1	1
1	0	0	0	0	1	0	0	1	0	1	1
0	1	0	1	1	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	1	1
1	0	0	0	0	0	0	0	1	0	1	1
1	0	0	1	1	0	0	1	0	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
0	1	0	1	1	0	1	1	0	1	0	0
0	1	0	1	1	0	1	1	0	0	0	0

Size of an instance

- In general, can think of input size as the total number of bits required to represent the input.

- For example, consider our LP problem:

- **LP**

- **Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational
  a rational number k

- **Question:** Is the optimal value of the LP (in standard form) $\leq k$?

- Input size is bounded by $2(mn + m + n + 1)\log L$, where L is the largest integer appearing in the numerator or denominator of any entry of A, b, c, k.

- Same idea holds for all other decision problems we introduced.

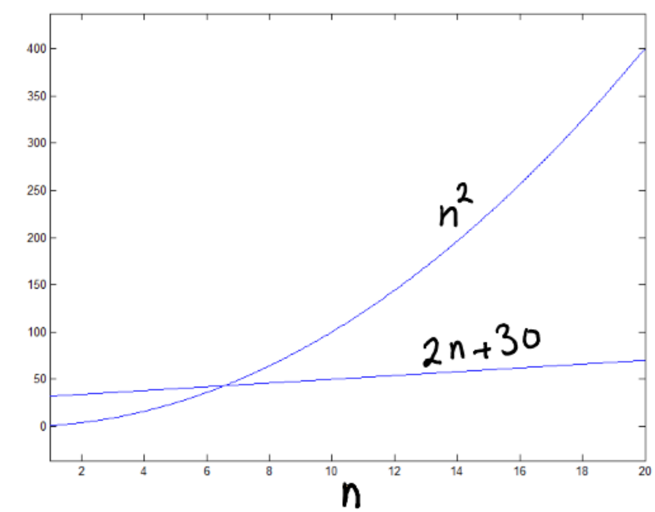
Useful notation for referring to running times

Definition. Let $f, g: \mathbb{R}_+ \to \mathbb{R}_+$. We write

- $f(n) = O\big(g(n)\big)$, if $\exists n_0, c > 0$, such that
$$f(n) \leq c g(n), \forall n \geq n_0.$$
- $f(n) = \Omega\big(g(n)\big)$, if $\exists n_0, c > 0$, such that
$$f(n) \geq c g(n), \forall n \geq n_0.$$
- $f(n) = \Theta\big(g(n)\big)$, if we have both $f(n) = O\big(g(n)\big)$ and $f(n) = \Omega\big(g(n)\big)$.

Examples.

- $5n^3 + 2n^2 + 40 = \Theta(n^3)$.
- $n \log n = O(n^2)$.
- $n \log n = \Omega(n)$.
- $\forall c, k > 0, 2^{cn} = \Omega\big(n^k\big)$.



$2n + 30 = O(n^2)$

Polynomial-time and exponential-time algorithms

- A polynomial-time algorithm is an algorithm whose running time as a function of the input size is $O(p(n))$ for some polynomial function p.
- Equivalent definition: Running time is $O(n^k)$ for some positive integer k.
- Note: this is the **worst-case running time** over all inputs of size n.

- An exponential-time algorithm is an algorithm whose running time as a function of the input size is $\Omega(2^{cn})$ for some positive constant c.
- Once again, when we talk about running time for a given input size n, we mean the worst-case running time over all inputs of size n.
- There are also algorithms with running time in between (e.g., $O(n^{\log n})$), but these also are perceived as slow.

Comparison of running times

Time complexity function	Size n					
	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.0001 second	.0004 second	.0009 second	.0016 second	.0025 second	.0036 second
n^3	.001 second	.008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minutes	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	2×10^8 centuries	1.3×10^{13} centuries

Image credit: [GJ79]

Can Moore's law come to rescue?

Size of Largest Problem Instance
Solvable in 1 Hour

Time complexity function	With present computer	With computer 100 times faster	With computer 1000 times faster
n	N_1	$100\ N_1$	$1000\ N_1$
n^2	N_2	$10\ N_2$	$31.6\ N_2$
n^3	N_3	$4.64\ N_3$	$10\ N_3$
n^5	N_4	$2.5\ N_4$	$3.98\ N_4$
2^n	N_5	$N_5+6.64$	$N_5+9.97$
3^n	N_6	$N_6+4.19$	$N_6+6.29$

Effect of improved technology on several polynomial and exponential time algorithms.

Image credit: [GJ79]

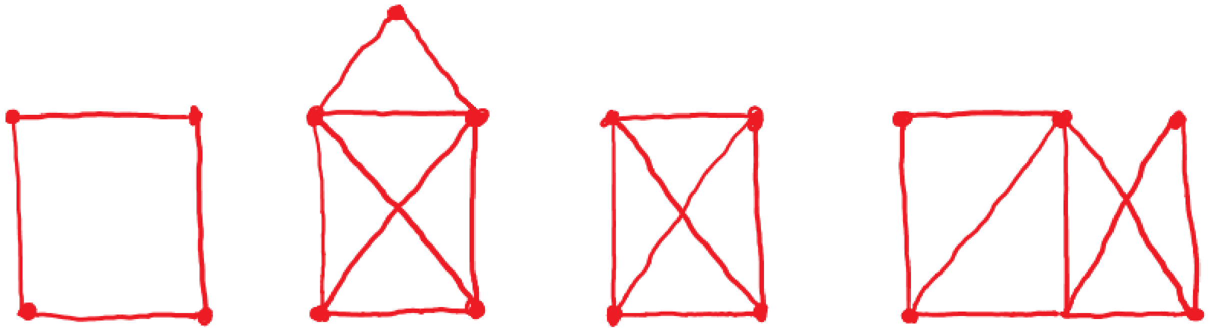
The complexity class P

- The class of all decision problems that admit a polynomial-time algorithm.

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,…

Example of a problem in P

- **PENONPAPER**
- **Input:** A connected undirected graph.
- **Question:** Can you draw it without lifting your pen from the paper?



- **Euler:** answer to PENONPAPER is YES if and only if "every node, with the possible exception of two nodes, has even degree."
- This condition can obviously be checked in polynomial time.
- Hence PENONPAPER∈P.

- **Peek ahead:** this problem is asking if there is a path that visits **every *edge* exactly once**.
- If we were to ask for a path that instead visits **every *node* exactly once**, we would have a completely different story in terms of complexity!

How to prove a problem is in P?

- Develop a poly-time algorithm from scratch! Can be **far from trivial** (examples below).
- Much easier: use a poly-time hammer somebody else has developed. (Reductions!)

- **LINEQ** (solve a system of linear equations)
  - Gaussian elimination -- $O(n^3)$
  - Can also use, e.g., the conjugate gradient algorithm -- $O(n^3)$
  - (Faster algorithms known: Google Strassen)
- **LP** (solve a system of linear inequalities)
  - Was open for a long time – simplex doesn't do it (at least, we don't know how to modify it so it does)
  - The ellipsoid algorithm (Khachiyan-1979)
  - Interior point algorithms (Karmarkar-1984)

The New York Times

BREAKTHROUGH IN PROBLEM SOLVING

By JAMES GLEICK
Published: November 19, 1984

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

- **PRIMES** (decide if a given integer is prime)
  - Was open for a long time -- Proved to be in P by Agrawal-Kayal-Saxena in 2002.
    - Kayal and Saxena were undergraduates!
  - Why doesn't the naïve algorithm work? "Given n, check all candidate divisors up to $\sqrt{n}$."

New Method Said to Solve Key Problem In Math

The New York Times

By SARA ROBINSON
Published: August 8, 2002

Three Indian computer scientists have solved a longstanding mathematics problem by devising a way for a computer to tell quickly and definitively whether a number is prime -- that is, whether it is evenly divisible only by itself and 1.

An aside: Factoring

- Despite knowing that PRIMES is in P, it is a major open problem to determine whether we can *factor* an integer in polynomial time.

```
RSA-1024 = 13506641086599522334960321627880596993888147560566702752448514385152651060
           48595338339402871505719094417982072821644715513736804197039641917430464965
           89274256239341020864383202110372958725762358509643110564073501508187510676
           59462920556368552947521350085287941637732853390610975054433499981115005697
           7236890927563
```

$100,000 prize money by RSA

```
RSA-2048 = 25195908475657893494027183240048398571429282126204032027777137836043662020
           70759555626401852588078440691829064124951508218929855914917618450280848912
           00728449926873928072877767359714183472702618963750149718246911650776133798
           59095700097330459748808428401797429100642458691817195118746121515172654632
           28221686998754918242243363725908514186546204357679842338718477444792073993
           42365848238242811981638150106748104516603773060562016196762561338441436038
           33904414952634432190114657544454178424020924616515723350778707749817125772
           46796292638635637328991215483143816789988504044536402352738195137863656439
           1212010397122822120720357
```

$200,000 prize money by RSA

- Google "RSA challenge"; was active until 2007.

Reductions

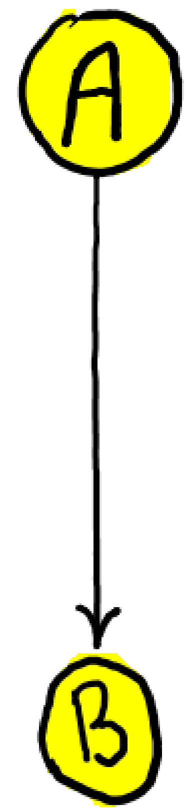
- Many new problems are shown to be in P via **a reduction to a problem that is already known to be in P.**
- What is a reduction?
    - Very intuitive idea -- **A reduces to B means: “If we could do B, then we could do A.”**
    - Being happy in life reduces to finding a good partner.
    - Passing the quals reduces to getting four A-’s.
    - Getting an A+ in ORF 523 reduces to finding the Shannon capacity of C7.
    - …
- Well-known joke - mathematician versus engineer boiling water:

- Day 1:

- Day 2:

Reductions

- A reduction from a decision problem A to a decision problem B is
  - **a "general recipe" (aka an algorithm)** for taking any instance of A and explicitly producing an instance of B, such that
  - the answer to the instance of A is YES if and only if the answer to the produced instance of B is YES.
    (OK for our purposes also if the YES/NO answer gets flipped.)
- This enables us to answer A by answering B.



MAXFLOW→LP

▪MAXFLOW

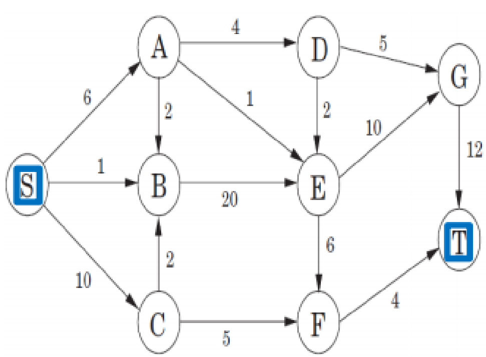
▪**Input:** A directed graph $G(V,E)$, nonnegative rational numbers c_i on each edge, a designated node S, a designated node T, a rational number k.

▪**Question:** Is there a flow of value $\geq k$ from S to T that respetcs the edge cost constraints and the conservation of flow constraints?

▪LP

▪**Input:** An $m \times n$ matrix A, an $m \times 1$ vector b, and an $n \times 1$ vector c, all rational a rational number k.

▪**Question:** Is the optimal value of the LP $\geq k$?



Poly-time reduction

(shown on once instance)

$$\max. \quad x_{SA} + x_{SB} + x_{SC}$$

s.t.

$$x_{SA},\ x_{AD},\ x_{BE},\ \dots,\ x_{GT} \geq 0$$

$$x_{SA} \leq 6,\ x_{AB} \leq 2,\ x_{EG} \leq 10,\ \dots,\ x_{GT} \leq 12$$

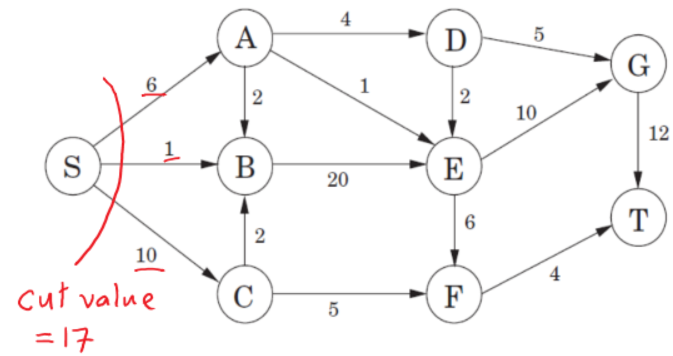
$$\left\{\begin{array}{l} x_{SA} = x_{AD} + x_{AB} + x_{AE} \\ x_{SC} = x_{CB} + x_{CF} \\ \vdots \\ x_{CF} + x_{EF} = x_{FT}. \end{array}\right.$$

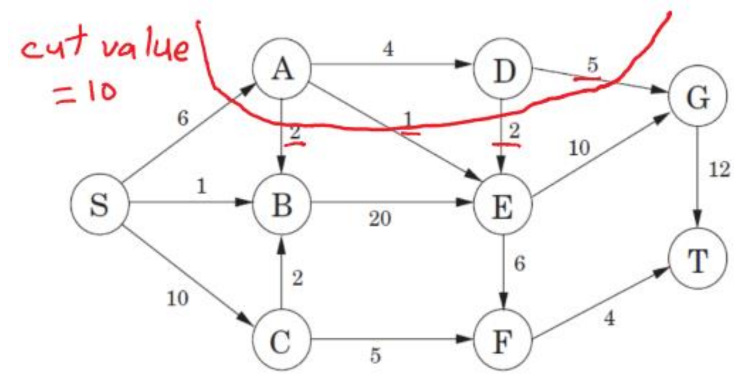
Polynomial time reductions

- So we say that “MAXFLOW reduces to LP”. (Notation: MAXFLOW→LP.)
- Since we know how to solve LP in polynomial time (e.g., via interior point methods), now we know how to solve MAXFLOW in polynomial time. So MAXFLOW∈P.
- This argument relies crucially on the fact that **the reduction is polynomial in length.**
    - Before we even solve the LP, we need to make sure its size is not too big (e.g., it doesn’t have too many decision variables, too many constraints, or data that takes an exponential number of bits to write down.)
    - What does “not too big” mean? The size needs to be polynomial in the size of the instance of the original problem (in this case MAXFLOW).
    - Without this constraint, one could give, e.g., a simple reduction from STABLE SET to LP (do you see how)? This should not happen (we’ll see why soon).

MINCUT

- A cut is a partition of the nodes of a graph into two (non-empty) sets U and $\overline{U}$.
- The value of a cut is the sum of edge weights going from U to $\overline{U}$.





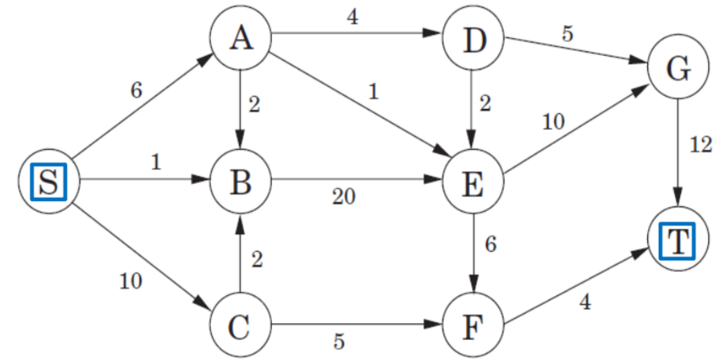
- **MINCUT**

- **Input:** A directed graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a rational number k.
- **Question:** Is there a cut of value $\leq k$?

  - Is MINCUT in P?

    - Yes! We'll reduce it to LP.

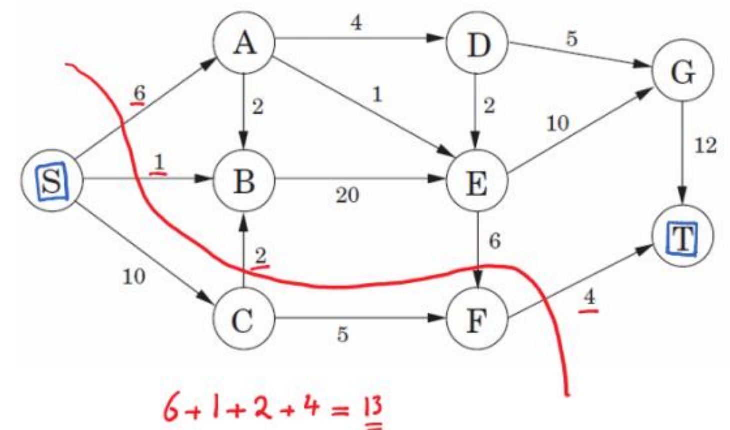
MIN S-T CUT



- **MIN S-T CUT**

- **Input:** A directed graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a rational number k, two designated nodes S and T.

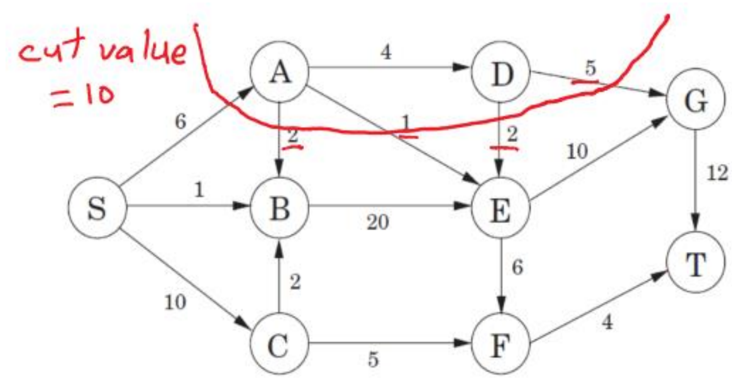
- **Question:** Is there a cut of value $\leq k$?
(that leaves S on one side and T on the other)



- Strong duality of linear programming implies the minimum S-T cut of a graph is exactly equal to the maximum flow that can be sent from S to T.

- Hence, MIN S-T CUT→MAXFLOW

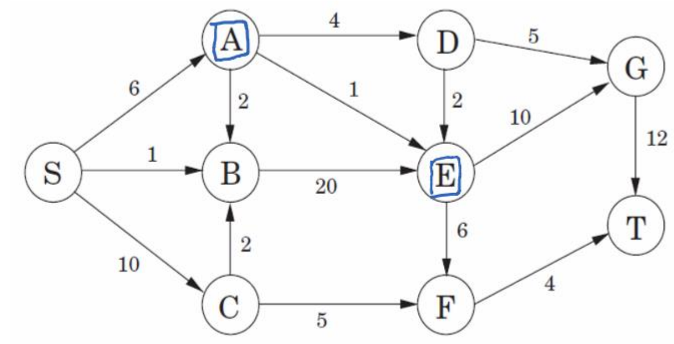
- We have already seen that MAXFLOW→ LP.

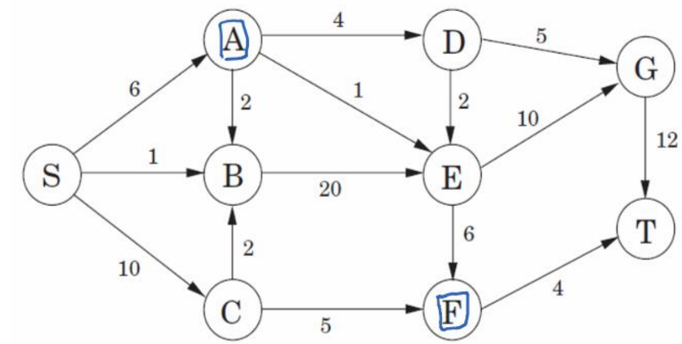


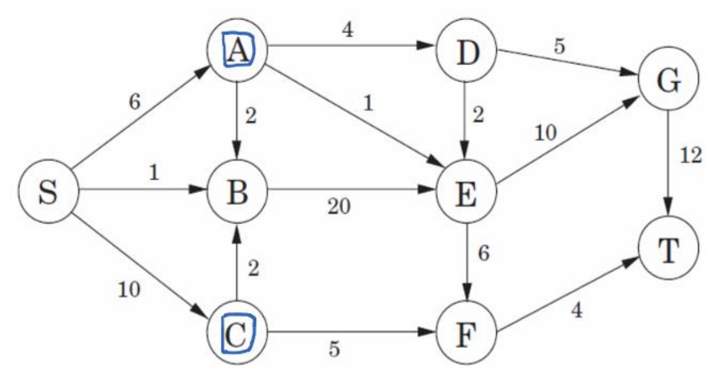
- But what about MINCUT? (without designated S and T)

MINCUT→MIN S-T CUT

- Pick a node (say, node A)
- Compute MIN S-T CUT from A to every other node
- Compute MIN S-T CUT from every other node to A
- Take the minimum over all these 2(|V|-1) numbers
- That's your MINCUT!
- The reduction is polynomial in length.







Overall reduction

- We have shown the following:

MINCUT→MIN S-T CUT→MAXFLOW→LP

- Polynomial time reductions compose (why?):

MINCUT→LP

- MINCUT∈P

- Unfortunately, we are not so lucky with all decision problems…

- Now comes the bad stuff…

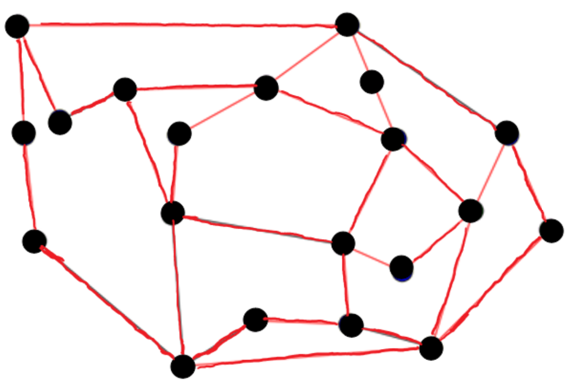
MAXCUT

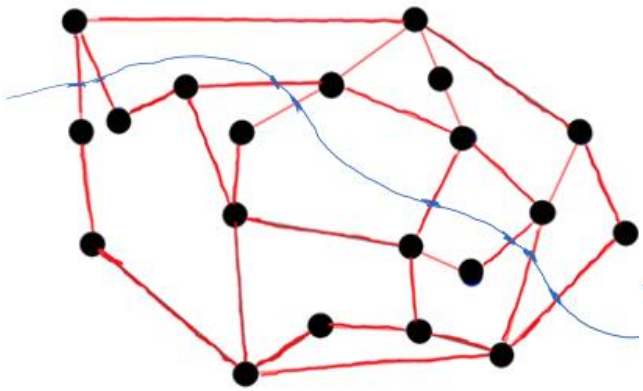
- **MAXCUT**

- **Input:** A graph $G(V, E)$, nonnegative rational numbers c_i on each edge, a rational number k.

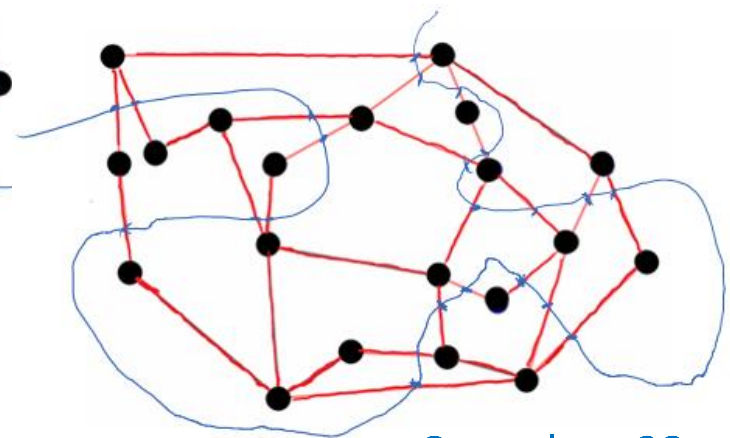
- **Question:** Is there a cut of value $\geq k$?

- Examples with edge costs equal to 1:





- Cut value=8



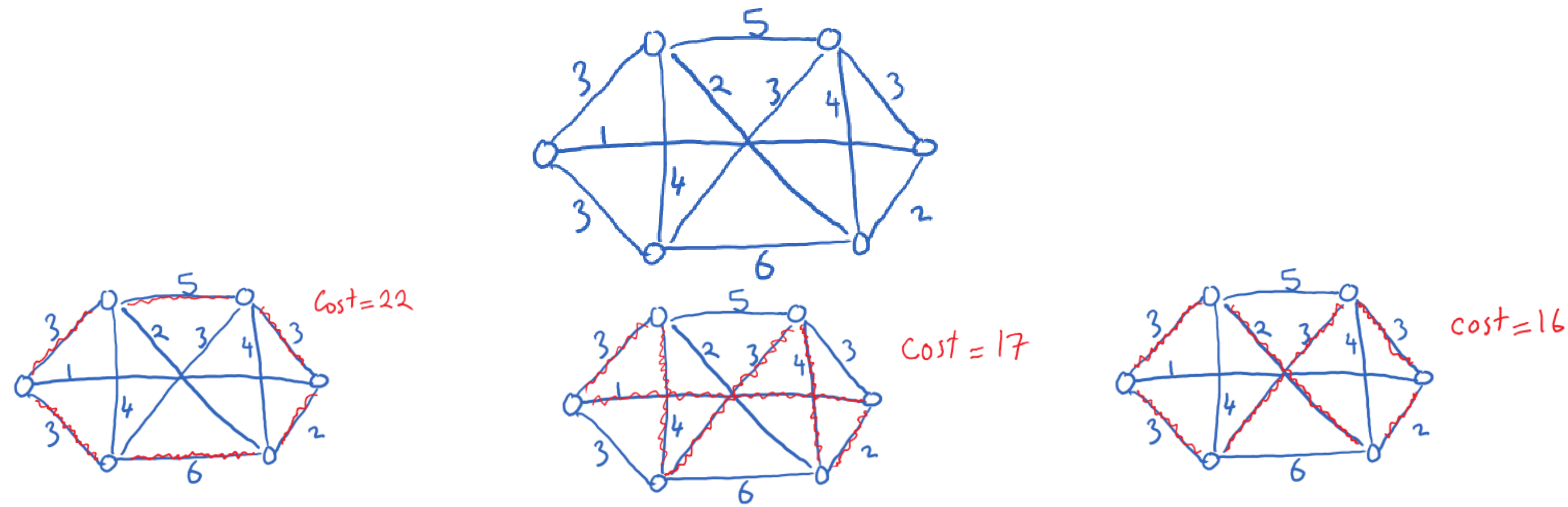
- Cut value=23 (optimal)

- To date, no one has come up with a polynomial time algorithm for MAXCUT.

- We want to understand why that is…

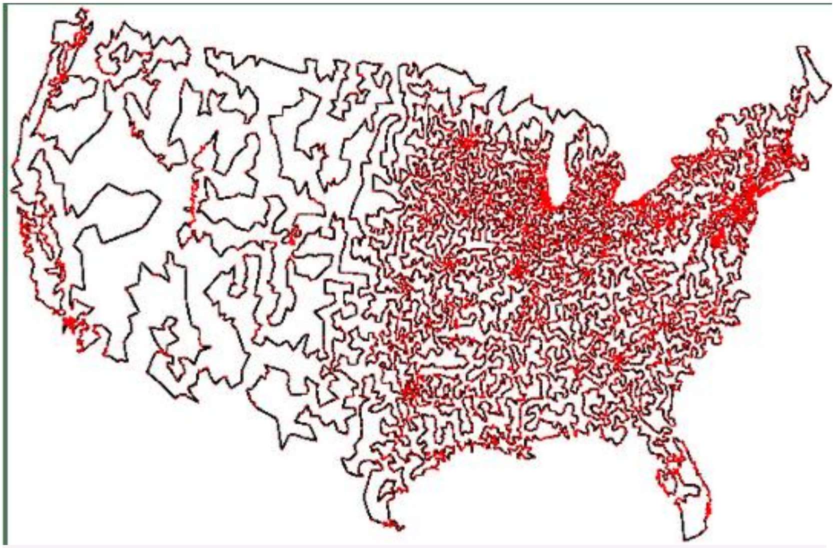
The traveling salesman problem (TSP)

- **TSP**
- **Input:** A graph $G(V,E)$, nonnegative rational numbers c_i on each edge, a rational number k.
- **Question:** Is there a tour of cost $\leq k$ that visits each node exactly once?



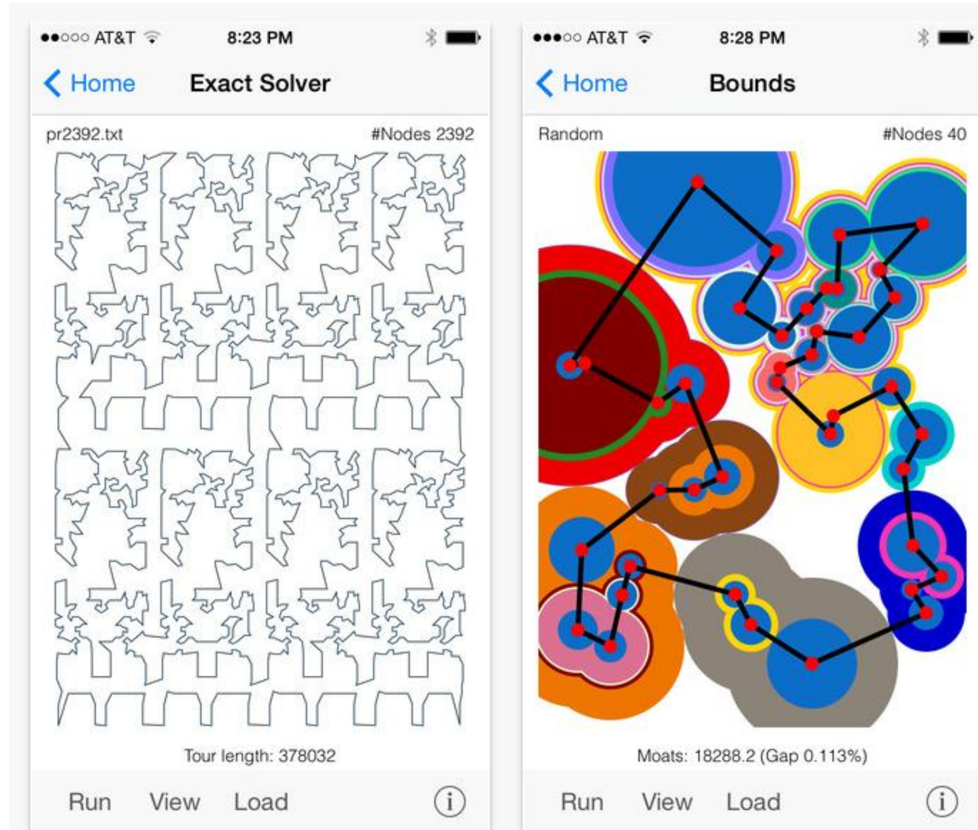
- Again, nobody knows how to solve this efficiently (over all instances).
- Note the sharp contrast with PENONPAPER.
- Amazingly, MAXCUT and TSP are in a precise sense "equivalent": there is a polynomial time reduction between them in either direction.

TSP



24,978 Cities in Sweden
Solved in 2004

15,112 Cities in Germany
Solved in 2001



Reference: http://www.math.uwaterloo.ca/tsp

The complexity class NP

- A decision problem belongs to the class **NP (Nondeterministic Polynomial time)** if every YES instance has a "certificate" of its correctness that can be verified in polynomial time.

- Examples: TSP, MAXCUT, PENONPAPER….what's the certificate in each case?

Remarks.

- A nondeterministic computer is a machine that can "guess" an answer and then verify it. It's a very unrealistic computer.
- NP does not mean "not polynomial"! There are many easy problems in NP (e.g., ADDITION, LINEQ).
- P⊆NP. (The poly-time algorithm itself is a certificate.)
- Note that for a given decision problem, it's not at all clear that a short certificate for the YES answer also implies a short certificate for the NO answer. (Think, e.g., of TSP.)

The complexity class NP

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,...

NP

- TSP
- MAXCUT
- STABLE SET
- SAT
- 3SAT
- PARTITION
- KNAPSACK
- IP
- COLORING
- VERTEXCOVER
- 3DMATCHING
- SUDOKU,...

NP-hard and NP-complete problems

Definition.

- A decision problem is said to be NP-hard if every problem in NP reduces to it via a polynomial-time reduction.
(roughly means "harder than all problems in NP.")

Definition.

- A decision problem is said to be NP-complete if

(i)It is NP-hard

(ii)It is in NP.

(roughly means "the hardest problems in NP.")

Remarks.

- NP-hardness is shown by a reduction from a problem that's already known to be NP-hard.
- Membership in NP is shown by presenting an easily checkable certificate of the YES answer.
- NP-hard problems may not be in NP (or may not be known to be in NP as is often the case.)

The complexity class NP

- ADDITION
- MULTIPLICATION
- LINEQ
- LP
- MAXFLOW
- MINCUT
- MATRIXPOS
- SHORTEST PATH
- SDP $_\epsilon$
- PRIMES
- ZEROSUMNASH
- PENONPAPER,...

- TSP
- MAXCUT
- STABLE SET
- SAT
- 3SAT
- PARTITION
- KNAPSACK
- IP
- COLORING
- VERTEXCOVER
- 3DMATCHING
- SUDOKU,...

$\in$P

NP-complete

The satisfiability problem (SAT)

- **SAT** (one of the most fundamental NP-complete problems.)

- **Input:** A Boolean formula in conjunctive normal form (CNF).

- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

$$\varphi = (x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

Formula

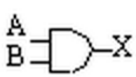
Clauses

Variables: x, y, z

$\vee$: OR, $\wedge$: AND, $\bar{x}$: NOT x

Literal: a variable or its complement.

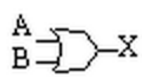
AND



$X = A \cdot B$

A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

OR



$X = A + B$

A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

The satisfiability problem (SAT)

- **SAT**

- **Input:** A Boolean formula in conjunctive normal form (CNF).
- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

$$(x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

YES $x=1, y=1, z=0.$

$$(x \vee y \vee z) \wedge (x \vee \bar{y}) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee z)$$

NO

3SAT

- **3SAT**

- **Input:** A Boolean formula in conjunctive normal form (CNF), where *each clause has exactly three literals.*

- **Question:** Is there a 0/1 assignment to the variables that satisfies the formula?

$$(x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee w \vee \bar{z}) \wedge (\bar{y} \vee w \vee z)$$

- There is a simple reduction from SAT to 3SAT.

- Hence, since SAT is NP-hard, then so is 3SAT. Moreover, 3SAT is clearly in NP (why?), so 3SAT is NP-complete.

ONE-IN-THREE 3SAT

- Has the same input as 3SAT.
- But asks whether there is a 0/1 assignment to the variables that in each clause **satisfies exactly one literal**.

$$\left(\overset{1}{x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}\right) \wedge \left(\overset{0}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}\right) \wedge \left(\overset{0}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3}\right)$$

(satisfiable)

$$x_1 = 1, x_2 = 1, x_3 = 0$$

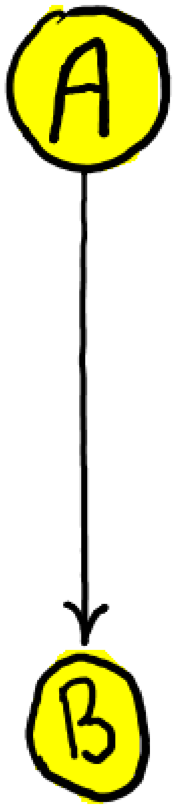
$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

(unsatisfiable)

- Reduction from 3SAT to ONE-IN-THREE 3SAT is on your homework.

Reductions (again)

- A reduction from a decision problem A to a decision problem B is
  - **a "general recipe" (aka an algorithm)** for taking any instance of A and explicitly producing an instance of B, such that
  - the answer to the instance of A is YES if and only if the answer to the produced instance of B is YES.
    (OK for our purposes also if the YES/NO answer gets flipped.)
- This enables us to answer A by answering B.



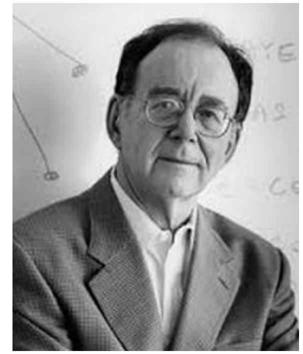
- This time we use the reduction for a different purpose:
  - If A is known to be hard, then B must also be hard.

The first 21 (official) reductions

REDUCIBILITY AMONG COMBINATORIAL PROBLEMS†

Richard M. Karp

University of California at Berkeley



Abstract: A large class of computational problems involve the determination of properties of graphs, digraphs, integers, arrays of integers, finite families of finite sets, boolean formulas and elements of other countable domains. Through simple encodings from such domains into the set of words over a finite alphabet

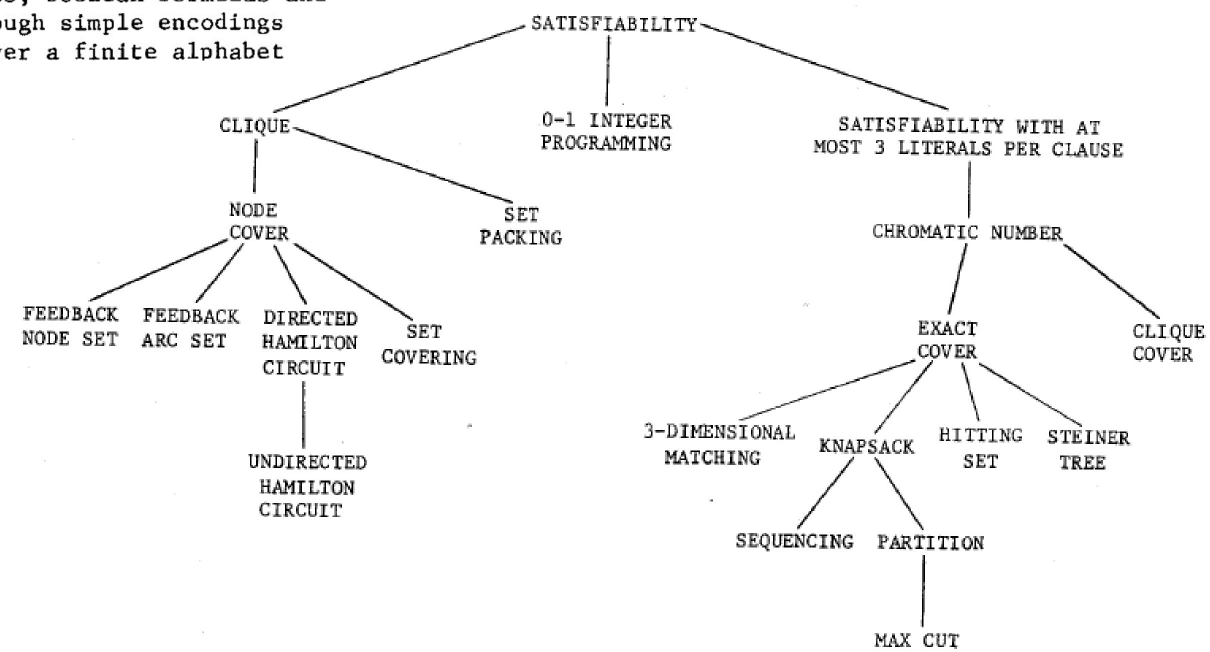
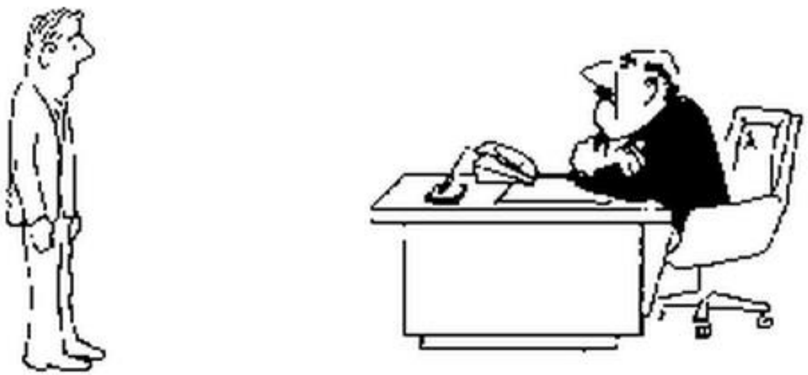


FIGURE 1 - Complete Problems

- Today we have thousands of NP-complete problems. In all areas of science and engineering.

The value of reductions



I can't find an efficient algorithm, I guess I'm just too dumb.

I can't find an efficient algorithm, because no such algorithm is possible

I can't find an efficient algorithm, but neither can all these famous people.

[Garey, Johnson]

Practice with reductions

I'll do a few reductions on the board:

- **3SAT→STABLE SET**
- **STABLE SET→ 0/1 IP** (trivial)
- **STABLE SET→ QUADRATIC EQS** (straightforward)
- **3SAT→POLYPOS (degree 6)**
- **ONE-IN-THREE 3SAT→POLYPOS (degree 4)**
- NP-hardness of testing local optimality

For homework you can do:

- **3SAT → ONE-IN-THREE 3SAT**
- **PARTITION→POLYPOS (degree 4)**

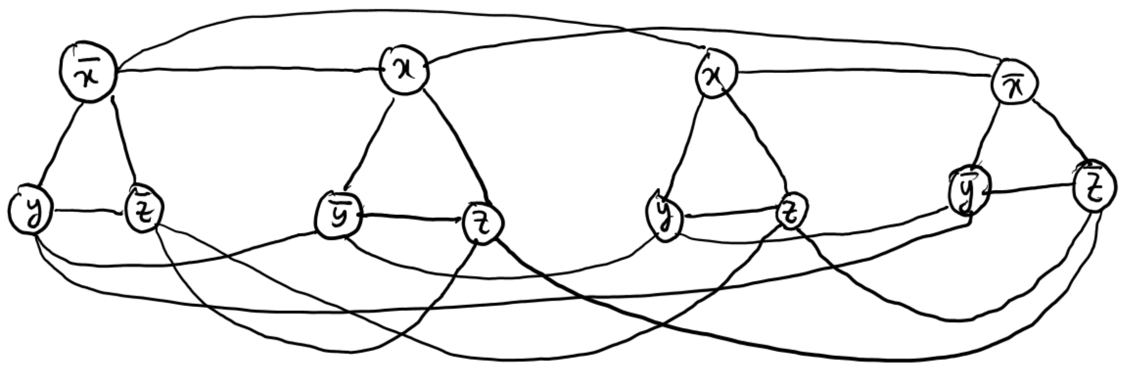
3SAT→STABLE SET

We show the reduction on an instance only. The pattern should be clear.

$$\varphi = (\bar{x} \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

(k clauses)

G:



Construction: For each clause create a "triangle". Across triangles connect each variable to its complement.

Claim: φ is satisfiable $\iff$ $\alpha(G) \geq k$.

STABLE SET → 0/1 Integer Programming

Given $G(V,E)$

$$\alpha(G) \geq K$$

$$\Updownarrow$$

$$\begin{cases} \sum_{i=1}^{n} x_i \geq K \\ x_i + x_j \leq 1 \quad \text{if } ij \in E \\ x_i \in \{0,1\} \quad i=1,\dots,n \end{cases}$$

is feasible.

STABLE SET → Feasibility of Quadratic Equations

Given $G(V,E)$

$$\alpha(G) \geq K$$

$$\Updownarrow$$

$$\begin{cases} \sum_{i=1}^{n} x_i - K = s^2 \\ x_i x_j = 0 \quad \text{if } i,j \in E \\ x_i(1-x_i) = 0 \quad i=1,\dots,n \end{cases}$$

is feasible.

3SAT→POLYPOS (degree 6)

We show the reduction on an instance only. The pattern should be clear.

Start with any instance of 3SAT, such as:

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4)$$

Construct p as

$$p(x) = \underbrace{\sum_{i=1}^{4} \left(x_i(1-x_i)\right)^2}_{\text{encoding 0/1 requirement}} + \Big[\overbrace{(x_1+(1-x_2)+x_3-1)\,(x_1+(1-x_2)+x_3-2)\,(x_1+(1-x_2)+x_3-3)}^{\text{encoding clause 1}}\Big]^2$$

$$+\Big[\overbrace{((1-x_1)+(1-x_2)+x_3-1)\,((1-x_1)+(1-x_2)+x_3-2)\,((1-x_1)+(1-x_2)+x_3-3)}^{\text{clause 2}}\Big]^2$$

$$+\Big[\overbrace{(x_1+x_2+x_4-1)\,(x_1+x_2+x_4-2)\,(x_1+x_2+x_4-3)}^{\text{clause 3}}\Big]^2.$$

Observe that the reduction is polynomial in length.

3SAT→POLYPOS (degree 6)

Claim: A general instance φ of 3SAT will be satisfiable

$$\Updownarrow$$

$\exists \bar{x} \in \mathbb{R}^n$ such that $p(\bar{x}) \leq 0$ (in fact $p(\bar{x}) = 0$), where p is constructed as above.

Pf. ($\Downarrow$) Take $\bar{x}$ to be the satisfying assignment of 3SAT.

All the terms of p vanish (why?)

($\Uparrow$) Suppose φ not satisfiable. Claim: $p(x) > 0 \ \forall x \in \mathbb{R}^n$.

p is a sum of squares $\Rightarrow p(x) \geq 0 \ \forall x \in \mathbb{R}^n$.

- If $x \notin \{0,1\}^n$, then $\sum (x_i(1-x_i))^2 > 0$ (why?)
- If $x \in \{0,1\}^n$, then at least one term out of the terms encoding the clauses will be positive. $\square$

ONE-IN-THREE 3SAT

- Has the same input as 3SAT.
- But asks whether there is a 0/1 assignment to the variables that in each clause **satisfies exactly one literal**.

$$(\overset{1}{x_1} \vee \overset{0}{\bar{x}_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{1}{x_2} \vee \overset{0}{x_3}) \wedge (\overset{0}{\bar{x}_1} \vee \overset{0}{\bar{x}_2} \vee \overset{1}{\bar{x}_3})$$

(satisfiable)

$$x_1 = 1, x_2 = 1, x_3 = 0$$

$$(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

(unsatisfiable)

- Reduction from 3SAT to ONE-IN-THREE 3SAT is on your homework.

ONE-IN-THREE-3SAT→POLYPOS (degree 4)

Almost the same construction as before, except ONE-IN-THREE-3SAT allows us to kill some terms and reduce the degree to 4. Nice!

Start with any instance of ONE-IN-THREE 3SAT, such as:

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4)$$

Construct p as

$$p(x) = \underbrace{\sum_{i=1}^{4} \left(x_i(1-x_i)\right)^2}_{\text{encoding 0/1 requirement}} + \Big[\overbrace{(x_1+(1-x_2)+x_3-1)\,(x_1+\cancel{(1-x_2)}+x_3-2)\,(x_1+\cancel{(1-x_2)}+x_3-3)}^{\text{encoding clause 1}}\Big]^2$$

$$+ \Big[\overbrace{((1-x_1)+(1-x_2)+x_3-1)\,((1-x_1)+\cancel{(1-x_2)}+x_3-2)\,((1-x_1)+\cancel{(1-x_2)}+x_3-3)}^{\text{clause 2}}\Big]^2$$

$$+ \Big[\overbrace{(x_1+x_2+x_4-1)\,(x_1+\cancel{x_2}+x_4-2)\,(x_1+\cancel{x_2}+x_4-3)}^{\text{clause 3}}\Big]^2.$$

Moral: Picking the tight problem for as the base problem of the reduction can make your life a lot simpler!

- An aside: Testing convexity of quartics is also NP-hard! [AOPT13]

The knapsack problem

- **KNAPSACK**

- **Input:** A list of item values $p_1, \ldots, p_n$, a list of weights on the same items $w_1, \ldots, w_n$, two rational numbers P, W.

- **Question:** Can the thief steal a set of items of total value $\geq P$ that fit in his knapsack of total weight W?

The partition problem

- **PARTITION**

- **Input:** A list of positive integers $a_1, \dots, a_n$.

- **Question:** Can you split them into to bags such that the sum in one equals the sum in the other?

$\{5,2,1,6,3,8,5,4,1,1,10\}$

$\{5,2,1,6,3,8,5,4,1,1,10\}$

- Note that the YES answer is easily verifiable.

- How would you efficiently verify a NO answer? (no one knows)

Testing polynomial positivity

- **POLYPOS**

- **Input:** A multivariate polynomial $p(x) := p(x_1, \dots, x_n)$ of degree four.
- **Question:** Is there an $x \in \mathbb{R}^n$ for which $p(x) \leq 0$?

- Example:

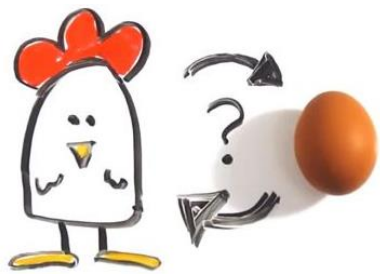
$$p(x) = x_1^4 + 2x_1^2 x_2^2 - 3x_1 x_3 + 5x_2^4 + 6x_1^2 x_2 - x_1 x_2 x_3 + 4x_3^4 + 100.$$

- A reduction from PARTITION to POLYPOS is on your homework.
- Is there an easy certificate of the NO answer? (the answer is believed to be negative)
- Is there an easy certificate of the YES answer? We don't know; the obvious approach doesn't work:

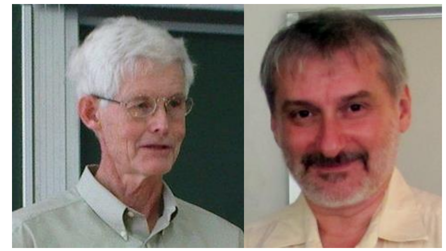
$$p(x) = (x_1 - 2)^2 + (x_2 - x_1^2)^2 + (x_3 - x_2^2)^2 + \dots + (x_n - x_{n-1}^2)^2$$

$$p(x) = 0 \implies x_n = 2^{2^n}.$$

But what about the first NP-complete problem?!!



▪The Cook-Levin theorem.



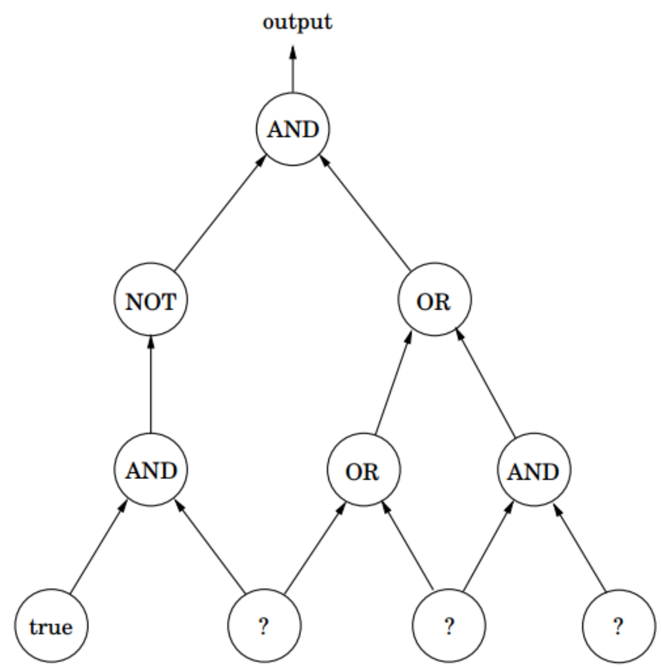
▪In a way a very deep theorem.

▪At the same time almost a tautology.

▪We argued in class how every problem in NP can be reduced to CIRCUIT SAT.

- See Chapter 8 of [DPV].

An instance of CIRCUIT SAT.

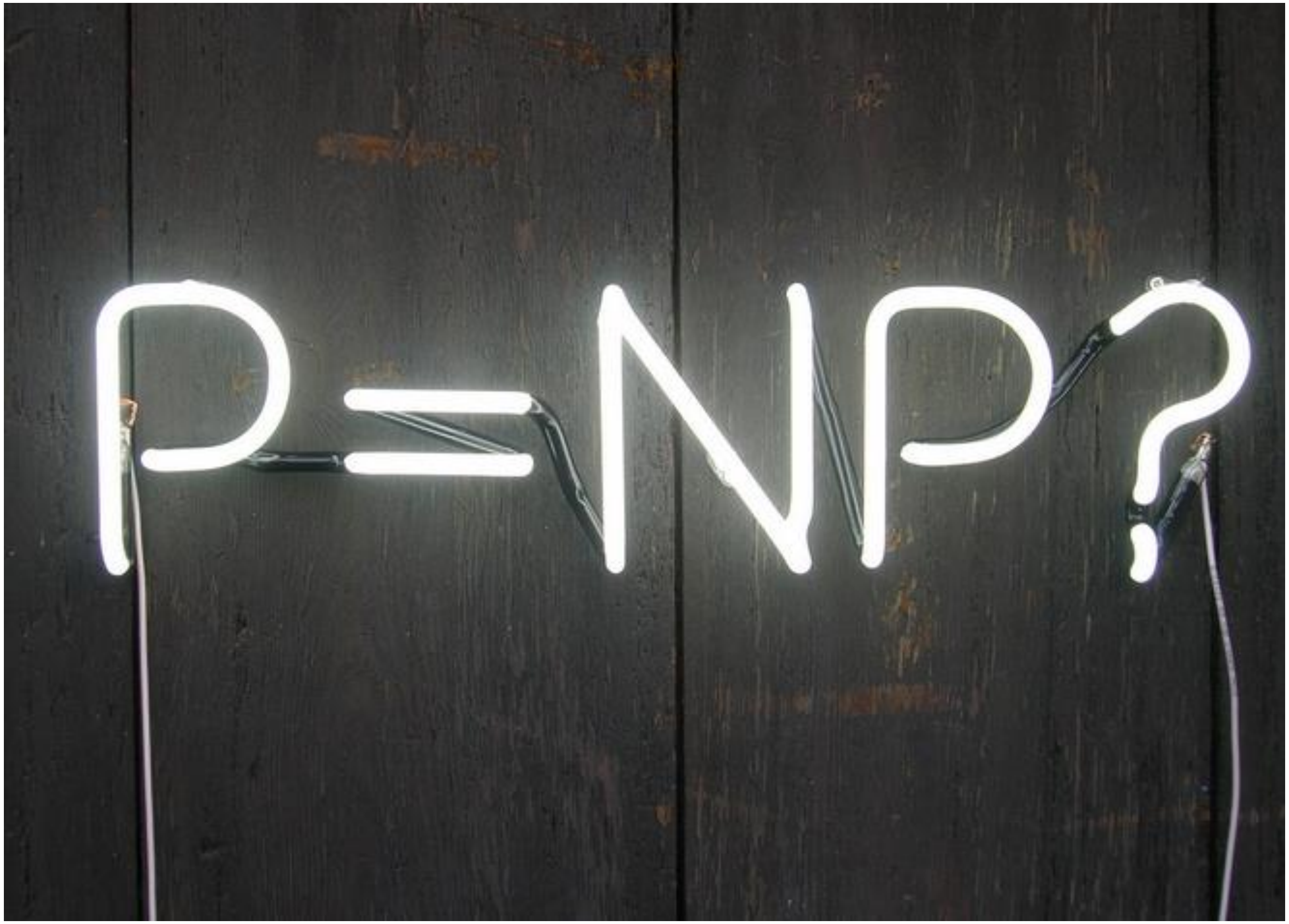


▪CIRCUIT SAT→ SAT→ 3SAT (easy reductions)

The domino effect

- All NP-complete problems reduce to each other!
- If you solve one in polynomial time, you solve ALL in polynomial time!

The $1M question!



- Most people believe the answer is NO!
- Philosophical reason: If a proof of the Goldbach conjecture were to fly from the sky, we could efficiently verify it. But should this imply that we can *find* this proof efficiently? P=NP would imply the answer is yes.

Nevertheless, there are believers too…

- Over 100 wrong proofs have appeared so far (in both directions)! See http://www.win.tue.nl/~gwoegi/P-versus-NP.htm

Main messages…

- Computational complexity theory beautifully classifies many problems of optimization theory as easy or hard
    - At the most basic level, easy means “in P”, hard means “NP-hard.”
- The boundary between the two is very delicate:
    - MINCUT vs. MAXCUT, PENONPAPER vs. TSP, LP vs. IP, ...
- Important: When a problem is shown to be NP-hard, it doesn’t mean that we should give up all hope. NP-hard problems arise in applications all the time. There are good strategies for dealing with them.
    - Solving special cases exactly
    - Heuristics that work well in practice
    - Using convex optimization to find bounds and near optimal solutions
    - Approximation algorithms – suboptimal solutions with worst-case guarantees
- P=NP?
    - Maybe one of you guys will tell us one day.

The remaining lectures

1. **The SOS relaxation**

   A very general and powerful framework for dealing with NP-hard problems

2. **Robust optimization**

   Dealing with uncertainty in the formulation of optimization problems

3. **Approximation algorithms**

   Guaranteeing quality of suboptimal solutions in the worst case

- References:

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