Cutting Planes for Mixed-Integer Programming: Theory and Practice

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Mathematical optimization

- *A generic mathematical optimization problem:*

$$\begin{aligned} \min: \quad & f(x) \\ \textit{subject to:} \quad & g_i(x) \le 0 \quad i = 1, \dots, m \\ & x \in X \end{aligned}$$

- *Computationally tractable cases:*

  – *If* $f(x)$ and all $g_i(x)$ *are linear, and* $X = \mathcal{R}_+^n \Rightarrow$ LP

  – If $f(x)$ *and all* $g_i(x)$ are linear, and $X = \mathcal{Z}_+^{n_1} \times R_+^{n_2} \Rightarrow$ *MILP*

  – *If* $f(x)$ is quadratic and all $g_i(x)$ *are linear, and* $X = \mathcal{R}_+^n \Rightarrow$ QP

  – If $f(x)$ *and all* $g_i(x)$ are quadratic, and $X = \mathcal{R}_+^n \Rightarrow$ *QCQP*

- *Only LP can be solved in polynomial time. Even Box QP is hard!*

$$\begin{aligned} \min: \quad & x^T Q x \\ \textit{subject to:} \quad & 1 \ge x \ge 0 \end{aligned}$$

Mixed-integer programming

- *A generic Mixed Integer Linear Program has the form:*

$$\min\{c^T x : Ax \geq b, x \geq 0,\ x_j \text{ integer},\ j \in \mathcal{I}\}$$

where matrix A does not necessarily have a special structure.

- *A very large number of practical problems can be modeled in this form:*
  - *Production planning,*
  - *Airline scheduling (routing, staffing, etc.)*
  - *Telecommunication network design,*
  - *Classroom scheduling,*
  - *Combinatorial auctions,*
  - *...*

- *In theory, MIP is NP-hard: not much hope for efficient algorithms.*

- *But in practice, even very large MIPs can be solved to optimality in reasonable time.*

Mixed-integer programming

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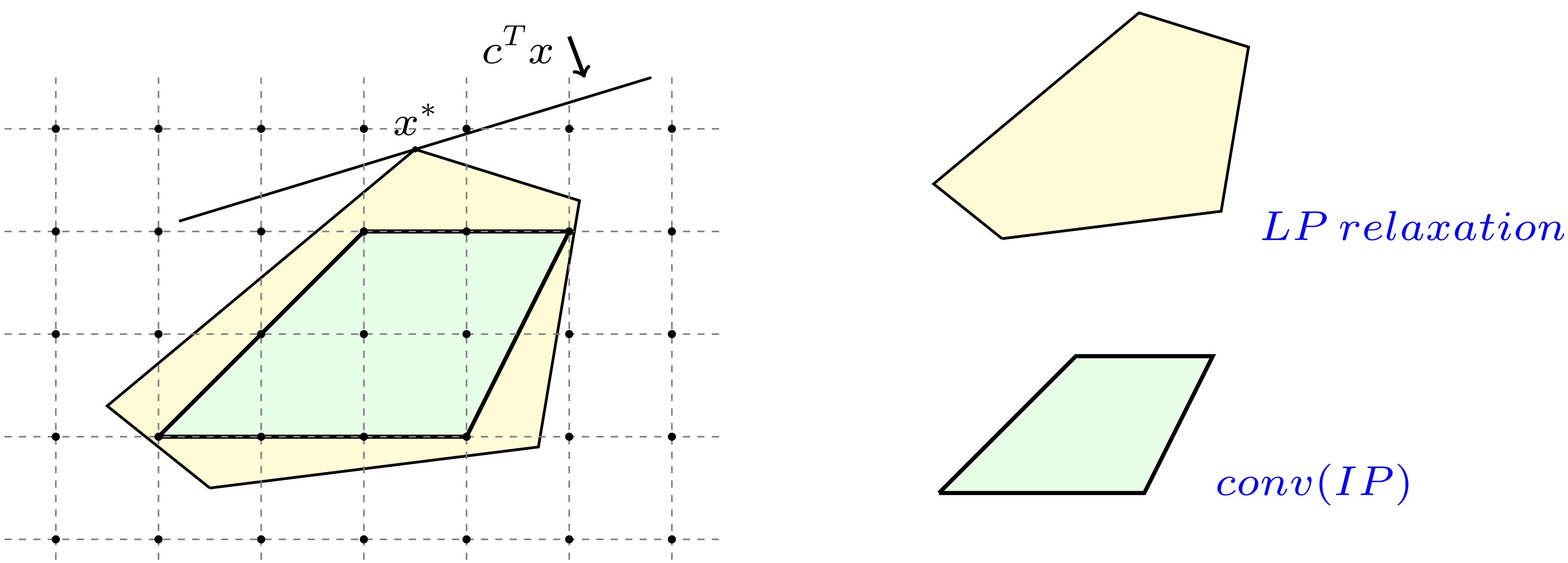


Overview of the talk

- *Introduction*
  - *Mixed-integer programming, branch-and-cut*
- *Commercial Software (Cplex)*
  - *Evolution, main components*
- *Cutting planes*
  - *Mixed-integer rounding*

- *A new approach to cutting planes*
  - *Lattice free cuts, multi-branch split cuts*

- *A finite cutting-plane algorithm*

Solving Mixed Integer Linear Programs

- *In practice MIPs are solved via enumeration:*
  - *The branch-and-bound algorithm, Land and Doig (1960)*
  - *The branch-and-cut scheme proposed by Padberg and Rinaldi (1987)*

- *Given an optimization problem* $z^* = \min \{f(x) : x \in P\}$,

(i) **Partitioning:** Let $P = \cup_{i=1}^{p} P_i$ *(division), then*

$$z^* = \min_i \{z_i\} \text{ where } z_i = \min \{f(x) : x \in P_i\},$$

(ii) **Lower bounding:** *For* $i = 1, \ldots, p$*, let* $P_i \subseteq P_i^R$ *(relaxation), then*

$$z_i \geq z_i^R = \min \{f(x) : x \in P_i^R\}, \text{ and } z^* \geq \min_i \{z_i^R\}.$$

(iii) **Upper bounding:** *If* $\bar{x} \in P_i \subseteq P$ *then* $f(\bar{x}) \geq z^*$*.*

[Same framework is used to solve non-convex QP's, for example.]

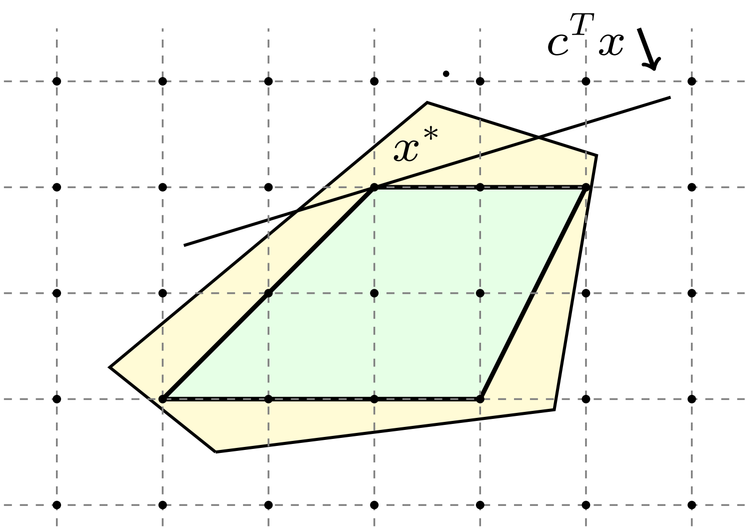
Relaxation step

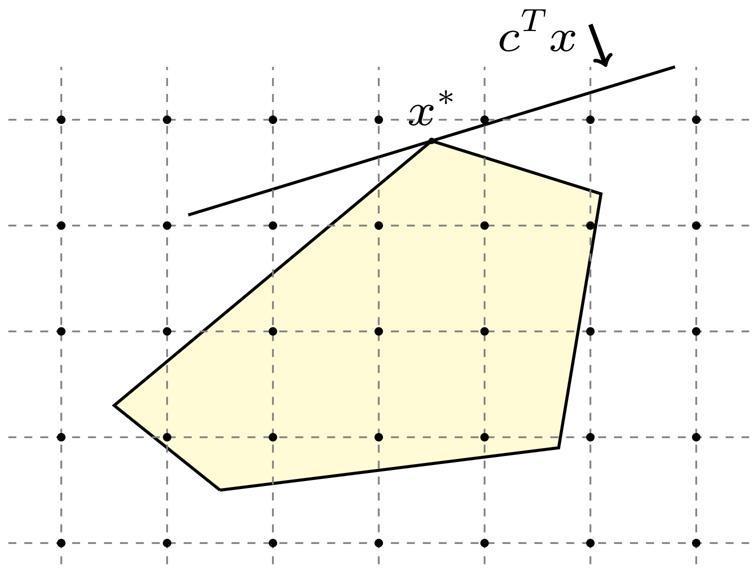
Mixed Integer Program:

$$\begin{aligned} &\min c^T x \\ Ax &\geq b \\ x &\geq 0 \\ x_j &\in \mathcal{Z} \text{ for } j \in \mathcal{I} \end{aligned}$$

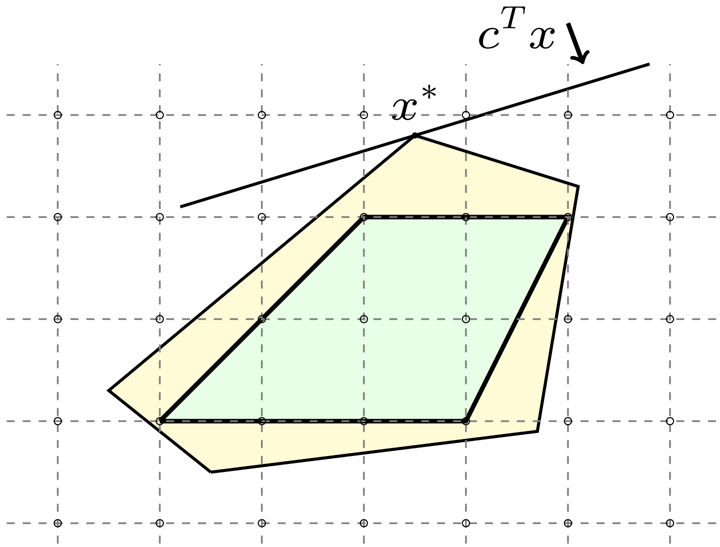
LP Relaxation:

$$\begin{aligned} &\min c^T x \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

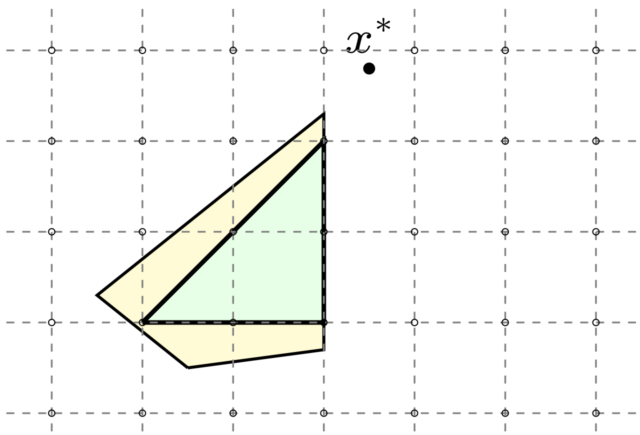




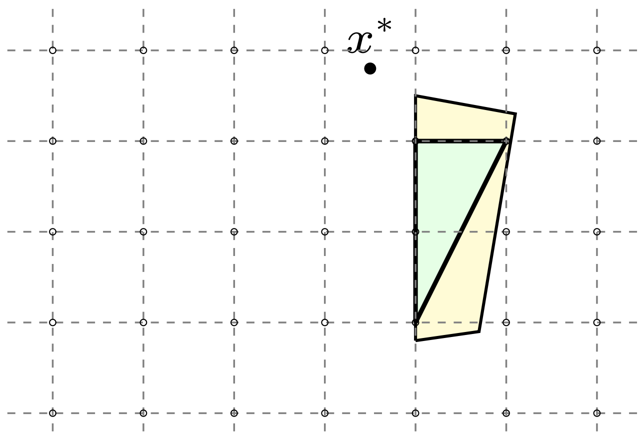
Partitioning step



Initial problem



Subproblem 1: $x_1 \leq \lfloor x_1^* \rfloor$



Subproblem 2: $x_1 \geq \lceil x_1^* \rceil$

Next: Commercial Solvers

MIP Evolution, early days

- *Early MIP solvers focused on developing fast and reliable LP solvers for branch-and-bound schemes. (eg. 10^6-fold improvement in Cplex from 1990 to 2004!).*
- *Remarkable exceptions are:*
  - *1983 Crowder, Johnson & Padberg: PIPX, pure 0/1 MIPs*
  - *1987 Van Roy & Wolsey: MPSARX, mixed 0/1 MIPs*
- *When did the early days end?*

  A crucial step has been the computational success of cutting planes for TSP
  - *Padberg and Rinaldi (1987)*
  - *Applegate, Bixby, Chvtal, and Cook (1994)*
- *In addition for general MIPs:*
  - *1994 Balas, Ceria & Cornuéjols: Lift-and-project*
  - *1996 Balas, Ceria, Cornuéjols & Natraj: Gomory cuts revisited*

Evolution of MIP Solvers by numbers

- *Bixby & Achterberg compared all Cplex versions (with MIP capability)*
- *1,734 MIP instances*
- *Computing times are geometric means normalized wrt Cplex 11.0*

Cplex versions	year	better	worse	time
11.0	2007	0	0	1.00
10.0	2005	201	650	1.91
9.0	2003	142	793	2.73
8.0	2002	117	856	3.56
7.1	2001	63	930	4.59
6.5	1999	71	997	7.47
6.0	1998	55	1060	21.30
5.0	1997	45	1069	22.57
4.0	1995	37	1089	26.29
3.0	1994	34	1107	34.63
2.1	1993	13	1137	56.16
1.2	1991	17	1132	67.90

- *The key feature of Cplex v. 6.5 was extensive cutting plane generation.*

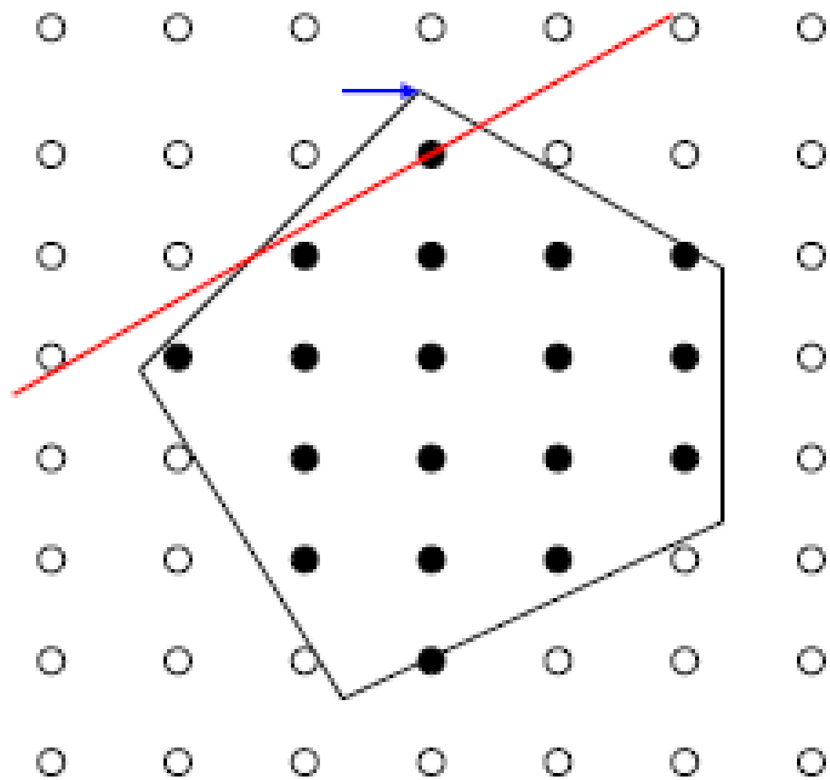
An example: p2756

- *This problem has 2756 binary variables and 755 constraints*
- *Hardest instance in Crowder, Johnson and Padberg (1983)*
- *Solving with Cplex 11:*
  - *without cuts it takes 3,414,408 nodes*
  - *with cuts it takes 11 nodes!*
- *Cplex reduces the root optimality gap from 13.5% to 0.2% with*
  - *22 Gomory mixed-integer cuts, and*
  - *23 cover inequalities*

    (both are "mixed-integer rounding" inequalities.)
- *This and many other MIPLIB instances are available at* `http://miplib.zib.de`

Strengthening the LP relaxation by cutting planes

- *Given the optimal solution $\bar{x}$ of the LP relaxation (not integral)*
- *Do not branch right away*
- *Find a valid inequality for the MIP $a^T x \geq b$ such that $a^T \bar{x} < b$.*



Branch and cut

Main components of a MIP solver

- **Preprocessing**
  - *Clean up the model (empty/implied rows, fixed variables, . . .)*
  - *Coefficient reduction (ex: p0033, all variables binary)*

$$-230x_{10} - 200x_{16} - 400x_{17} \leq -5 \Longrightarrow x_{10} + x_{16} + x_{17} \geq 1$$

- **Cutting plane generation***:*
  Gomory Mixed Integer cuts, MIR inequalities, cover cuts, flow covers, . . .
- **Branching strategies***:*
  strong branching, pseudo-cost branching, (not most fractional!)
- **Primal heuristics***:*
  rounding heuristics, diving heuristics, local search, . . .
- **Node selection strategies:**
  a combination of best-bound and diving.

Some features of a good MIP solver

- *Solving a MIP to optimality is only one aspect for many applications (sometimes not the most important one)*
  - *Detect infeasibility in the model early on and report its source to help with modeling.*
  - *Feasible (integral) solutions*
    - *Find good solutions quickly*
    - *Find many solutions and store them*
- *Not all MIPs are the same*
  - *Recognize problem structure and adjust parameters/strategies accordingly (there are too many parameters/options for hand-tuning.)*
  - *Deal with both small and very large scale problems*
  - *Handle numerically difficult instances with care (**very important**)*
- *Not all users are the same*
  - *Allow user to take over some of the control (callbacks)*

Beyond MIP

- *Not all non-convex optimization problems are MIPs :)*
- *But it is possible to extend the capability of the MIP framework. For example:*

  1. **Bonmin** *(Basic Open-source Nonlinear Mixed INteger programming, [Bonami et. al.])*
     - *For Convex MINLP within the framework of the MIP solver* **Cbc** *[Forrest].*
  2. **GloMIQO** *(Global mixed-integer quadratic optimizer, [Misener])*
     - *Spatial branch-and-bound algorithm for non-convex QP.*
  3. **Couenne** *(Convex Over and Under ENvelopes for Nonlinear Estimation, [Belotti])*
     - *Spatial and integer branch-and-bound algorithm for non-convex MINLP.*
  4. **SCIP** *(Solving Constraint Integer Programs, [Achterberg et. al.])*
     - *Tight integration of CP and SAT techniques within a MIP solver.*
     - *Significant recent progress for non-convex MINLP.*

- *All codes are open source and can be obtained free of charge.*

Next: Cutting planes

Cutting planes for IP

Integer Program:

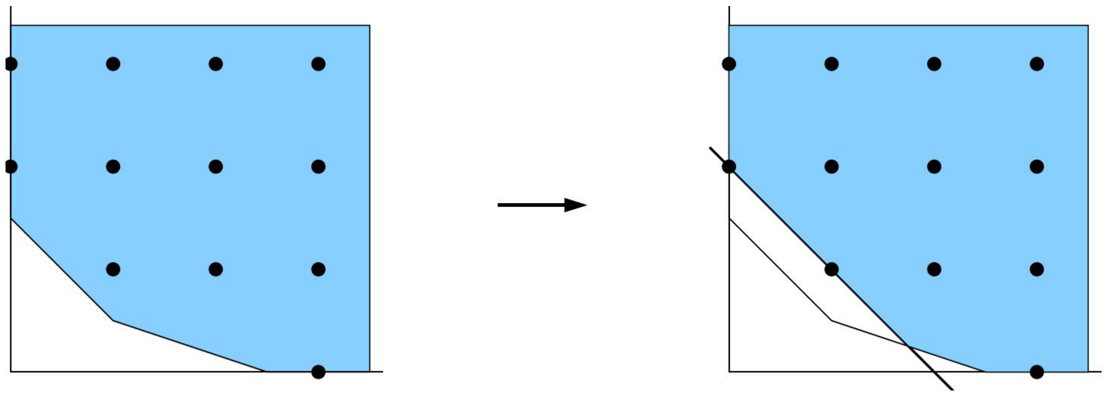
$$\min c^T x$$
$$Ax \geq b,$$
$$x \geq 0,$$
$$x \text{ integral}$$

LP Relaxation:

$$\min c^T x$$
$$Ax \geq b$$
$$x \geq 0$$

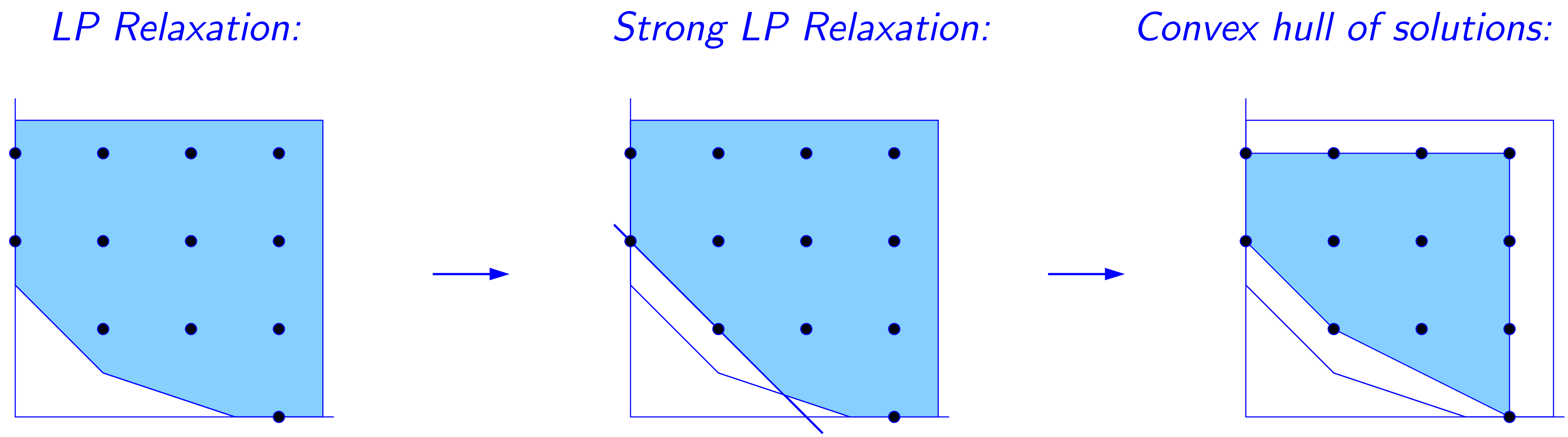
Tighten:

$$\min c^T x$$
$$Ax \geq b$$
$$x \geq 0$$
$$\alpha_1 x \geq d_1$$
$$\alpha_2 x \geq d_2$$
$$\vdots$$



Convex hull of mixed-integer sets

- *Any MIP can be solved by linear programming (**without branching**) by finding the "right" cuts (i.e. by finding the convex hull)*



- *Gomory proposed a finite cutting plane algorithm for pure IPs (1958).*
- *Dash, Dobbs, Gunluk, Nowicki,and Swirszcz, did the same for MIPs (2014).*
- *In practice,*
  - *These algorithms are hopeless except some very easy cases.*
  - *But, getting closer to the convex hull helps.*

Gomory cuts

Let

$$Q^0 = \left\{ y \in Z \ : \ y \ \geq \ b_1, \ y \ \leq \ b_2 \right\}$$

then, the following inequalities:

$$y \geq \lceil b_1 \rceil \quad \textit{and} \quad y \leq \lfloor b_2 \rfloor$$

are valid for Q^0 *and*

$$conv(Q^0) = \left\{ y \in R \ : \ \lfloor b_2 \rfloor \ \geq \ y \ \geq \ \lceil b_1 \rceil \ \right\}.$$

- y *can be replace with any integer expression to obtain a valid cut.*
- *These cuts are also called Chvatal-Gomory cuts*

Basic mixed-integer rounding set (Wolsey '98)

Let

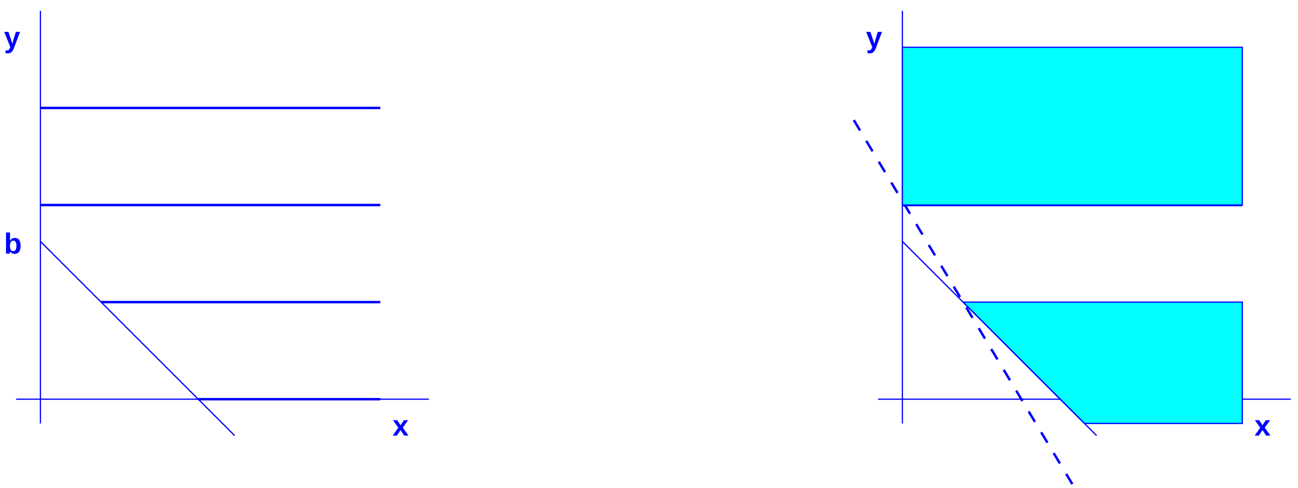
$$Q^1 = \left\{ v \in R,\ y \in Z\ :\ v + y\ \geq\ b,\ \ v \geq 0 \right\}$$

then, MIR Inequality:

$$v \geq \hat{b}(\lceil b \rceil - y)$$

where $\hat{b} = b - \lfloor b \rfloor$*, is valid for* Q^1 *and*

$$conv(Q^1) = \left\{ v, y \in R\ :\ v + y\ \geq\ b,\ \ v + \hat{b}y \geq \hat{b}\lceil b \rceil,\ \ v \geq 0 \right\}.$$



Basic mixed-integer rounding set – example

Let

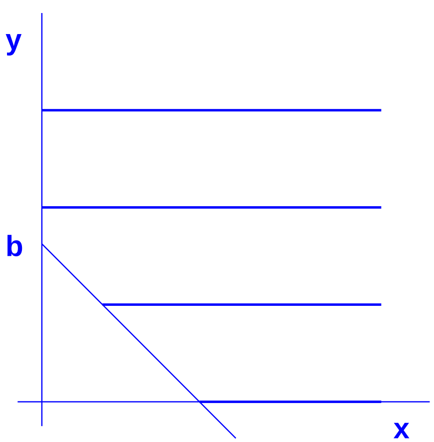
$$Q^1 = \left\{ v \in R,\ y \in Z\ :\ v + y\ \geq\ 7.3,\ \ v \geq 0 \right\}$$

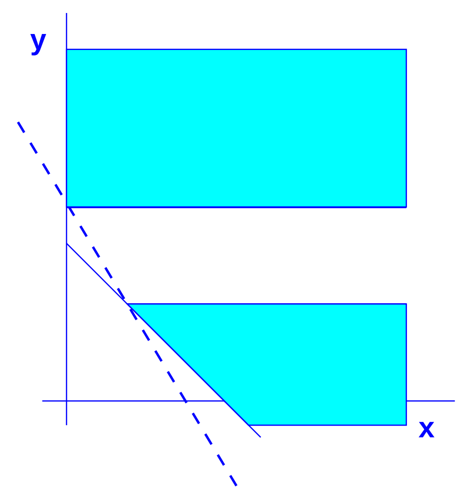
then, MIR Inequality:

$$v \geq 0.3(8 - y)$$

where $0.3 = 7.3 - 7$, *is valid for* Q^1 *and*

$$conv(Q^1) = \left\{ v, y \in R\ :\ v + y\ \geq\ 7.3,\ \ v + 0.3y \geq 0.3 \times 8,\ \ v \geq 0 \right\}.$$





Next: MIR Inequalities

MIR inequalities for single constraint sets

Let

$$P^1 = \left\{ v \in R^{|C|},\ y \in Z^{|I|}\ :\ \sum_{j \in C} c_j v_j + \sum_{j \in I} a_j y_j\ \geq\ b,\ \ v,\ y \geq 0 \right\}$$

Re-write:

$$\sum_{c_j<0} c_j v_j + \sum_{c_j>0} c_j v_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j + \sum_{\hat{a}_j \geq \hat{b}} \hat{a}_j y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j\ \geq\ b\ =\ \hat{b} + \lfloor b \rfloor$$

Relax:

$$\underbrace{\sum_{c_j>0} c_j v_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j}_{\geq 0}\quad + \quad \underbrace{\sum_{\hat{a}_j \geq \hat{b}} y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j}_{\in Z}\quad \geq\ b$$

MIR cut:

$$\sum_{c_j>0} c_j v_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j + \hat{b} \left(\sum_{\hat{a}_j \geq \hat{b}} y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j \right)\ \geq\ \hat{b} \lceil b \rceil$$

(Applying MIR to the simplex tableau rows gives the Gomory mixed-integer cut)

MIR inequalities for multiple constraint sets

Let

$$P = \left\{ v \in R^{|C|},\ y \in Z^{|I|}\ :\ Cv + Ay\ \geq\ d,\ \ v,\ y \geq 0 \right\}$$

where $C \in R^{m\times|C|},\ A \in R^{m\times|I|},\ d \in R^m$.

- *Obtain a "base" inequality using* $\lambda \in R^m_+$ *:* $\quad \lambda C v + \lambda A y\ \geq\ \lambda d$
- *Write the corresponding MIR inequality:*

$$\sum_{j\in C} (\lambda C_j)^+ v_j + \hat{b} \sum_{j\in I} \lfloor \lambda A_j \rfloor y + \sum_{j\in I} \min\{\lambda A_j - \lfloor \lambda A_j \rfloor, \hat{b}\} y\ \geq\ \hat{b} \lceil \lambda d \rceil$$

where $\hat{b} = \lambda d - \lceil \lambda d \rceil$.

Better MIR inequalities for multiple constraint sets

- *Add (non-negative) slack variables to the defining inequalities:*

$$Cv + Ay - Is \; = \; d$$

- *Obtain a "base" equation using $\lambda \in R^m$:*

$$\lambda Cv + \lambda Ay - \lambda Is \; = \; \lambda d$$

- *Write the corresponding MIR inequality:*

$$\sum_{j \in C} (\lambda C_j)^+ v_j + \hat{b} \sum_{j \in I} \lfloor \lambda A_j \rfloor y + \sum_{j \in I} \min\{\lambda A_j - \lfloor \lambda A_j \rfloor, \hat{b}\} y + \sum_{\lambda_i < 0} |\lambda_i| s_i \; \geq \; \hat{b} \lceil \lambda d \rceil$$

- *Substitute out slacks to obtain*

$$\sum_{j \in C} (\lambda C_j)^+ v_j + \hat{b} \sum_{j \in I} \lfloor \lambda A_j \rfloor y + \sum_{j \in I} \min\{\ldots, \hat{b}\} y + \sum_{\lambda_i < 0} |\lambda_i| (Cv + Ax - d)_i \; \geq \; \hat{b} \lceil \lambda d \rceil$$

Example

Consider the set

$$T = \{v \in R,\ x \in Z\ :\ -v - 4x \geq -4,\ \ -v + 4x \geq 0,\ \ v, x \geq 0\}$$

Any base inequality generated by λ_1, λ_2 *has the form*

$$(-\lambda_1 - \lambda_2)v + (-4\lambda_1 + 4\lambda_2)x \geq -4\lambda_1$$

If $\lambda_1, \lambda_2 \geq 0$, v *has a negative coefficient and does not appear in the cut.*

Using multipliers $\lambda = [-1/8, 1/8]$

$$\begin{array}{rcl} -\frac{1}{8}\Big(-v - 4x - s_1 = -4\Big) & + & \frac{1}{8}\Big(-v + 4x - s_2 = 0\Big) \\ & \Downarrow & (Base\ inequality) \\ x + s_1/8 - s_2/8 & \geq & 1/2 \\ & \Downarrow & (MIR) \\ 1/2x + s_1/8 & \geq & 1/2\ \Rightarrow\ -v/8 \geq 0\ \Rightarrow\ v \leq 0 \end{array}$$

This inequality defines the only non-trivial facet of T.

Computational performance of MIR inequalities

			MIR (DGL)				CG (FL)		Split (BS)	
instance	$\lvert I\rvert$	$\lvert J\rvert$	# iter	# cuts	% gap	time	% gap	time	% gap	time
10teams	1,800	225	338	3341	100.00	3,600	57.14	1,200	100.00	90
arki001	538	850	14	124	33.93	3,600	28.04	1,200	83.05*	193,536
bell3a	71	62	21	166	98.69	3,600	48.10	65	65.35	102
bell5	58	46	105	608	93.13	3,600	91.73	4	91.03	2,233
blend2	264	89	723	3991	32.18	3,600	36.40	1,200	46.52	552
dano3mip	552	13,321	1	124	0.10	3,600	0.00	1,200	0.22	73,835
danoint	56	465	501	2480	1.74	3,600	0.01	1,200	8.20	147,427
dcmulti	75	473	480	4527	98.53	3,600	47.25	1,200	100.00	2,154
egout	55	86	37	324	100.00	31	81.77	7	100.00	18,179
fiber	1,254	44	98	408	96.00	3,600	4.83	1,200	99.68	163,802
fixnet6	378	500	761	4927	94.47	3,600	67.51	43	99.75	19,577
flugpl	11	7	11	26	93.68	3,600	19.19	1,200	100.00	26
gen	150	720	11	127	100.00	16	86.60	1,200	100.00	46
gesa2	408	816	433	1594	99.81	3,600	94.84	1,200	99.02	22,808
gesa2_o	720	504	131	916	97.74	3,600	94.93	1,200	99.97	8,861
gesa3	384	768	464	1680	81.84	3,600	58.96	1,200	95.81	30,591
gesa3_o	672	480	344	1278	69.74	3,600	64.53	1,200	95.20	6,530
khb05250	24	1,326	65	521	100.00	113	4.70	3	100.00	33

Table 1: MIPs of the MIPLIB 3.0.

Numerical and other practical issues

When implementing these ideas to solve mixed integer programs one has to be careful:

- *How to obtain the base inequality?*
  - *Formulation rows*
  - *Simplex tableau rows*
  - *Aggregate formulation rows using different heuristics*

- *Numerical issues*
  - *LP-solvers are not numerically exact.*

$$b = 5.00001 \quad \Longrightarrow \quad \lceil b \rceil = 6 \text{ and } \hat{b} = 0.00001$$
$$b = 4.99999 \quad \Longrightarrow \quad \lceil b \rceil = 5 \text{ and } \hat{b} = 0.99999$$

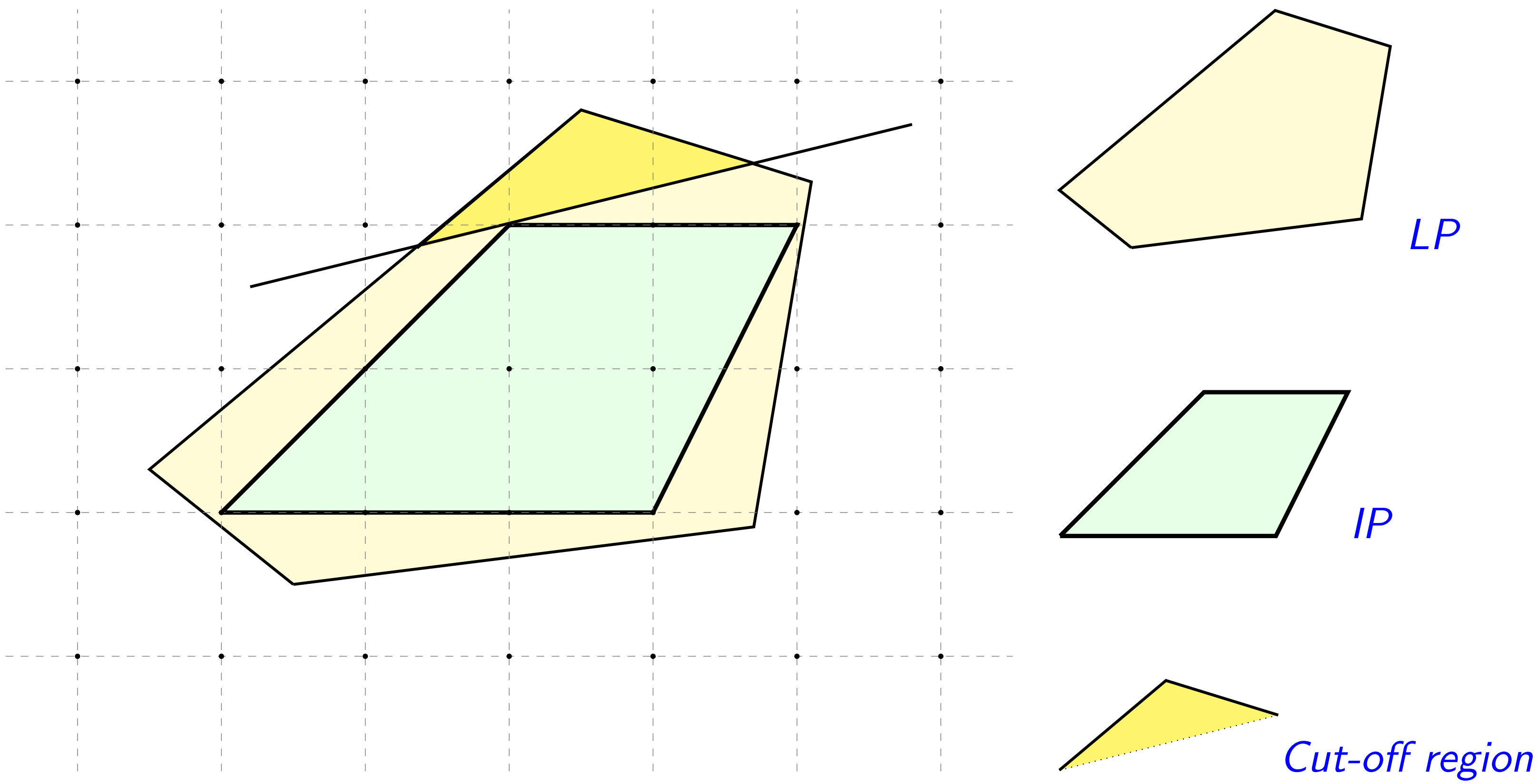
  - *Avoid large numbers:* $1000000x_1 - 10000000x_2 \geq 0.3$ *is not a good cut.*
  - *Avoid dense rows*

Next:

Beyond MIR Inequalities: Lattice free cuts, multi-branch split cuts

. *(joint work with Dash, Dobbs, Nowicki, and Świrszcz)*

Cutting Planes for MILP

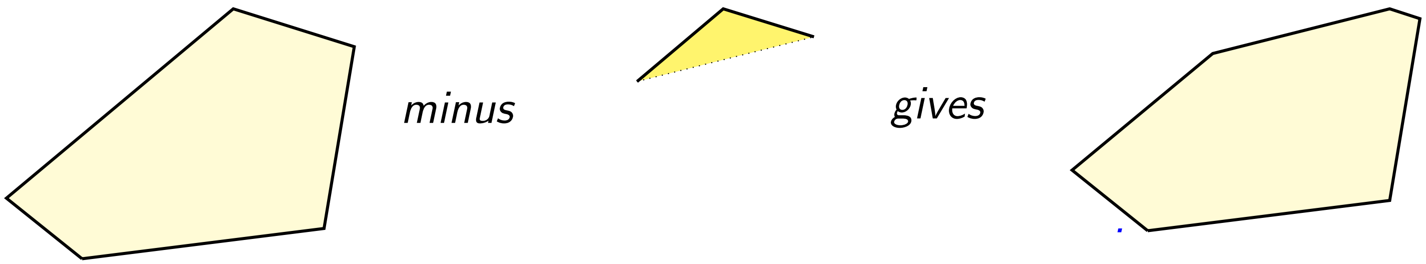


- *The region cut-off by the valid inequality is always strictly lattice-free*

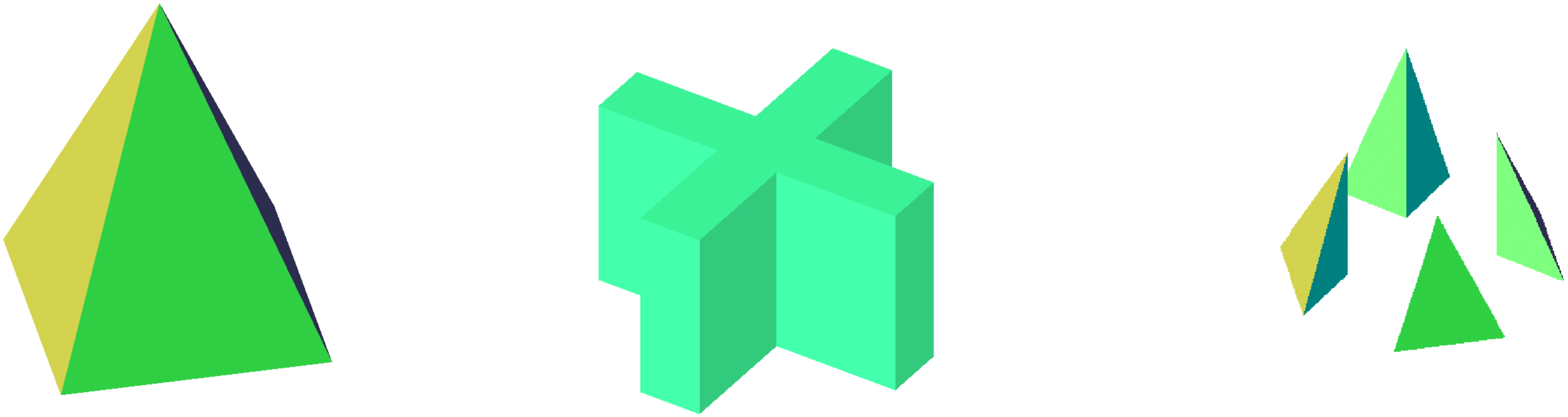
Generating Cutting Planes Using Lattice Free Sets

- *Relaxation minus a strictly lattice-free (convex) set gives a tighter relaxation.*

Ex:



- *We can also use non-convex lattice-free sets:*



(but then need to convexify afterwards to obtain a nice relaxation)

Disjunctive cuts [Balas '79]

- *Let* $D = \cup_{i \in K} D_i$ *where*

$$D_i = \{(x, y) \in \mathcal{R}^{n+l} : A^i x \le b^i\}$$

- $D \subseteq \mathcal{R}^{n+l}$ *is called a* **disjunction** *if* $\mathcal{Z}^n \times \mathcal{R}^l \subseteq D$ *(clearly* $D = D^n \times \mathcal{R}^l$*)*

- *Let* $P = P^{LP} \cap (\mathcal{Z}^n \times \mathcal{R}^l)$ *where*

$$P^{LP} = \{(x, v) \in \mathcal{R}^n \times \mathcal{R}^l : Ax + Cv \ge d\}$$

- *The* **disjunctive hull** *of* P *with respect to* D *is*

$$P_D = \text{conv}\left(P^{LP} \cap D\right) = \text{conv}\Big(\bigcup_{k \in K} (P^{LP} \cap D_k)\Big)$$

- *Notice that* $P_D = \text{conv}\left(P^{LP} \setminus B\right)$ *where* $B = \mathcal{R}^{n+l} \setminus D$ *is* **strictly lattice-free**.

All valid inequalities are disjunctive cuts

Let $c^T x + d^T y \geq f$ *be a valid inequality for* P *and*

$$V = \{(x, y) \in P^{LP} \; : \; c^T x + d^T y < f\}.$$

Clearly $V \cap (\mathcal{Z}^n \times \mathcal{R}^l) = \emptyset$, *i.e. strictly lattice-free.*

Jörg (2007) observes that $V_x \subseteq \mathrm{int}(B_x)$ *where*

- $V_x \subseteq \mathcal{R}^n$ *is the orthogonal projection of* V *in the space of the integer variables*
- $B_x \subseteq \mathcal{R}^n$ *is a polyhedral lattice-free set defined by rational (integral) data*

$$B_x = \{x \in \mathcal{R}^n \; : \; \pi_i^T x \geq \gamma_i, \; i \in K\}$$

Therefore the cut is valid for

$$\textit{conv}\left(P^{LP} \setminus (\mathrm{int}(B_x) \times \mathcal{R}^l)\right) \subseteq \textit{conv}\left(P^{LP} \setminus (V^x \times \mathcal{R}^l)\right).$$

Based on this observation, Jörg then argues that $|K| \leq 2^n$ *and*

$$D \;=\; \bigcup_{i \in K} \{(x, y) \in \mathcal{R}^{n+l} \; : \pi_i^T x \leq \gamma_i\}$$

is a valid disjunction and $c^T x + d^T y \geq f$ *can be derived from this disjunction.*

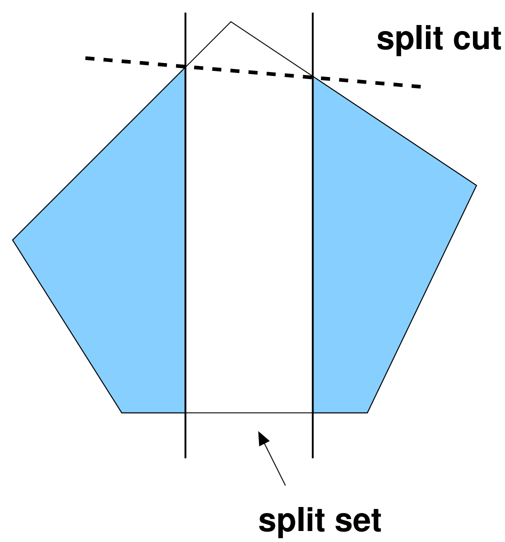
Split cuts

- *Let $\pi \in \mathcal{Z}^n$ and $\gamma \in \mathcal{Z}$ and consider the split set*

$$S(\pi, \gamma) = \{(x, y) \in \mathcal{R}^{n+l} \; : \; \gamma < \pi^T x < \gamma + 1\}$$

  *(which is **strictly** lattice-free)*

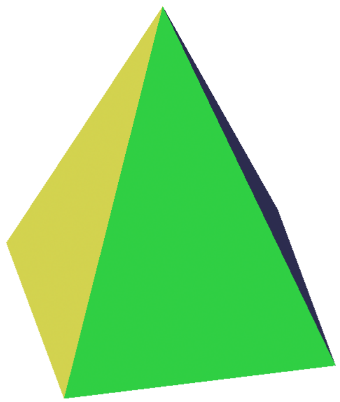
- *A **split cut** is an inequality valid for $P^{LP} \setminus S(\pi, \gamma)$:*



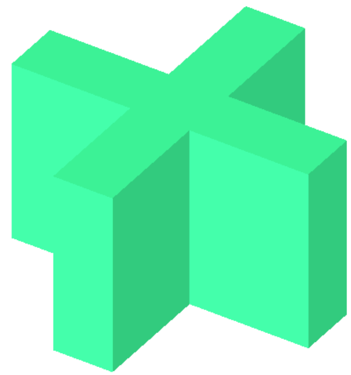
- *Split cuts are disjunctive cuts $D_1 = \{\pi^T x \leq \gamma\}$ and $D_2 = \{\pi^T x \geq \gamma + 1\}$*

- *MIR cuts are split cuts with $\pi = \lceil \lambda A \rfloor$ and $\gamma = \lfloor \lambda d \rfloor$.*

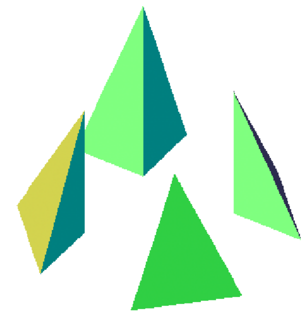
A Generalization of Split Cuts: Cross cuts



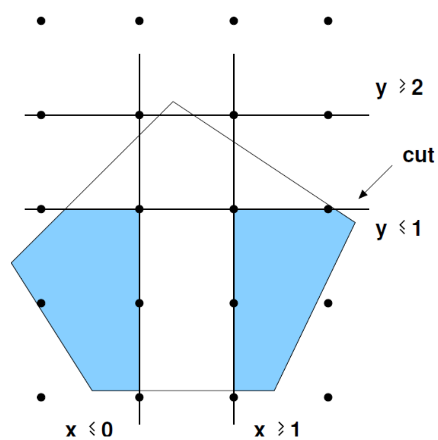
P^{LP}



the cross set



$P^{LP} \setminus$ *cross set*



Computational experiments with cross cuts

problem	GMI	DG	Split	Cross	Best Split	Improvement
bell5	14.5	25.9	86.2	99.8	93.0	97.1
cap6000	41.7	64.6	65.2	67.2	65.2	5.7
gesa3	45.9	93.3	95.1	97.4	95.8	38.1
gesa3o	50.6	93.1	95.3	99.0	95.2	79.2
gt2	67.7	96.6	96.7	99.2	98.4	50.0
mas74	6.7	11.4	14.3	15.7	14.0	2.0
mas76	6.4	16.1	25.1	34.5	26.5	10.9
mkc	1.2	4.1	52.4	55.3	49.3	11.8
modglob	15.1	90.8	94.0	99.0	92.2	87.2
p0033	54.6	84.1	86.2	100.0	87.4	100.0
p0201	18.2	71.5	74.0	98.4	74.9	93.6
pp08a	52.9	96.0	96.6	98.4	97.0	46.7
pp08aCUTS	30.1	93.3	94.7	96.5	95.8	16.7
qiu	2.0	21.8	78.1	78.4	77.5	4.0
set1ch	38.1	88.0	88.7	98.6	89.7	86.4
vpm2	12.6	72.0	76.5	81.7	81.0	3.7
Average	25.5	61.3	76.1	82.2	77.3	45.8

Table 2: Some MIPLIB Problems – 16 out of 32

. *(joint work with Dash and Vielma)*

Multi-branch split cuts

- *Let*
$$P = \{(x, v) \in \mathcal{Z}^n \times \mathcal{R}^l \; : \; Ax + Cv \; = \; d, \;\; v \geq 0\}$$
be rational and let P^{LP} *denote its continuous relaxation.*

- *Let* $\pi_i \in \mathcal{Z}^n$ *and* $\gamma_i \in \mathcal{Z}$ *for* $i = 1, \ldots, t$ *and consider the split sets*
$$S(\pi_i, \gamma_i) = \{(x, y) \in \mathcal{R}^{n+k} \; : \; \gamma_i < \pi_i^T x < \gamma_i + 1\}$$

- *A* **multi-branch split cut** *is an inequality valid for*
$$P^{LP} \setminus \bigcup_i S(\pi_i, \gamma_i)$$
(Li/Richard ('08) call these cuts **t-branch** *split cuts)*

- *2-branch split cuts are* **cross** *cuts.*
- *Multi-branch split cuts are disjunctive cuts [Balas '79].*

Are all valid inequalities multi-branch split cuts?

Let π_i *and* γ_i *be integral for* $i = 1, \ldots, t$ *and consider the split sets*

$$S(\pi_i, \gamma_i) = \{(x, y) \in \mathcal{R}^{n+k} \ : \ \gamma_i < \pi_i^T x < \gamma_i + 1\}$$

A **multi-branch split cut** *is an inequality valid for* $P^{LP} \setminus \bigcup_i S(\pi_i, \gamma_i)$

The corresponding disjunction is

$$D = \bigcup_{S \subseteq \{1,\ldots,t\}} \{(x, y) \in \mathcal{R}^{n+k} \ : \pi_i^T x \leq \gamma_i \text{ if } i \in S, \ \pi_i^T x \geq \gamma_i + 1 \text{ if } i \notin S\}$$

Question : *Are all facet defining inequalities t-branch split cuts for finite* t*?*

Remember the points cut off by the valid inequality $c^T x + d^T y \geq f$

$$V = \{(x, y) \in P^{LP} \ : \ c^T x + d^T y < f\}.$$

Fact : *Let* $\mathcal{S} = \cup S_i$ *be a collection of split sets in* $\mathcal{R}^{n+k}$*. If* $V \subseteq \mathcal{S}$*, then* $c^T x + d^T y \geq f$ *is a multi-branch split cut obtained from* $\mathcal{S}$*.*

Lattice width

- *Given a closed, bounded, convex set (or convex* body*)* $B \subseteq \mathcal{R}^n$ *and a vector* $c \in \mathcal{Z}^n$*,*

$$w(B, c) = \max\{c^T x : x \in B\} - \min\{c^T x : x \in B\}.$$

is the lattice width *of* B *along the direction* c*.*

- *The* lattice width *of* B *is*

$$w(B) = \min_{c \in \mathcal{Z}^n \setminus \{\mathbf{0}\}} w(B, c)$$

(If the set is not closed, we define its lattice width to be the lattice width of its closure)

- **Khinchine's flatness theorem:** *there exists a function* $f(\cdot) : \mathcal{Z}_+ \to \mathcal{R}_+$ *such that for any strictly lattice-free bounded convex set* $B \subseteq \mathcal{R}^n$*,*

$$w(B) \leq f(n)$$

where $f(\cdot)$ *depends on the dimension of* B *(not on the complexity of* B*)*

- *Lenstra uses this result to construct a finite enumeration tree to solve the integer feasibility problem.*

Bounding the lattice width

- *Given a lattice free convex body* $B \subseteq \mathcal{R}^n$ *the lattice width is*

$$w(B) = \min_{c \in \mathcal{Z}^n \setminus \{\mathbf{0}\}} w(B, c) \leq f(n)$$

- *Lenstra (1983) showed that* $f(n) \leq 2^{n^2}$

- *Kannan and Lovász (1988) showed that* $f(n) \leq c_0(n+1)n/2$ *for some constant* c_0 *(*$c_0 = \max\{1, 4/c_1\}$ *where* c_1 *is another constant defined by Bourgain and Milman)*

- *Banaszczyk, Litvak, Pajor, and Szarek (1999) showed that* $O(n^{3/2})$

- *Rudelson (2000) showed that* $O(n^{4/3} \log^c n)$ *for some constant* c*.*

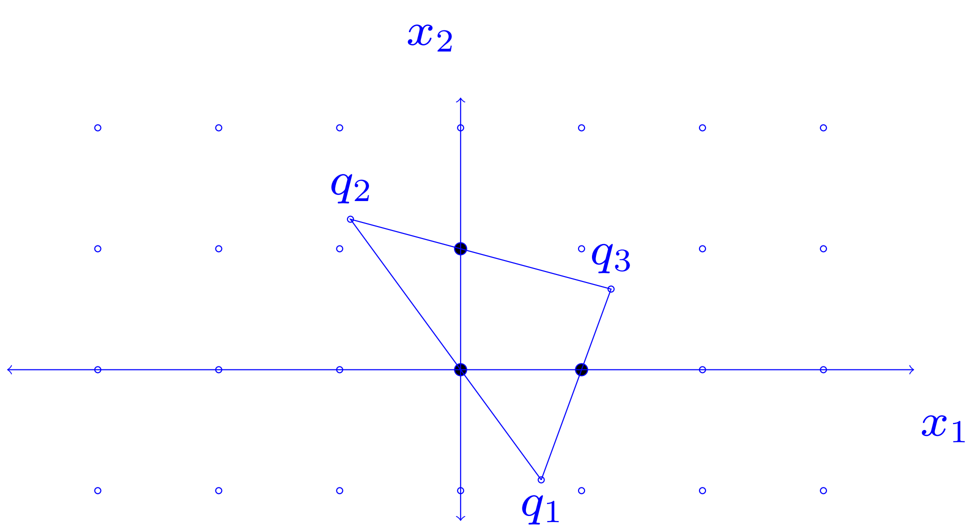
Lattice-free sets in $\mathcal{R}^2$

Theorem : *[Hurkens (1990)] If* $B \in \mathcal{R}^2$, *then* $w(B) \leq 1 + \frac{2}{\sqrt{3}} \approx 2.1547$. *Furthermore* $w(B) = 1 + \frac{2}{\sqrt{3}}$ *if and only if* B *is a triangle with vertices* q_1, q_2, q_3 *such that:*

$$\frac{1}{\sqrt{3}}\, q_i + (1 - \frac{1}{\sqrt{3}})\, q_{i+1} = b_i, \text{ for } i = 1, 2, 3.$$

where $b_i \in \mathcal{Z}^2$ *for* $i = 1, 2, 3$. *(and* $q_4 := q_1$*)*

The lattice-free triangle T *when* $b_1 = (0,0)^T$, $b_2 = (0,1)^T$, *and* $b_3 = (1,0)^T$

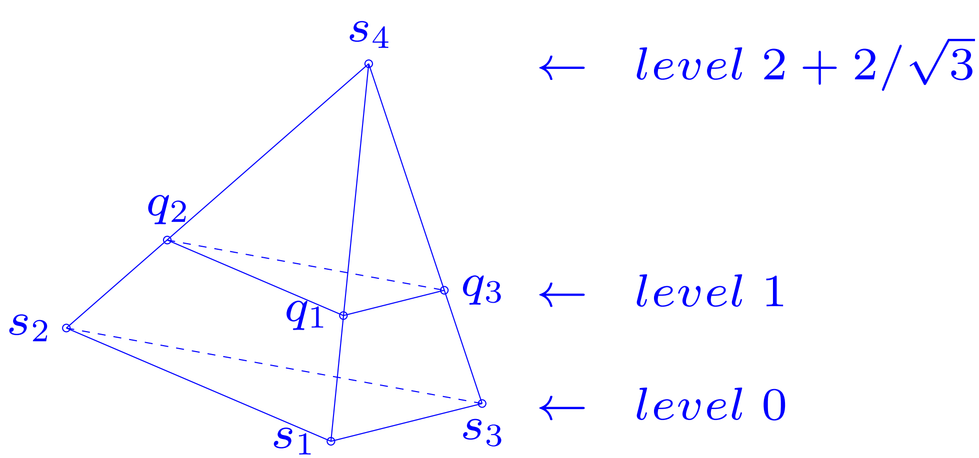


(this is called a type 3 triangle)

Lattice-free sets in $\mathcal{R}^3$

Averkov, Wagner and Weismantel (2011) enumerated all maximal lattice-free bodies in $\mathcal{R}^3$ that are integral. These sets have the lattice width ≤ 3.

There exists a tetrahedron H with lattice width $2 + 2/\sqrt{3} \approx 3.1547$:



where $s_4 = (0, 0, 2 + 2/\sqrt{3})$, and $q_1, \ldots, q_3 \in \mathcal{R}^2$ are the vertices of Hurken's triangle.

We can also show that $f(3) \leq 4.25$.

Next: A finite cutting-plane algorithm for mixed-integer programming

Can MIP's be solved only using cutting planes (without branching)?

History of finite cutting plane algorithms:

- *Gomory (1958) developed the first finite cutting plane algorithm for pure IPs.*
- *Later, (1960) he extended this to MIPs with integer objective.*
- *Cook/Kannan/Schrijver (1990) gave an example in* $\mathcal{Z}^2 \times \mathcal{R}$ *which cannot be solved in finite time using split cuts.*
- *Later Dash and Gunluk (2013) generalized this to examples in* $\mathcal{Z}^n \times \mathcal{R}$ *that cannot be solved in finite time using* $(n-1)$*-branch split cuts.*
- *For bounded polyhedra Jörg (2008) gave a finite cutting plane algorithm for MIPs.*
- *Using multi-branch split cuts, we recently gave a finite cutting plane algorithm for MIPs without assuming boundedness or integer objective.*
  ("This algorithm is of purely theoretical interest, and is highly impractical".)

Recap

- *A* **multi-branch split cut** *is an inequality valid for* $P^{LP} \setminus \bigcup_i S(\pi_i, \gamma_i)$ *where*

$$S(\pi_i, \gamma_i) = \{(x, y) \in \mathcal{R}^{n+k} \; : \; \gamma_i < \pi_i^T x < \gamma_i + 1\}$$

and π_i *and* γ_i *are integral.*

- *Let* $c^T x + d^T y \geq f$ *be a valid inequality for* P *and*

$$V = \{(x, y) \in P^{LP} \; : \; c^T x + d^T y < f\}.$$

be the set points cut off by it. ($V \cap (\mathcal{Z}^n \times \mathcal{R}^l) = \emptyset$*)*

If $V \subseteq \mathcal{S}$ *where* $\mathcal{S} = \cup_{i=1}^t S_i$*, then* $c^T x + d^T y \geq f$ *is a* t*-branch split cut.*

Claim: *All such* V *can be covered by a bounded number of split sets.*

Lemma : *Let B be a bounded, strictly lattice-free convex set in $\mathcal{R}^n$. Then B is contained in the union of at most $h(n)$ split sets.*

Proof : *By Khinchine's flatness result.*

- *There is an integer vector $a \in \mathcal{Z}^n$ such that $f(n) \geq u - l$ where*

$$u = \max\{a^T x : x \in B\} \text{ and } l = \min\{a^T x : x \in B\}$$

- *Therefore, $B \subseteq \{x \in \mathcal{R}^n : \lfloor l \rfloor \leq a^T x \leq \lceil u \rceil\}$.*
- *Let U be the collection of the split sets $S(a, b)$ for $b \in W = \{\lfloor l \rfloor, \ldots, \lceil u \rceil - 1\}$*

$$B \setminus \bigcup_{b \in W} S(a,b) = \bigcup_{b \in \bar{W}} \{x \in B : a^T x = b\}$$

  where $\bar{W} = \{\lceil l \rceil, \ldots, \lfloor u \rfloor\}$.

- *All $\{x \in B : a^T x = b\}$ are strictly lattice-free and have dimension at most $n - 1$*
- *Repeating the same argument proves the claim. ($h(n) \approx \Pi_{i=1}^n (2 + \lceil f(i) \rceil)$)* ∎

Lemma : *Let B be a strictly lattice-free, convex, unbounded set in $\mathcal{R}^n$ which is contained in the interior of a maximal lattice-free convex set. Then B can be covered by $h(n)$ split sets.*

Proof :

- *Let B' be a maximal lattice free set containing B in its interior.*
- *Lovász (1989) and Basu, Conforti, Cornuejols, Zambelli (2010) showed that*

$$B' = Q + L$$

  where Q is a polytope and L a rational linear space.
- *Let $dim(Q) = d$ and $dim(L) = n - d > 0$.*
- *After a unimodular transformation, $Q \subset \mathcal{R}^d$ and $L = \mathcal{R}^{n-d}$*
- *Use the result for the bounded case and the result follows.* ■

Combining the two cases

Theorem : *Every facet-defining inequality for P is a $h(n)$-branch split cut.*

- *Let $c^T x + d^T y \geq f$ be valid for conv$\,(P)$ but not for P^{LP},*
- *Let $V \subseteq \mathcal{R}^{n+l}$ be the set cut off by $c^T x + d^T y \geq f$ and let V^x be its the projection on the space of the integer variables.*
- *V^x is strictly lattice-free, and is non-empty.*
- *Jörg (2007) showed that V^x is contained in the interior of a lattice-free rational polyhedron and therefore in the interior of a maximal lattice-free convex set.*
- *Depending on whether V^x is bounded or unbounded, we can use either of the previous two lemmas to prove the claim.* ∎

Note :

- *Jörg already observed that every facet-defining inequality is a disjunctive cut.*
- *We show that they can be derived as structured disjunctive cuts.*

Theorem : *The mixed-integer program*

$$\min\{c^T x + d^T y \ : \ (x, y) \in \mathcal{Z}^n \times \mathcal{R}^l \ , \ Ax + Gy \geq b\}$$

where the data is rational, can be solved in finite time via a pure cutting-plane algorithm which generates only t-branch split cuts.

Proof : *Let $t = h(n) \approx \Pi_{i=1}^{n}(2 + \lceil f(i) \rceil)$.*

- *Represent any t-branch split disjunction $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ by $v \in \mathcal{Z}^{(n+1)t}$.*
- *Let $\Omega = \mathcal{Z}^{(n+1)t}$ and arrange its members in a sequence $\{\Omega_i\}$, (by increasing norm)*
- *Let D_i be the t-branch split disjunction defined by Ω_i.*
- *Any facet-defining inequality of conv (P), is a t-branch split cut defined by the disjunction D_k for some (finite) k.*
- *Let k^* be the largest index of a disjunction associated with facet-defining inequalities.*
- *Solve the relaxation of the MIP for $P_i = P_{i-1} \cap$ conv $(P_0 \cap D_i)$ for $i = 1, 2, \ldots$* ■

Note: Validity of a given inequality can also be checked by changing the termination criterion. Similarly, conv (P) can also be computed the same way.

How finite is this algorithm?

Previous Theorem : *The mixed-integer program*

$$\min\{c^T x + d^T y \ : \ (x,y) \in \mathcal{Z}^n \times \mathcal{R}^l \ , \ Ax + Gy \geq b\}$$

can be solved in finite time via a pure cutting-plane algorithm.

Proof : *The algorithm cannot run forever* ■

Stronger result: *The runtime of this algorithm is bounded.*

Proof : P^{LP} *has bounded facet complexity (#bits to represent facet defining inequalities)*

⇒ *Therefore conv* (P) *has bounded facet complexity.*

⇒ *Therefore* V *(points cut-off by a facet) has bdd complexity.*

⇒ V *has a "thin" direction along an integer vector of bdd complexity.*
(we prove this by formulating the lattice width problem as an IP with bdd complexity)

⇒ *Therefore the multi-branch disjunction needed to generate a facet has bdd complexity.*

⇒ *It is possible to make a list of relevant split disjunctions in advance.* ■

Next: How finite is t?

Bounding t

- *We showed that every facet-defining inequality for P is a multi-branch split cut that uses at most $h(n) \approx \Pi_{k=1}^{n}(2 + \lceil f(k) \rceil)$ split sets.*

  [best know bound $f(k) \leq O(k^{4/3} \log^c k)$ by Rudelson, 2000].

- *Are there examples where one has to use a large number of split sets?*
  - *It is easy to show that $t \geq \Omega(n)$*
  - *With some work, we can also show that $t \geq \Omega(2^n)$*

Theorem : *For any $n \geq 3$ there exists a nonempty rational mixed-integer polyhedral set in $\mathcal{Z}^n \times \mathcal{R}$ with a facet-defining inequality that cannot be expressed as a $3 \times 2^{n-2}$-branch split cut.*

.

Proof : *(outline)*

- *Construct a full-dimensional rational, lattice-free polytope $B \subset \mathcal{R}^n$ such that*
  - *Its interior cannot be covered by $3 \times 2^{n-2}$ split sets*
  - *The integer hull of $B \subset \mathcal{R}^n$ has dimension n*
- *Define a mixed-integer polyhedral set P_B as follows:*

$$P_B = \{(x, y) \in \mathcal{Z}^n \times \mathcal{R} : (x, y) \in B'\}.$$

where

$$B' = conv((B \times \{-1\}) \cup (B \times \{0\}) \cup (\bar{x} \times \{1/2\}))$$

and $\bar{x}$ is a point in the interior of B.

- *$y \leq 0$ is a facet-defining inequality for conv (P_B)*
- *To cover $V = \{(x, y) \in P_B^{LP} \ : \ y > 0\}$, one needs at least $(3 \times 2^{n-2}) + 1$ split sets.* ■

How to construct the lattice-free polytope $B \subset \mathcal{R}^n$

For $\Delta \in \{0, \dots, 2^{n-2} - 1\}$*, let* $T_\Delta \in \mathcal{R}^2$ *be a (rational) lattice-free triangle*

Let $\Delta = \sum_{l=1}^{n-2} \delta_l 2^{l-1}$ *with* $\delta_l \in \{0, 1\}$

$$\mathbf{T}_\Delta = \{(\delta_1, \dots, \delta_{n-2}, x, y) \in R^n | (x, y) \in T_\Delta\}$$

Define

$$B_\varepsilon = \textit{conv}\Big(\bigcup_{\Delta=0}^{2^{n-2}-1} (\mathbf{T}_\Delta \cup \{p_{\varepsilon,\Delta}\}) \Big)$$

where

$$p_{\varepsilon,\Delta} = (\delta_1, \dots, \delta_{n-2}, \mathrm{cent}(T_\Delta)) + ((2\delta_1 - 1)\varepsilon, \dots, (2\delta_{n-2} - 1)\varepsilon, 0, 0)$$

(For example, $p_{\varepsilon,0} = (-\varepsilon, \dots, -\varepsilon, \bar{x}, \bar{y})$ *where* $(\bar{x}, \bar{y}) = \mathrm{cent}(T_0)$*.)*

Fact : B_ε *is full dimensional, rational, lattice-free and* $rel.int(\mathbf{T}_\Delta) \subset int(B_\varepsilon)$*.*

How to construct the triangles $\mathbf{T}_\triangle$

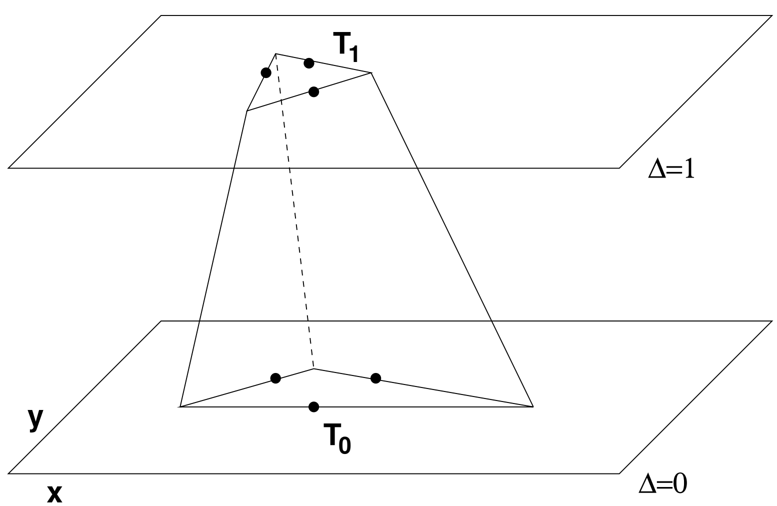
- $\mathbf{T}_0 \in \mathcal{R}^2$ *is a rational Hurken's triangle with* $w(\mathbf{T}_0) \geq 2.15$ *that needs at least* 3 *split sets to cover.*
- *For* $\Delta \in \{1, \ldots, 2^{n-2} - 1\}$,

$$T_\triangle = M_\triangle \mathbf{T}_0$$

where $M_\triangle$ *is a* 2×2 *unimodular matrix with the property that:*

* *If a split set is useful in covering some* $T_\triangle$*, it is not useful for* $T'_\triangle$ *unless* $\Delta = \Delta'$

when $n = 3$



1. **Useful split sets are finite**.

   For any compact set $K \subset \mathcal{R}^n$ and any number $\varepsilon > 0$, the collection of split sets $S(a,b)$ such that $vol(K \cap S(a,b)) \geq \varepsilon$ is finite.

2. **Useful split sets are really necessary for $\mathbf{T}_0$**.

   For any fixed $l \geq 0$, there exists a finite collection of split sets Σ_l such that whenever some l split sets cover $\mathbf{T}_0$, then at least 3 of them are contained in Σ_l.

3. **Bending the triangles**.

   Given any two finite sets of vectors $V, W \subseteq \mathcal{Z}^2 \setminus \{\mathbf{0}\}$, there exists an unimodular matrix M such that $MW \subseteq \mathcal{Z}^2 \setminus \{\mathbf{0}\}$ and $MW \cap V = \emptyset$.

   Proof : *Let $q = \max_{v \in W} ||v||_\infty$ then*

$$M = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + 1 \end{pmatrix} \text{ where } \mu = 3q$$

■

Putting it together

- *Start with* 2^{n-2} *copies of the rational Hurken's triangle in* $\mathcal{R}^2$.
- *Bend the* k*th copy so that split sets useful for* $\mathbf{T}_0, \ldots, \mathbf{T}_{k-1}$ *are not useful for* $\mathbf{T}_k$.
- *Extend the corners of a hypercube in* $\mathcal{R}^{n-2}$ *with the triangles to* $\mathcal{R}^n$.
- *Add apexes to make the triangles in the interior of* B.
- *To cover the interior of* B, *one needs to cover the triangles*
- *Last two coordinates of a split set in* $\mathcal{R}^n$ *gives a split set in* $\mathcal{R}^2$.
- *At least* $3 \cdot 2^{n-2}$ *split sets are necessary to cover* B.
- *To show that* $y \leq 0$ *is valid for* $\text{conv}\,(P_B)$

$$P_B = \{(x, y) \in \mathcal{Z}^n \times \mathcal{R} : (x, y) \in B'\}.$$

where

$$B' = \text{conv}\,((B \times \{-1\}) \cup (B \times \{0\}) \cup (\bar{x} \times \{1/2\}))$$

one needs at least $(3 \times 2^{n-2})$ *split sets.* ■

thank you...