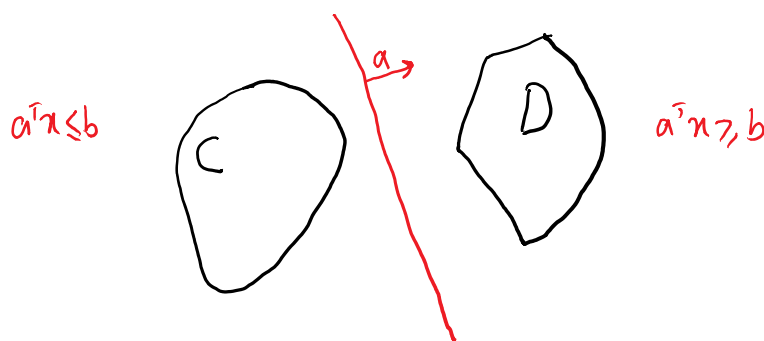## Separating Hyperplane Theorems and Applications

This Lecture:
- Separation of convex sets with hyperplanes
- The Farkas lemma
- Strong duality of linear programming.

Separating hyperplane theorems

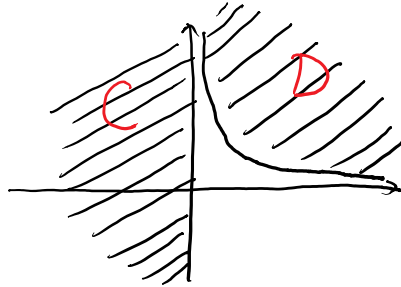The following is one of the most fundamental theorems about convex sets:

__Thm1__. Let C and D be two convex sets in $\mathbb{R}^n$ that do not intersect (i.e. $C \cap D = \phi$). Then, there exists $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$, such that

$$a^T x \leq b \quad \forall x \in C \quad \text{and} \quad a^T x \geq b \quad \forall x \in D.$$

This is called a "separation". In the case of the picture, we in fact have strict separation; i.e, $a^T x < b \quad \forall x \in C$ and $a^T x > b \quad \forall x \in D$.

Strict separation may not always be possible, even when both C and D are closed:
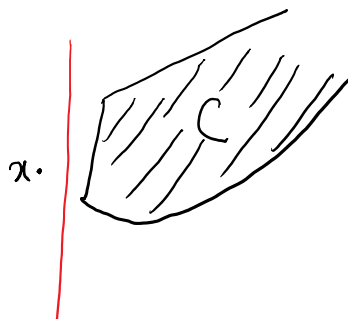

We will prove a special case of Thm 1, which will be good enough for our purposes. (And we will prove strict separation in this special case.) Extending this proof to a proof of Thm 1 is left as an exercise.

__Thm 2__. Let C and D be two closed convex sets in $\mathbb{R}^n$ with at least one of them bounded. Then, $\exists a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$ s.t.

$$a^T x > b \quad \forall x \in D \quad \text{and} \quad a^T x < b \quad \forall x \in C.$$

The following is an important corollary.

__Corollary 3__. Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $x \in \mathbb{R}^n$ a point not in C. Then, x and C can be strictly separated by a hyperplane.
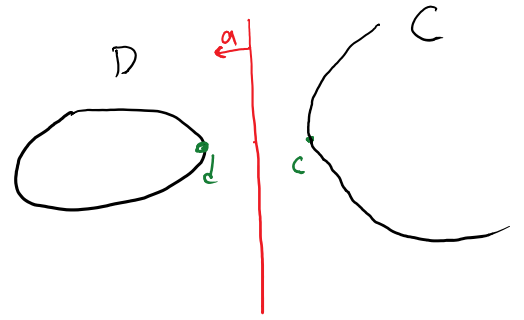
Proof of Thm 2.

Our proof follows [BV04] with some minor

deviations. Define

$$\text{dist}(C, D) = \inf \|u - v\|$$
$$\text{s.t.} \quad u \in C$$
$$v \in D$$

This infimum is acheived (why?) and is positive (why?). Let $c \in C$ and $d \in D$

be the points that acheive it. Let

$$a = d - c, \qquad b = \frac{\|d\|^2 - \|c\|^2}{2}. \qquad \text{(Note } a \neq 0)$$

Our separating hyperplane will be a function $f(x) = a^T x - b$. We claim that

$$f(x) > 0, \quad \forall x \in D \qquad \text{and} \quad f(x) < 0 \quad \forall x \in C.$$

If you are wondering why b is chosen as above, observe that

$$f\left(\frac{c+d}{2}\right) = (d-c)^T \left(\frac{c+d}{2}\right) - \frac{\|d\|^2 - \|c\|^2}{2} = 0.$$

We show that $f(x) > 0$ for all $x \in D$. The proof that $f(x) < 0 \ \forall x \in C$ is identical.

Suppose for the sake of contradiction that $\exists \ \bar{d} \in D$ with $f(\bar{d}) \leq 0$.

$$\Rightarrow (d-c)^T \bar{d} - \frac{\|d\|^2 - \|c\|^2}{2} \leq 0. \qquad \text{①}$$

Define $g(x) = \|x - c\|^2$. We claim that $\bar{d} - d$ is a descent direction for g at d.

Indeed, $\qquad g(x) = (x-c)^T(x-c) = x^T x - 2c^T x + c^T c \Rightarrow \nabla g(x) = 2x - 2c$

$$\nabla g^T(d) (\bar{d} - d) = (2d - 2c)^T (\bar{d} - d)$$

$$= 2\left(-\|d\|^2 + d^T \bar{d} - c^T \bar{d} + c^T d\right) = 2\left(-\|d\|^2 + (d-c)^T \bar{d} + c^T d\right)$$

$$\overset{①}{\leq} 2\left(-\|d\|^2 + \frac{\|d\|^2 - \|c\|^2}{2} + c^T d\right) = -\|d\|^2 - \|c\|^2 + 2c^T d$$

$$= -\|d - c\|^2 < 0.$$
$$\underset{d \neq c}{\uparrow}$$

Hence $\exists \bar{\alpha} > 0$ s.t. $\forall \alpha \in (0, \bar{\alpha})$

$$g(d + \alpha(d - \bar{d})) < g(d)$$

i.e., $\left\| d + \alpha(d - \bar{d}) - c \right\|^2 < \left\| d - c \right\|^2$.

But this contradicts that d was the closest point to c. $\square$

Farkas' Lemma

$\underline{\text{Theorem (or Lemma!)}}$. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Then exactly one of the following sets must be empty: (i) $\{x \mid Ax = b, x \geq 0\}$

(ii) $\{y \mid A^T y \leq 0, b^T y > 0\}$.

A few comments:

○ Systems (i) and (ii) are called $\underline{\text{strong alternatives}}$, meaning that exactly one of them can be feasible. $\underline{\text{Weak alternatives}}$ are systems where at most one can be feasible.

○ This theorem is particularly useful for proving infeasibility of an LP via an explicit and easily-verifiable certificate. If somebody gives you a y as in (ii), then you are convinced immediately that (i) is infeasible.

○ Geometric interpretation of the Farkas lemma:

The geometric interpretation of the Farkas lemma illustrates the connection to the separating hyperplane theorem and makes the proof straightforward.

We need first a few definitions.

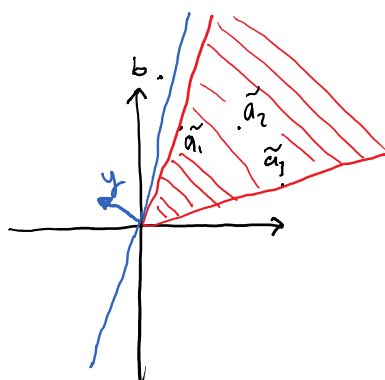Defn. A set $C \subseteq \mathbb{R}^n$ is called a **cone** if

$$x \in C, \; \alpha \in \mathbb{R}, \alpha \geq 0 \implies \alpha x \in C.$$

Defn. The **conic hull** of a set $S \subseteq \mathbb{R}^n$, denoted by Cone(S), is the set of all **conic combinations** of the points in S; i.e,

$$\text{Cone}(S) = \left\{ \sum_{i=1}^{m} \lambda_i x_i \;\middle|\; x_i \in S, \; \lambda_i \geq 0 \text{ for } i = 1, \dots, m \right\}$$

Cone(S)

S

o

○ Geometric interpretation of Farkas' lemma: Let $\tilde{a}_1 \dots, \tilde{a}_n$ denote the columns of A and cone $\{\tilde{a}_1, \dots, \tilde{a}_n\}$ the cone of all their nonnegative combinations. If $b \notin$ cone $\{a_1, \dots, a_n\}$ then we can prove this with a separating hyperplane.

b.

$\tilde{a}_2$

$\tilde{a}_1$

$\tilde{a}_7$

y

Proof (of Farkas Lemma):

First the easy direction: (ii) feasible $\Rightarrow$ (i) infeasible.

Suppose the contrary: $\exists\, x \geq 0,\ Ax = b$. Then $0 < b^T y = x^T A^T y$. But $x \geq 0, A^T y \leq 0$

$\Rightarrow x^T A^T y \leq 0$. Contradiction.

Now we prove (i) infeasible $\Rightarrow$ (ii) feasible.

Let $\tilde{a}_1, \ldots, \tilde{a}_n$ be the columns of A. Let $C := \text{Cone}\{\tilde{a}_1, \ldots, \tilde{a}_n\} = \{\sum_{i=1}^{n} \alpha_i \tilde{a}_i, \ \alpha_i \geq 0\}$. Note that

C is convex (why?) and closed (why*?). Furthermore, $b \notin C$ by the assumption that (i)

is infeasible. By Corollary 3, b and C can be (even strictly) separated:

$\quad \exists\, y \in \mathbb{R}^m, y \neq 0, r \in \mathbb{R}$ s.t. $\quad y^T z \leq r \quad \forall z \in C$ and $y^T b > r$.
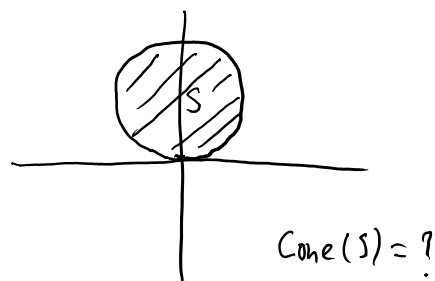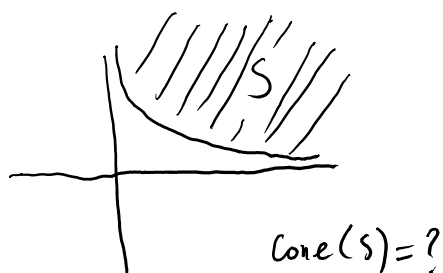
Since $0 \in C$, we must have $r \geq 0$. If $r > 0$, we can replace it by $r' = 0$: Indeed, if

$\exists z \in C$ s.t. $y^T z > 0$, then $y^T(\alpha z)$ can be arbitrarily large as $\alpha \to \infty$ and $\alpha z \in C$.

So, $\quad y^T z \leq 0 \quad \forall z \in C$ and $y^T b > 0$.

Since $\tilde{a}_1, \ldots, \tilde{a}_n \in C$, we see that $A^T y \leq 0$. $\qquad \square$

($*$) The fact that C is closed takes some thought. Note that conic hulls of

closed (or even compact) sets may not be closed.

Cone(S)=?   Cone(S)=?

In class, we argued why if S is a finite set of points, then Cone(S) is closed.

We remark that the Farkas lemma can be directly proven from strong duality of linear programming. The converse is also true! We will show these facts next. Note that there are other proofs of LP strong duality; e.g, based on the simplex algorithm. However the simplex-based proof does not generalize to broader classes of convex programs, whereas the separating hyperplane based proofs do.

Farkas Lemma from LP strong duality

Consider the primal-dual LP pair:

$$(P) \quad \min. \quad 0 \qquad\qquad (D) \quad \max. \quad b^T y$$
$$Ax = b \qquad\qquad\qquad A^T y \leq 0$$
$$x \geq 0$$

Note that (D) is trivially feasible (set $y = 0$), so if (P) is infeasible, then (D) must be unbounded or else strong duality would imply that the two optimal values should match, which is impossible since (P) is by assumption infeasible.

But (D) unbounded $\Rightarrow \exists y$ s.t. $A^T y \leq 0$, $b^T y > 0$. $\square$

LP strong duality from Farkas lemma

Strong duality theorem: Consider a primal-dual LP pair:

$$(P) \quad \min c^T x \qquad\qquad (D) \quad \max b^T y$$
$$Ax = b \qquad\qquad\qquad A^T y \leq c$$
$$x \geq 0$$

If (P) has a finite optimal value, then so does (D) and the two values match.

Remark. If you don't recall how to write down the dual of an LP, look up the first few pages of Chapter 5 of [BV04]. The derivation there works more broadly (not just LP).

To prove strong duality from Farkas, it is useful to first prove a variant of the Farkas lemma. This variant comes handy when you want prove infeasibility of an LP in inequality form.

Lemma (Farkas Variant). Let $A \in \mathbb{R}^{m \times n}$.

$$\{x \mid Ax \leq b\} \text{ is empty} \iff \exists \lambda \geq 0 \text{ s.t. } \lambda^T A = 0, \lambda^T b < 0.$$

Proof: ($\Leftarrow$) Easy (why?)

($\Rightarrow$) Rewrite the LP in standard form and apply the (standard) Farkas lemma:

$$Ax \leq b \longleftrightarrow \left[\begin{array}{l} A(x^+ - x^-) + s = b \\ \quad s \geq 0 \\ \quad x^+ \geq 0 \\ \quad x^- \geq 0 \end{array} \right. \longleftrightarrow (A \mid -A \mid I) \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b$$

$$x^+ \geq 0, \ x^- \geq 0, \ s \geq 0.$$

$$\overset{\text{Farkas}}{\Rightarrow} \quad \exists \lambda \text{ s.t. } b^T y < 0, \quad \begin{pmatrix} A^T \\ -A^T \\ I \end{pmatrix} \lambda \geq 0 \ \overset{\exists \lambda \text{ s.t.}}{\Rightarrow} \ b^T y < 0, \ A^T \lambda = 0, \ \lambda \geq 0. \ \square$$

Proof of LP strong duality from Farkas lemma :

Consider the primal dual pair :

(P) min. $c^T x$ 　　　　(D) max. $b^T y$

　　　s.t. $Ax = b$ 　　　　　　s.t. $A^T y \leq c$.

　　　　　$x \geq 0$

Assume the optimal value of (P) is finite and equal to p^*. We would be done if we prove that the following inequalities are feasible :

$$\begin{bmatrix} y^T b \geq p^* \\ A^T y \leq c \end{bmatrix}. \quad ①$$

Indeed, any y satisfying $A^T y \leq c$ must also satisfy $y^T b \leq p^*$ by weak duality (whose proof is trivial), so we would get that $y^T b = p^*$.

Let's rewrite ① a bit : $\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -p^* \end{pmatrix}$. Suppose these inequalities

were infeasible. Then "the Farkas lemma variant" would imply

$\exists \lambda := \begin{pmatrix} \tilde{\lambda} \\ \lambda_0 \end{pmatrix} \geq 0$ s.t. $\tilde{\lambda}^T A^T - \lambda_0 b^T = 0$ and $\tilde{\lambda}^T c - \lambda_0 p^* < 0$

$\Rightarrow A\tilde{\lambda} = \lambda_0 b, \quad c^T \tilde{\lambda} < \lambda_0 p^*$.

We consider two cases :

Case 1 : $\lambda_0 = 0 \Rightarrow A\tilde{\lambda} = 0, \quad c^T \tilde{\lambda} < 0$.

Let $\bar{x}$ be an optimal solution to P. Consider $x = \bar{x} + \tilde{\lambda}$. Then $Ax = A\bar{x} + A\tilde{\lambda} = b$ and $x = \bar{x} + \tilde{\lambda} \geq 0$, so x is primal feasible. But $c^T x = c^T \bar{x} + c^T \tilde{\lambda} < p^*$, contradicting optimality of $\bar{x}$.

Case 2: $\lambda_0 > 0$. Let $x = \tilde{\lambda}/\lambda_0$. Then $Ax = \frac{1}{\lambda_0} A\tilde{\lambda} = \frac{1}{\lambda_0} \lambda_0 b = b$, $x \geq 0$, and

$c^T x = c^T \frac{\tilde{\lambda}}{\lambda_0} < \frac{1}{\lambda_0} \lambda_0 p^* = p^*$. This again contradicts p^* being the primal optimal value. □

Notes

Further reading for this lecture can include Chapter 2 of [BV04].

References

○ [BV04] S. Boyd and L. Vandenberghe. Convex Optimization

Cambridge Press, 2004.