When in doubt on the accuracy of these notes, please cross check with the instructor's notes, on **aaa.princeton.edu/orf523**. *Any typos should be emailed to gh4@princeton.edu.*

In this lecture, we will cover an application of LP strong duality to combinatorial optimization:

- Bipartite matching

- Vertex covers

- König's theorem

- Totally unimodular matrices and integral polytopes.

1   Bipartite matching and vertex covers

Recall that a *bipartite* graph $G = (V, E)$ is a graph whose vertices can be divided into two disjoint sets such that every edge connects one node in one set to a node in the other.

Definition 1 (Matching, vertex cover). *A* matching *is a disjoint subset of edges, i.e., a subset of edges that do not share a common vertex.*
A vertex cover *is a subset of the nodes that together touch all the edges.*
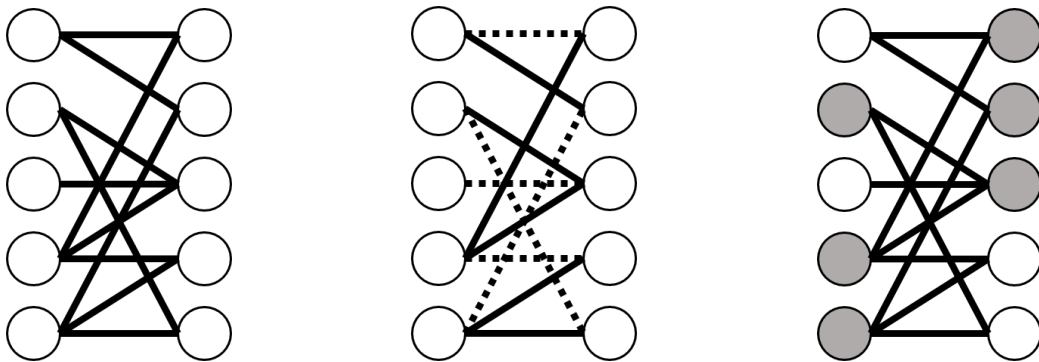
(a) An example of a bipartite graph

(b) An example of a matching (dotted lines)

(c) An example of a vertex cover (grey nodes)

Figure 1: Bipartite graphs, matchings, and vertex covers

Lemma 1. *The cardinality of any matching is less than or equal to the cardinality of any vertex cover.*

This is easy to see: consider any matching. Any vertex cover must have nodes that at least touch the edges in the matching. Moreover, a single node can at most cover one edge in the matching because the edges are disjoint.

As it will become clear shortly, this property can also be seen as an immediate consequence of weak duality in linear programming.

Theorem 1 (König)**.** *If G is bipartite, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.*

Remark: The assumption of bipartedness is needed for the theorem to hold (consider, e.g., the triangle graph).

Proof: One can rewrite the cardinality M of the maximum matching as the optimal value of the integer program

$$M := \max \sum_{i=1}^{|E|} x_i$$
$$\text{s.t. } \sum_{i \sim v} x_i \leq 1, \ \forall v \in V \tag{1}$$
$$x_i \in \{0, 1\}, i = 1, \ldots, |E|,$$

where the second sum is over the edges adjacent to vertex v, as denoted by the notation $i \sim v$. This problem can be *relaxed* to a linear program, whose optimal value is denoted by M_{LP}

$$M_{LP} := \max \sum_{i=1}^{|E|} x_i$$
$$\text{s.t. } \sum_{i \sim v} x_i \leq 1, \ \forall v \in V \tag{2}$$
$$0 \leq x_i \leq 1, i = 1, \ldots, |E|.$$

Notice that $M \leq M_{LP}$. Indeed, any feasible solution to integer program (1) is feasible to linear program (2). We now rewrite (2) in matrix notation. Let A be the incidence matrix of G, i.e., an $|V| \times |E|$ zero-one matrix whose $(i, j)^{th}$ entry is a 1 if and only if node i is adjacent to edge j. Note that each column of A has two 1's (as any edge joins two nodes)

and each row of A has as many 1's as the degree of the corresponding node. Problem (2) can then be written as[1]

$$M_{LP} = \max \sum_{i=1}^{|E|} x_i$$

$$\text{s.t. } Ax \leq \mathbf{1} \tag{3}$$

$$x_i \geq 0, i = 1 \ldots, |E|.$$

We take the dual of this LP (we will soon observe that this coincides with the LP relaxation of vertex cover):

$$\min \sum_{i=1}^{|V|} y_i$$

$$\text{s.t. } A^T y \geq \mathbf{1} \tag{4}$$

$$y \geq 0.$$

Consider now an integer programming formulation of the minimum vertex cover problem:

$$VC := \min \sum_{i=1}^{|V|} y_i$$

$$\text{s.t. } y_i + y_j \geq 1, \text{ if } (i,j) \in E \tag{5}$$

$$y_i \in \{0,1\}, i = 1, \ldots, |V|.$$

This can be relaxed to the LP[2]

$$VC_{LP} := \min \sum_{i=1}^{|V|} y_i$$

$$\text{s.t. } y_i + y_j \geq 1, \text{ if } (i,j) \in E$$

$$y_i \geq 0, i = 1, \ldots, |V|,$$

which exactly corresponds to (4) once we rewrite it in matrix form using the incidence matrix.

An outline of the rest of the proof is given in Fig 2. We have shown that $M_{LP} = VC_{LP}$ using strong duality. It remains to be shown that $VC = VC_{LP}$ and $M = M_{LP}$ to conclude the proof. This will be done by introducing the concept of *totally unimodular matrices.*

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[1]Note that there is no harm in dropping the constraints $x_i \leq 1, i = 1, \ldots, |E|$ (why?).

[2]Note that again there is no harm in dropping the constraints $y_i \leq 1, i = 1, \ldots, |V|$ (why?).
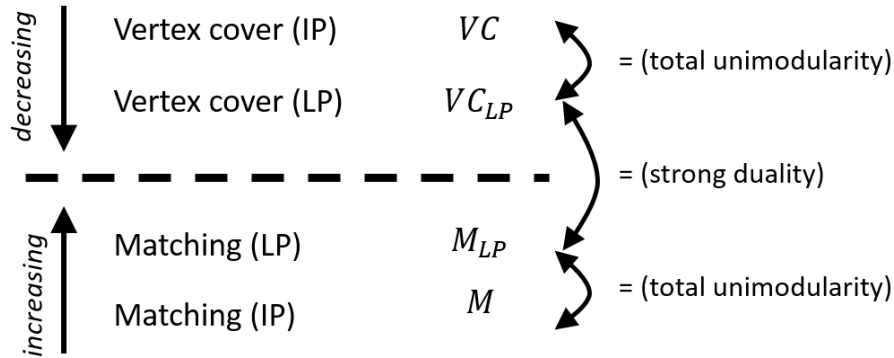
Figure 2: Outline of the proof of Theorem 1

2 Totally unimodular matrices

Definition 2. *A matrix A is* totally unimodular *(TUM) if the determinant of any of its square submatrices belongs to $\{0, -1, +1\}$.*

Definition 3.

- *Given a convex set P, a point $x \in P$ is* extreme *if $\nexists y, z \in P, y \neq x, z \neq x$, such that $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in [0, 1]$.*

- *Given a polyhedron $P = \{x \mid Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, a point x is a* vertex *of P if*

  (1) $x \in P$

  (2) $\exists n$ linearly independent tight constraints at x. (Recall that a set of constraints is said to be linearly independent if the corresponding rows of A are linearly independent.)

Theorem 2 (see, e.g., Lecture 11 of ORF363 [1])**.** *A point x is an extreme point of $P = \{x \mid Ax \leq b\}$ if and only if it is a vertex.*

Theorem 3 (see, e.g., Lecture 11 of ORF363 [1])**.** *Consider the LP:*

$$\min_{x \in P} c^T x$$

where $P = \{x \mid Ax \leq b\}$. Suppose P has at least one extreme point. Then, if there exists an optimal solution, there also exists an optimal solution which is at a vertex.

Definition 4. *A polyhedron* $P = \{x \mid Ax \leq b\}$ *is* integral *if all of its extreme points are integral points (i.e., have integer coordinates).*

Theorem 4. *Consider a polyhedron* $P = \{x \mid Ax \leq b\}$ *and assume that* A *is TUM. Then if* b *is integral, the polyhedron* P *is integral.*

<u>Proof:</u> Let x be an extreme point. By definition, n constraints out of the total m are tight at x. Let $\tilde{A} \in \mathbb{R}^{n \times n}$ (resp. $\tilde{b} \in \mathbb{R}^n$) be the matrix (resp. vector) constructed by taking the n rows (resp. elements) corresponding to the tight constraints. As a consequence

$$\tilde{A}x = \tilde{b}.$$

By Cramer's rule,

$$x_i = \frac{det(\tilde{A}_i)}{det(\tilde{A})},$$

where $\tilde{A}_i$ is the same as $\tilde{A}$ except that its ith column is swapped for $\tilde{b}$. Since $\tilde{A}$ is nonsingular, total unimodularity of A implies that $det(\tilde{A}) \in \{-1, 1\}$. Moreover, $det(\tilde{A}_i)$ is some integer. $\square$

Theorem 5. *The incidence matrix* A *of a bipartite graph is TUM.*

<u>Proof:</u> Let B be a $t \times t$ square submatrix of A We prove that $det(B) \in \{-1, 0, +1\}$ by induction on t. If $t = 1$, the determinant is either 0 or 1 as any element of an incidence matrix is either 0 or 1. Now, let t be any integer $\leq n$. We consider three cases.

(1) *Case 1:* B has a column which is all zeros. In that case, $det(B) = 0$.

(2) *Case 2:* B has a column which has exactly one 1. In this case, the matrix B (after possibly shuffling the columns) can be written as:

$$B = \begin{bmatrix} 1 & b' \\ 0 & \\ \vdots & B' \\ 0 & \end{bmatrix}.$$

Then, $det(B) = det(B')$ and $det(B') \in \{-1, 0, 1\}$ by induction hypothesis.

(3) *Case 3:* All columns of B have exactly two 1's. Because our graph is bipartite, after possibly shuffling the rows of B, we can write

$$B = \begin{bmatrix} B' \\ B'' \end{bmatrix}$$

where each column of B' has exactly one 1 and each column of B'' also has exactly one 1. If we add up the rows of B' or the rows of B'' we get the all ones vector. Hence the rows of B are linearly dependent and $det(B) = 0$. $\square$

We now finish the proof of König's theorem. Recall that the linear programming relaxation of the matching problem is given by

$$M_{LP} = \max \sum_{i=1}^{|E|} x_i$$

$$\text{s.t. } Ax \leq \mathbf{1}$$

$$x \geq 0.$$

The feasible set of this problem is the polyhedron

$$Q = \{x \mid Ax \leq \mathbf{1}, x \geq 0\}. \tag{6}$$

We have just shown that the incidence matrix A of our graph G is TUM. Furthermore, $b = \mathbf{1}$ is integral. However, we are not quite able to apply Theorem 4 as Q is not exactly in the form given by the theorem. The following theorem will fix this problem.

Theorem 6. *If $A \in \mathbb{R}^{m \times n}$ is TUM, then for any integral $b \in \mathbb{R}^m$, the polyhedron $P = \{x \mid Ax \leq b, x \geq 0\}$ is integral*[3].

Proof: We rewrite the polyhedron P as

$$P = \{x \mid \tilde{A}x \leq \tilde{b}\}, \text{ where } \tilde{A} = \begin{bmatrix} A \\ -I \end{bmatrix} \text{ and } \tilde{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

One can check that $\tilde{A}$ is TUM (convince yourself that if we concatenate a TUM matrix with a standard basis vector, the result is still TUM). As $\tilde{b}$ is integral, we can apply Theorem 4 and conclude that P is integral. $\square$

We now know that the polyhedron Q in (6) is integral. This by definition means that all extreme points of Q are integral. As there exists an optimal solution which is an extreme point (from Theorem 3), we conclude that there exists an optimal solution to LP (3) which is integral. This solution is also an optimal solution to integer program (1). Hence, $M_{LP} = M$.

In a similar fashion, one can show that $VC_{LP} = VC$. This follows from the fact that if a matrix E is TUM, then E^T is also TUM.

The equalities $M_{LP} = M$ and $VC_{LP} = VC$ combined with the fact that $VC_{LP} = M_{LP}$ enable us to conclude that $M = VC$, i.e., König's theorem.

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[3]It turns out that the converse of this result is also true.

References

[1] ORF363 lectures notes. `http://www.aaa.princeton.edu/orf363`. Accessed: 2016-02-26.