Princeton University

ORF 523
Final Exam, Spring 2021

Monday, May 3, 8am EST, to Friday, May 14, 12pm EST

Instructor:
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AIs:
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1. Please write out and sign the following pledge on top of the first page of your exam: "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this exam."

2. The exam is not to be discussed with *anyone* except possibly the professor and the AIs. You can only ask clarification questions as *public* (and preferably non-anonymous) questions on Piazza. No emails.

3. You are allowed to consult the lecture notes and videos, your own notes, the problem sets and their solutions (yours and ours), the midterm exam and its solution (yours and ours), the practice exams and their solutions, past Piazza questions and answers, but *nothing else.* You can only use the Internet in case you run into problems related to software. (There should be no need for that either hopefully.)

4. For all problems involving a coding element, show your code. The output that you present should come from your code.

5. The exam is to be submitted on Gradescope before Friday, May 14, at 12 PM EST.

6. Some problems might be harder than others and there is no particular order. Please be rigorous, brief, and to the point in your answers. Good luck!

Problem 1: A tale of two hardworking TAs (32 pts)

A professor has a set T of tasks to assign to the two teaching assistants of his course. Suppose the number of tasks $|T|$ is an even number. In order to be fair to the two teaching assistants, he would like to split the set of tasks into two subsets T_1, T_2 of equal size. However, some tasks depend on each other and are best handled if assigned to the same teaching assistant. The objective of the professor is therefore to minimize the number of dependencies between tasks that are assigned to different teaching assistants, subject to the fairness constraint. The decision version of this problem can be formally stated as follows.

FAIR-SPLIT

Input: A set of tasks $T = \{t_1, \ldots, t_n\}$ with n even; a symmetric $n \times n$ matrix D with zero diagonal and entry D_{ij} equal to 1 if tasks t_i and t_j depend on each other, and equal to 0 otherwise; a positive integer ℓ.

Question: Can the professor split T into two subsets T_1, T_2 of equal size such that the set $\big\{\{t_i, t_j\} \,:\, t_i \in T_1, t_j \in T_2, D_{ij} = 1\big\}$ has at most ℓ elements?

1. Show that FAIR-SPLIT is NP-complete. (Hint: You may want to consider a reduction from the MAX-CUT problem.)

2. Let J denote the $n \times n$ matrix of all ones, and for $m \geq 1$, let e_m denote the $m \times 1$ vector of all ones. Let

$$\begin{aligned} \eta^* \quad := \quad & \min_{Z \in \mathbb{R}^{n\times 2}} && \frac{1}{2}\mathrm{Tr}\big(D(J - ZZ^T)\big) \\ & \text{s.t.} && Ze_2 = e_n \\ & && Z^T e_n = (n/2 \;\; n/2)^T \\ & && Z_{ij} \in \{0,1\} \text{ for } i = 1, \ldots, n \text{ and } j = 1, 2. \end{aligned} \tag{1}$$

   Argue that the answer to FAIR-SPLIT is yes if and only if $\eta^* \leq \ell$.

3. Show that the following semidefinite program gives a lower bound on η^* (i.e., $\lceil \eta^*_{SDP} \rceil \leq \eta^*$):

$$\begin{aligned} \eta^*_{SDP} \quad := \quad & \min_{X \in S^{n\times n}} && \frac{1}{2}\mathrm{Tr}\big(D(J - X)\big) \\ & \text{s.t.} && \mathrm{diag}(X) = e_n \\ & && 2\mathrm{Tr}(JX) = n^2 \\ & && 2X - J \succeq 0. \end{aligned} \tag{2}$$

4. Solve (2) for the matrix D given in `dependency_matrix.mat`. Is the SDP bound tight? If so, recover an optimal split T_1, T_2 from your optimal solution of (2).

5. Present an instance of FAIR-SPLIT where the answer is no but $\lceil \eta^*_{SDP} \rceil \leq \ell$.

Problem 2: Spectral radius and convexity (15 pts)
Recall that the spectral radius $\rho(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is the largest absolute value of the eigenvalues of A. Prove or disprove the following statements:

1. The function $\rho : \mathbb{R}^{n\times n} \to \mathbb{R}$ is quasiconvex.
2. The function $\rho : S^{n\times n} \to \mathbb{R}$ is convex. (Here, $S^{n\times n}$ denotes the set of real symmetric $n \times n$ matrices.)

Problem 3: Eigenvalues and semidefinite programming (30 pts)
Given matrices $A_0, A_1, \ldots, A_m \in S^{n\times n}$, let the function $A : \mathbb{R}^m \to S^{n\times n}$ be defined as $A(x) = A_0 + \sum_{i=1}^m x_i A_i$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$ be the eigenvalues of $A(x)$.

1. Suppose there exists a vector $\bar{x} \in \mathbb{R}^m$ such that $A(\bar{x})$ is positive definite. Show that the problem minimizing $\lambda_1(x)/\lambda_n(x)$ subject to $A(x)$ being positive semidefinite can be formulated as a semidefinite program. Will the optimal value always be attained?
2. Show that the problem of minimizing $|\lambda_1(x)|+|\lambda_2(x)|+\cdots+|\lambda_n(x)|$ can be formulated as a semidefinite program. Will the optimal value always be attained?

Problem 4: Difference of SOS decomposition (23 pts)
Recall that a polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is a *sum of squares* (SOS) if it can be written as $p(x) = \sum_i q_i^2(x)$ for some polynomials $q_i(x)$.

1. Show that any polynomial $p(x)$ can be written as $p(x) = s_1(x) - s_2(x)$, where $s_1(x)$ and $s_2(x)$ are SOS polynomials whose degrees are at most one higher than the degree of $p(x)$. Is this decomposition unique?
2. Write the Motzkin polynomial

$$M(x) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$$

as $M(x) = s_1(x) - s_2(x)$, where $s_1(x)$ and $s_2(x)$ are SOS polynomials of degree at most 6 and the 2-norm of the coefficients of $s_2(x)$ is as small as possible. Report the coefficients of s_2 and its 2-norm.