ORF 523	Problem set 2	Spring 2021, Princeton University
Instructor: A.A. Ahmadi		
AIs: A.Z. Chaudry, C. Dibek	UCAs: C. Becker, K. Leung	
Due on March 4, 2021, at 1:30pm EST, on Gradescope		

Problem 1: True or False?

Specify whether each of the following statements is true or false and provide either a proof or a counterexample depending on your answer. Let S be a set in $\mathbb{R}^n$.

1. The convex hull of S is the intersection of all convex sets that contain S.

2. If S is closed, then the convex hull of S is closed.

3. If S is bounded, then the convex hull of S is bounded.

4. If S is compact, then the convex hull of S is compact.
(You may want to use the following fact from analysis: the image of a compact set under a continuous mapping is compact.)

5. The sum of two quasiconvex functions is quasiconvex.

6. A quadratic function $f(x) = x^TQx + b^Tx + c$ is convex if and only if it is quasiconvex.
(You can use the fact that f is convex if and only if $Q \succeq 0$ if you need to.)

7. Any closed convex set $\Omega \subseteq \mathbb{R}^n$ can be written as $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ for some convex function $g : \mathbb{R}^n \to \mathbb{R}$.

8. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex on a convex set $S \subseteq \mathbb{R}^n$, then f is continuous on S.

9. Suppose $P \in \mathbb{R}^{n\times n}$ is a matrix with nonnegative entries whose columns each sum up to one. Then, there exists $x \in \mathbb{R}^n$ such that $Px = x$, $x \geq 0$, and $\sum_{i=1}^n x_i = 1$.

10. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the midpoint convexity property
$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) \;\forall x, y \in \mathbb{R}^n$$
is convex.

Problem 2: CVX warmup / Minimum fuel optimal control

(Boyd&Vandenberghe, Problem 4.16)

We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1,$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e. $x(0) = 0$. The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

(a) Formulate the minimum fuel optimal control problem as a linear program, i.e., a convex optimization problem with affine objective and constraint functions.

(b) Solve the minimum fuel optimal control problem using CVX for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 30.$$

Plot the actuator signal $u(t)$ as a function of time t using the Matlab function `stairs`. You are allowed to let CVX formulate the LP for you, but it's a good idea to check the answer against the LP that you formulated in the previous part.

Problem 3: Theory-applications split in a course. (Courtesy of Stephen Boyd)
A professor teaches a course with 24 lectures, labeled $i = 1, \ldots, 24$. The course involves some interesting theoretical topics, and many practical applications of the theory. The professor must decide how to split each lecture between theory and applications. Let T_i and A_i denote the fraction of the ith lecture devoted to theory and applications, for $i = 1, \ldots, 24$. (We have $T_i \geq 0$, $A_i \geq 0$, and $T_i + A_i = 1$.)
A certain amount of theory has to be covered before the applications can be taught. We model this in a crude way as

$$A_1 + \cdots + A_i \leq \phi(T_1 + \cdots + T_i), \quad i = 1, \ldots, 24, \tag{1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a given nondecreasing function. We interpret $\phi(u)$ as the cumulative amount of applications that can be covered, when the cumulative amount of theory covered is u. We will use the simple form $\phi(u) = a \max\{0, u - b\}$ with $a, b > 0$, which means that no applications can be covered until b lectures of the theory is covered; after that, each lecture of theory covered opens the possibility of covering a lectures on applications.
The theory-applications split affects the emotional state of students differently. We let s_i denote the emotional state of a student after lecture i, with $s_i = 0$ meaning neutral, $s_i > 0$ meaning happy, and $s_i < 0$ meaning unhappy. Careful studies have shown that s_i evolves via a linear recursion (dynamics)

$$s_i = (1 - \theta)s_{i-1} + \theta(\alpha T_i + \beta A_i), \quad i = 1, \ldots, 24,$$

with $s_0 = 0$. Here α and β are parameters (naturally interpreted as how much the student likes or dislikes theory and applications, respectively), and $\theta \in [0, 1]$ gives the emotional volatility of the student (i.e., how quickly he or she reacts to the content of recent lectures). The student's *cumulative emotional state* (CES) is by definition $s_1 + \cdots + s_{24}$. This is a measure of his/her overall happiness throughout the semester.
Now consider a specific instance of the problem, with course material parameters $a = 2$, $b = 3$, and three groups of students, with emotional dynamics parameters given as follows:

	Group 1	Group 2	Group 3
θ	0.05	0.1	0.3
α	-0.1	0.8	-0.3
β	1.4	-0.3	0.7

Your job is to plan (four different) theory-applications splits that respectively maximize the CES of the first group, the CES of the second group, the CES of the third group, and, finally, the minimum of the cumulative emotional states of all three groups. Report the numerical values of the CES for each group, for each of the four theory-applications splits (i.e., fill out the following table):

	Group 1	Group 2	Group 3
Plan 1			
Plan 2			
Plan 3			
Plan 4			

For each of the four plans, plot T_i as well as the emotional state s_i for all three groups, versus i. (So you should have four figures with four curves on each.) These plots show you how the emotional states of the students change as the amount of theory varies.