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Problem 1: A nuclear program for peaceful reasons

A popular matrix norm in machine learning these days is the so-called *nuclear norm*. The nuclear norm of a matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$\|A\|_* := \sum_{i=1}^{\min\{m,n\}} \sigma_i(A),$$

where σ_i is the i -th singular value of A . There is considerable interest in this norm partly because it serves as the convex envelope of the function $\text{rank}(A)$ over the set $\{A \in \mathbb{R}^{m \times n} \mid \|A\|_2 \leq 1\}$.¹ There are numerous application areas where one would like to minimize the rank of a matrix; routine examples are collaborative filtering (<http://www.netflixprize.com/>) or nonconvex quadratic programming.

1. Show that the dual norm of the spectral norm is the nuclear norm.
2. Plot the unit ball of the nuclear norm for symmetric 2×2 matrices.
3. Show that the problem of minimizing the nuclear norm of a matrix subject to arbitrary affine constraints can be cast as a semidefinite program.

Problem 2: Distance geometry

You are given a list of distances d_{ij} for $(i, j) \in \{1, \dots, m\} \times \{1, \dots, m\}$. You would like to know whether there are points $x_i \in \mathbb{R}^n$, for some value of n , such that

$$\|x_i - x_j\|_2 = d_{ij}, \forall i, j.$$

1. Show that this problem can be formulated as that of checking whether a fixed matrix whose entries depend on d_{ij} is positive semidefinite. If this test passes, how would you recover n and the points x_i ?
2. Give an example of a set of distances that respect the triangle inequality but for which there does not exist an embedding in any dimension.

¹What does this statement simplify to in the case where A is diagonal?

Problem 3: Stability of a pair of matrices

Recall that the spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\rho(A)$, is the maximum of the absolute values of its eigenvalues. We call a matrix “stable” if $\rho(A) < 1$. Let us call a pair of real $n \times n$ matrices $\{A_1, A_2\}$ stable if $\rho(\Sigma) < 1$, for any finite product Σ out of A_1 and A_2 . (For example, Σ could be $A_2A_1, A_1A_2, A_1A_1A_2A_1$, and so on.)

1. Does stability of A_1 and A_2 imply stability of the pair $\{A_1, A_2\}$?
2. Prove (possibly using optimization) that the pair $\{A_1, A_2\}$ with

$$A_1 = \frac{1}{4} \begin{pmatrix} -1 & -1 \\ -4 & 0 \end{pmatrix}, A_2 = \frac{1}{4} \begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix}$$

is stable.

Problem 4: SDPs with rational data but no rational feasible solution

Give an example of symmetric $n \times n$ matrices A_1, \dots, A_m with rational entries and rational numbers b_1, \dots, b_m such that the set

$$S := \{X \in S^{n \times n} \mid \text{Tr}(A_i X) = b_i, i = 1, \dots, m, X \succeq 0\}$$

is non-empty, but only contains matrices that have at least one irrational entry. Here, $S^{n \times n}$ denotes the set of symmetric $n \times n$ matrices with real entries and Tr stands for the trace operation. You can pick any value for n and m that you like as long as the above requirements are met.²

²This exercise shows why it is in general difficult for an SDP solver to return an exact feasible solution. By contrast, the situation for LPs is much nicer as a feasible LP with rational data always has a rational feasible solution.