Princeton University

ORF 523

Midterm Exam, Spring 2021

March 19, 2021, 8AM EST - March 22, 2021, 8AM EST

The exam must be taken in a single continuous 120-minute session (including submission time to Gradescope).

Instructor:
Amir Ali Ahmadi

AIs:
Abraar Chaudhry
Cemil Dibek
Cole Becker (UCA)
Kathryn Leung (UCA)

Please read the exam rules below before you start.

1. The exam should be submitted on Gradescope within two hours of the time you receive it. We prefer a single PDF file, but pictures of solutions to individual problems are acceptable as well. Only the latest version submitted before your deadline will be graded.

2. Please remember to write your name on the first page of your solutions. Right next to it, please write out and sign the following pledge: *"I pledge my honor that I have not violated the honor code or the rules specified by the instructor during this examination."*

3. You cannot communicate with anyone during the exam.

4. You can only use the Internet for submission of the exam.

5. You can cite results shown in lecture or on problem sets without proof.

Problem 1: Inclusion Relations among Convex Sets (32 pts)
For two sets $A, B \subseteq \mathbb{R}^n$, define their *Minkowski sum*, $A + B$, as follows:

$$A + B := \{x + y \mid x \in A, y \in B\}.$$

Let $\text{conv}(\cdot)$ denote the convex hull operation. Prove or disprove the following statements:

1. $\text{conv}(A \cap B) \subseteq \text{conv}(A) \cap \text{conv}(B)$
2. $\text{conv}(A \cap B) \supseteq \text{conv}(A) \cap \text{conv}(B)$
3. $\text{conv}(A + B) \subseteq \text{conv}(A) + \text{conv}(B)$
4. $\text{conv}(A + B) \supseteq \text{conv}(A) + \text{conv}(B)$

Problem 2: Difference of Convex Functions (18 pts)
Show that any quadratic function $x^T Q x + b^T x + c$ can be written as the difference of two strictly convex quadratic functions. (Here, Q is a symmetric $n \times n$ matrix, b is a vector in $\mathbb{R}^n$, and c is a scalar.)

Problem 3: Separation of a Point from a Convex Set (20 pts)
Prove or disprove the following statement:
If a point $x \in \mathbb{R}^n$ does not belong to a convex and bounded set $C \subseteq \mathbb{R}^n$, then there exists a nonzero vector $y \in \mathbb{R}^n$ and a scalar r such that $y^T x \leq r$ and $y^T z > r$ for all $z \in C$.

Problem 4: LP under Uncertainty (30 pts)
Consider the linear (but infinitely-constrained) optimization problem

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^T x \\ \text{s.t.} & a_i^T x \leq b_i \quad \forall a_i \in U_{a_i} \quad i = 1, \ldots, m, \end{array}$$

where the sets $U_{a_i} \subseteq \mathbb{R}^n$ are defined as follows:

$$U_{a_i} := \{\bar{a}_i + P_i u \mid ||u||_2 \leq 1\}.$$

The data to this problem is a vector $c \in \mathbb{R}^n$, scalars $b_1 \ldots, b_m \in \mathbb{R}$, matrices $P_1, \ldots, P_m \in \mathbb{R}^{n \times n}$, and vectors $\bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$. Reformulate this optimization problem as a second-order cone program (SOCP). (That is, formulate an SOCP that finds the optimal value and an optimal solution to the above problem, whenever an optimal solution exists.)