**Name:**________________________________

Princeton University

# ORF 523
# Midterm Exam 1, Spring 2018

March 15, 2018, from 1:30 pm to 2:50 pm


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PLEASE DO NOT TURN THIS PAGE. START THE EXAM ONLY WHEN YOU ARE INSTRUCTED TO DO SO.

1. You are allowed a single sheet of A4 paper, double sided, hand-written or typed.
2. No electronic devices are allowed (e.g., cell phones, calculators, laptops, etc.).
3. All questions should be answered in the booklet. Please write out and sign the following pledge on the booklet: "I pledge my honor that I have not violated the honor code during this examination."
4. Each problem has 25 points. You can cite results shown in lecture or on problem sets without proof.

**Problem 1:**
Let $A$ and $B$ be two compact sets in $\mathbb{R}^n$. Show that there exists a nonzero vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that

$$a^T x - b \leq -1 \ \forall x \in A \text{ and } a^T x - b \geq 1 \ \forall x \in B,$$

if and only if the intersection of the convex hull of $A$ and the convex hull of $B$ is empty.

**Problem 2:**
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function that satisfies $f(0) = 0$ and $f(x) > 0$ for all $x \neq 0$.

(a) Prove that $x^T \nabla f(x) > 0$ for all $x \neq 0$. (Hint: you might want to consider the univariate function $g(t) = f(tx)$.)

(b) Prove that $f$ is coercive, i.e., $f(y) \to \infty$ as $||y|| \to \infty$. (Hint: use the result in part (a) with an appropriate choice for $x$.)

**Problem 3:**
An $n \times n$ real symmetric matrix $Q$ is said to be *copositive* if $x^T Q x \geq 0$ for all $x \in \mathbb{R}^n$ such that $x \geq 0$. (The inequality on $x$ is elementwise.)

(a) Prove that the set of $n \times n$ copositive matrices is convex. Show that the set of $n \times n$ noncopositive matrices is nonconvex unless $n = 1$.

(b) Give an example of a matrix that is copositive but neither positive semidefinite nor elementwise nonnegative. (You have to prove all claims about the example that you produce.)

**Problem 4:**
A real $n \times n$ matrix $Q$ is said to be *doubly stochastic* if its entries are nonnegative and its rows and columns all sum up to 1. We say that $Q$ is a *permutation* matrix if it has exactly one 1 in every row and every column and zeros everywhere else. Show that every doubly stochastic matrix is a convex combination of permutation matrices. (Hint: you can use the fact that any point in a bounded polyhedron is a convex combination of its vertices.)