Problem 1: The Lovász sandwich theorem

The Lovász sandwich theorem states that for any graph \( G(V, E) \), with \( |V| = n \), we have

\[
\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G})
\]

where

- \( \alpha(G) \) is the stability number of \( G \) (i.e., the size of its largest independent set(s)),
- \( \vartheta(G) \) is the Lovász theta number; i.e., the optimal value of the SDP

\[
\vartheta(G) := \max_{X \in S^{n \times n}} \text{Tr}(JX)
\]

s.t. \( \text{Tr}(X) = 1 \),

\[
X_{i,j} = 0, \quad \text{if} \quad \{i, j\} \in E
\]

\( X \succeq 0 \),

- \( \chi(H) \) is the coloring number of \( H \), that is the minimum number of colors needed to color the nodes of a graph \( H \) such that no two adjacent nodes get the same color, and
- \( \bar{G} \) is the complement graph of \( G \), i.e., a graph on the same node set which has an edge between two nodes if and only if \( G \) doesn’t.

1. We proved inequality (1) in class. Prove inequality (2).

   \textit{Hint:} You may want to first show that the optimal value of the following SDP also gives \( \vartheta(G) \):

\[
\min_{Z \in S^{(n+1) \times (n+1)}} Z_{n+1,n+1}
\]

s.t. \( Z_{n+1,i} = Z_{i,i} = 1, \quad i = 1, \ldots, n \)

\[
Z_{ij} = 0 \quad \text{if} \quad \{i, j\} \in \bar{E}
\]

\( Z \succeq 0 \).

2. Given an example of a graph \( G \) where neither inequality (1) nor inequality (2) is tight.
Problem 2: Comparison of LP and SDP relaxations

For a graph \( G(V, E) \), with \(|V| = n\), we saw in class that an SDP-based upperbound for the stability number \( \alpha(G) \) of the graph is given by \( \vartheta(G) \) (as defined in Problem 1). We also saw that alternative upperbounds on the stability number can be obtained through the following family of LP relaxations:

\[
\eta^k_{LP} := \max \sum_{i=1}^{n} x_i
\]

s.t. \( 0 \leq x_i \leq 1, \quad i = 1, \ldots, n \)

\( C_2, \ldots, C_k \)

where \( C_k \) contains all clique inequalities of order \( k \), i.e. the constraints

\[
x_{i_1} + \ldots + x_{i_k} \leq 1
\]

for all \( \{i_1, \ldots, i_k\} \in V \) defining a clique of size \( k \).

1. Show that for any graph \( G \), we have \( \vartheta(G) \leq \eta^k_{LP} \forall k \geq 2 \).

   \textit{Hint:} You may want to show that \( \vartheta(G) \) can also be obtained as the optimal value of the following optimization problem:

\[
\max_{Y \in S^{(n+1) \times (n+1)}} \sum_{i=1}^{n} Y_{ii}
\]

s.t. \( Y \succeq 0 \),

\( Y_{n+1,n+1} = 1 \),

\( Y_{n+1,i} = Y_{ii}, \quad i \in V \),

\( Y_{ij} = 0 \), if \((i, j) \in E\).

2. The file \texttt{Graph.mat} contains the adjacency matrix of a graph \( G \) with 50 nodes (depicted above). Compute \( \vartheta(G) \), \( \eta^2_{LP} \), \( \eta^3_{LP} \), \( \eta^4_{LP} \) and \( \alpha(G) \) for this graph. You can directly load the data file in MATLAB. In Python, you can use the following code to do this.

   \begin{verbatim}
   1 import scipy
   2 mat = scipy.io.loadmat('Graph.mat')
   3 G = mat['G']
   \end{verbatim}

3. Present a stable set of maximum size. Prove or disprove the claim that this graph has a unique maximum stable set.
Problem 3: Shannon capacity of graphs

1. Consider two graphs $G_A$ and $G_B$ (with possibly a different number of nodes) and denote their adjacency matrices by $A$ and $B$ respectively. Express the adjacency matrix of their strong graph product $G_A \otimes G_B$ in terms of $A$ and $B$.

2. Compute the Shannon capacity of the graph given in Problem 2.2.