ORF 523	PROBLEM SET 5	Spring 2024, Princeton University
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Due on April 18, 2024, at 1:30pm EST, on Gradescope		

Problem 1: The Lovász sandwich theorem

The Lovász sandwich theorem states that for any graph $G(V, E)$, with $|V| = n$, we have

$$\alpha(G) \underset{(1)}{\leq} \vartheta(G) \underset{(2)}{\leq} \chi(\bar{G})$$

where

- $\alpha(G)$ is the stability number of G (i.e., the size of its largest independent set(s)),
- $\vartheta(G)$ is the Lovász theta number; i.e., the optimal value of the SDP

$$\begin{aligned} \vartheta(G) := \max_{X \in S^{n\times n}} \; & \mathrm{Tr}(JX) \\ \text{s.t. } & \mathrm{Tr}(X) = 1, \\ & X_{i,j} = 0, \text{ if } \{i, j\} \in E \\ & X \succeq 0, \end{aligned}$$

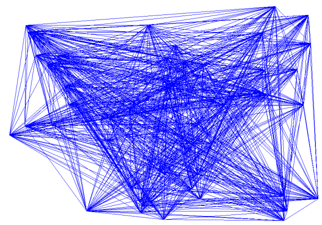
- $\chi(H)$ is the coloring number of H, that is the minimum number of colors needed to color the nodes of a graph H such that no two adjacent nodes get the same color, and
- $\bar{G}$ is the complement graph of G, i.e., a graph on the same node set which has an edge between two nodes if and only if G doesn't.

1. We proved inequality (1) in class. Prove inequality (2).
   Hint: You may want to first show that the optimal value of the following SDP also gives $\vartheta(G)$:

$$\begin{aligned} \min_{Z \in S^{(n+1)\times(n+1)}} \; & Z_{n+1,n+1} \\ \text{s.t. } & Z_{n+1,i} = Z_{ii} = 1, \; i = 1, \ldots, n \\ & Z_{ij} = 0 \text{ if } \{i, j\} \in \bar{E} \\ & Z \succeq 0. \end{aligned}$$

2. Given an example of a graph G where neither inequality (1) nor inequality (2) is tight.

Problem 2: Comparison of LP and SDP relaxations



For a graph $G(V, E)$, with $|V| = n$, we saw in class that an SDP-based upperbound for the stability number $\alpha(G)$ of the graph is given by $\vartheta(G)$ (as defined in Problem 1). We also saw that alternative upperbounds on the stability number can be obtained through the following family of LP relaxations:

$$\begin{aligned}\eta_{LP}^k := \max \sum_{i=1}^n x_i& \\ \text{s.t. } 0 \leq x_i \leq 1,\ i = 1, \ldots, n& \\ C_2 \ldots, C_k,&\end{aligned}$$

where C_k contains all clique inequalities of order k, i.e. the constraints

$$x_{i_1} + \ldots + x_{i_k} \leq 1$$

for all $\{i_1, \ldots, i_k\} \in V$ defining a clique of size k.

1. Show that for any graph G, we have $\vartheta(G) \leq \eta_{LP}^k \ \forall k \geq 2$.
   Hint: You may want to show that $\vartheta(G)$ can also be obtained as the optimal value of the following optimization problem:

$$\begin{aligned}\max_{Y \in S^{(n+1)\times(n+1)}} \sum_{i=1}^n Y_{ii}& \\ \text{s.t. } Y \succeq 0,& \\ Y_{n+1,n+1} = 1,& \\ Y_{n+1,i} = Y_{ii},\ i \in V,& \\ Y_{ij} = 0\ ,\ \text{if } (i,j) \in E.&\end{aligned}$$

2. The file `Graph.mat` contains the adjacency matrix of a graph G with 50 nodes (depicted above). Compute $\vartheta(G)$, η_{LP}^2, η_{LP}^3, η_{LP}^4 and $\alpha(G)$ for this graph. You can directly load the data file in MATLAB. In Python, you can use the following code to do this.

```
import scipy
mat = scipy.io.loadmat('Graph.mat')
G = mat['G']
```

3. Present a stable set of maximum size. Prove or disprove the claim that this graph has a unique maximum stable set.

Problem 3: Shannon capacity of graphs

1. Consider two graphs G_A and G_B (with possibly a different number of nodes) and denote their adjacency matrices by A and B respectively. Express the adjacency matrix of their strong graph product $G_A \otimes G_B$ in terms of A and B.

2. Compute the Shannon capacity of the graph given in Problem 2.2.