ECO 519. Homework #2
(Due: First Wednesday after Spring Break)

1. Suppose $y_i = x_i'\beta_0 + \varepsilon_i$ where ε_i is independent of x_i and ε_i is continuously distributed with strictly increasing CDF $F_\varepsilon(\epsilon)$ and corresponding density function $f_\varepsilon(\epsilon)$. Suppose also that the τ^{th} quantile of ε_i is zero, i.e: $F_\varepsilon(0) = \tau$ and $f_\varepsilon(0) > 0$.

   (a) For $\tau \in (0,1)$ define $\rho_\tau(z) = z(\tau - \mathbb{1}\{z < 0\})$. Show that if the previous assumptions are satisfied, then

   $$\underset{\beta}{\text{argmin}}\; E\big[\rho_\tau(y_i - x_i'\beta)\big|x_i\big] = \beta_0$$

   (b) Suppose we observe an iid sample $(y_i, x_i)_{i=1}^n$ (no truncation or censoring). Based on the previous part, propose an estimator for β and sketch any additional conditions needed for consistency and asymptotic normality. Characterize precisely the asymptotic distribution of the proposed estimator if all your assumptions are satisfied. Show that the LAD estimator is a special case. Hint: for asymptotic normality, focus on Huber (1967).

2. Assume the exact same setup of the Censored LAD estimator. Consider a real-valued function $g(\cdot)$ that satisfies: (i) $g(z)$ is strictly increasing for all $z \geq 0$, (ii) $g(0) = 0$, $g'(0) = 0$ and $g''(0) = 0$.

   (a) Show that under the censored-LAD assumptions,

   $$\text{median}\big(g(y_i)\big|x_i\big) = g\big(\max\{0, x_i'\beta_0\}\big)$$

   (b) Consider an estimator $\widetilde{\beta}$ such that

   $$\widetilde{\beta} = \underset{\beta}{\text{argmin}}\; \frac{1}{n}\sum_{i=1}^{n}\Big|g(y_i) - g\big(\max\{0, x_i'\beta\}\big)\Big|$$

   characterize the asymptotic properties of $\widetilde{\theta}$, making any necessary modification to Powell's censored LAD assumptions. *Which of Powell's assumptions can be relaxed now?* (Hint: It's a nontrivial one).

3. (a) Consider some abstract class of estimators indexed by $\tau \in \mathbb{T}$ such that each member of this class has an asymptotically linear representation of the form

$$\widehat{\theta}(\tau) = \theta_0 + \frac{1}{n}\sum_{i=1}^{n} \psi_i(\tau) + o_p(n^{-1/2})$$

where θ_0 denotes the "true" parameter value (whatever this means in this abstract setting) and the influence function satisfies

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \psi_i(\tau) \xrightarrow{d} \mathcal{N}(0, V(\tau)) \quad \text{for each } \tau \in \mathbb{T}$$

show that if there exists a $\overline{\tau} \in \mathbb{T}$ such that $E\big[(\psi_i(\tau) - \psi_i(\overline{\tau}))\psi_i(\overline{\tau})'\big] = 0$ for all $\tau \in \mathbb{T}$, then $\widehat{\theta}(\overline{\tau})$ is asymptotically efficient in the class $\{\widehat{\theta}(\tau) : \tau \in \mathbb{T}\}$.

(b) Suppose there exists a function ϕ such that $E\big[\phi(z, \theta_0)\big|x\big] = 0$. Assume that $\theta \in \mathbb{R}^k$ and $\phi \in \mathbb{R}^r$. Consider a family of estimators $\widehat{\theta}(\tau)$ all of which are asymptotically linear with influence functions of the form

$$\underbrace{\psi_i(\tau)}_{k\times 1} = \underbrace{E\big[\tau(x)\nabla_\theta \phi(z, \theta_0)\big]}_{k\times k} \tau(x_i)\phi(z_i, \theta_0)$$

where $\tau(x)$ is a $k \times r$ matrix. Apply Theorem 5.3 in Newey and McFadden to find the τ that yields the smallest variance within this class of estimators. What is its asymptotic variance?

4. (a) Let $\widehat{\gamma} = \gamma_0 + \frac{1}{n}\sum_{i=1}^{n}\psi_i + o_p(n^{-1/2})$, where $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_i \xrightarrow{d} \mathcal{N}(0, \Sigma)$, $\gamma \in \mathbb{R}^p$ and Σ is a $p \times p$ matrix. Now consider an M-estimator $\widehat{\theta}$ given by:

$$\widehat{\theta} = \underset{\theta\in\Theta}{\operatorname{argmin}}\ \frac{1}{n}\sum_{i=1}^{n} q(z_i, \theta, \widehat{\gamma})$$

Assume that $E[z, \theta, \gamma]$ is uniquely minimized at (θ_0, γ_0). Provide conditions under which $\widehat{\theta}$ is $\sqrt{n}$-consistent, asymptotically normal. Characterize its asymptotic variance (assume q is $\mathcal{C}^2$).

(b) Consider the CMD estimator $\widehat{\theta}$ given by:

$$\widehat{\theta} = \underset{\theta\in\Theta}{\operatorname{argmin}}\ \big(\widehat{\pi} - h(\theta)\big)'\widehat{W}\big(\widehat{\pi} - h(\theta)\big)$$

where $\widehat{\pi} = \pi_0 + \frac{1}{n}\sum_{i=1}^{n}\psi_i + o_p(n^{-1/2})$, and $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_i \xrightarrow{d} \mathcal{N}(0,\Sigma)$. Let $W = \text{plim}\,\widehat{W}$ and suppose that $\pi_0 = h(\theta_0)$ (by definition of CMD). Provide conditions under which $\widehat{\theta}$ is $\sqrt{n}$-consistent, asymptotically normal. Characterize its asymptotic variance (assume h is $\mathcal{C}^2$).