

Department of Economics
Princeton University
S500, First Half.
Homework 1

1. Using mathematical induction, show the following:

- (a) $\sum_{j=1}^n j^2 = (2n^3 + 3n^2 + n)/6$ for all $n \in \mathbb{N}$.
- (b) Take a set X with n elements. Show that its power set 2^X has 2^n elements (Hint: Recall that the empty set \emptyset is always a subset of any set).
- (c) All numbers of the form $7^n - 2^n$, with $n \in \mathbb{N}$ are divisible by 5.
- (d) $|\sin nx| \leq n|\sin x|$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ (Hint: Recall the trigonometric identity $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ and the triangle inequality.)

2. Consider the following real-valued function:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

- (a) Show that (\mathbb{R}, d) is a metric space.
 - (b) Characterize an open ball in (\mathbb{R}, d) .
 - (c) Find the set of interior points and the set of boundary points for \mathbb{N} and \mathbb{Q} (each separately) using this metric space. Compare with the answer you would obtain if you had used the metric space $(\mathbb{R}, |\cdot|)$ (i.e, the absolute value metric) .
3. (a) Suppose (S_1, d_1) and (S_2, d_2) are metric spaces. Show that $(S_1 \times S_2, d)$ is a metric space, where

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

for all $x_1, y_1 \in S_1$ and all $x_2, y_2 \in S_2$.

- (b) Consider the set X of all real-valued bounded sequences $s = (s_1, s_2, s_3, \dots)$, and let $d(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \dots\}$ for any pair of sequences x and y in X . Show that (X, d) is a metric space.

4. Use the results for a sequence s_n such that $\lim_{n \rightarrow \infty} |s_{n+1}/s_n|$ exists to solve the following:
- Prove the following:
 - $\lim_{n \rightarrow \infty} \left[\frac{2^n}{n^2} + (-1)^n \right] = +\infty$
 - $\lim_{n \rightarrow \infty} \left[\frac{3^n}{n^3} - \frac{3^n}{n!} \right] = +\infty$
 - Use mathematical induction to show that $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for all $a \neq 1$
 - Find $\lim_{n \rightarrow \infty} (1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.
5. (a) Let $f(x) = \sqrt{4-x}$ for $x \leq 4$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.
- Are the composite functions $g \circ f$ and $f \circ g$ equal?
 - Characterize the domains for $f + g$, fg , $g \circ f$ and $f \circ g$.
- (b) Let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$ and $g(x) = 9 - x^2$ for all $x \in \mathbb{R}$.
- Characterize the functions $f + g$, fg , $g \circ f$ and $f \circ g$.
 - Which of the functions $f + g$, fg , $g \circ f$ and $f \circ g$ is continuous?
- (c) Show that the absolute value function $|x|$ is continuous for all $x \in \mathbb{R}$.
- (d) Let f and g be two real-valued functions.
- Show that $\min\{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.
 - Show that if f and g are continuous at a point $x \in \mathbb{R}$, then $\min\{f, g\}$ is also continuous at x .
- (e) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as: $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Show that f is continuous at $x = 0$ and is discontinuous at all other points in \mathbb{R} .
6. Let c_1, c_2, \dots, c_n be positive real numbers, and let $u = (u_1, u_2, \dots, u_n)'$ and $v = (v_1, v_2, \dots, v_n)'$ be two vectors in \mathbb{R}^n .

- (a) Show that the function

$$\langle u, v \rangle = c_1(u_1v_1) + c_2(u_2v_2) + \cdots + c_n(u_nv_n)$$

is a well-defined inner product in \mathbb{R}^n .

- (b) Characterize the norm and metric functions induced by this inner product.

7. Prove that

(a) $\sum_{k=1}^n \frac{1}{\sqrt{k}} \geq \sqrt{n}$.

(b) $\sum_{n=m}^{\infty} (k-1)k^{-n} = k^{1-m}$.

8. In any inner product space \mathcal{V} and $\forall x, y \in \mathcal{V}$

(a) Prove the Schwarz inequality $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$, equality holds iff $x = \lambda y$.

(b) Can $\langle x, x \rangle^{\frac{1}{2}}$ be used as a norm ? Prove your claim.

(c) Show that the inner product $\langle x, y \rangle$ is jointly continuous, i.e. continuous in x for fixed y , and vice versa.