Department of Economics

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ECO S500 (First Half). Fall 2005.

Suggested Answers for Homework 1

1. (a) Take $n = 1$. Then $\sum_{j=1}^{n} j^2 = 1 = \big(2n^3 + 3n^2 + n\big)/6 = 6/6$. This establishes the <u>basis for induction</u>. Now for an arbitrary n suppose $\sum_{j=1}^{n} j^2 = \big(2n^3 + 3n^2 + n\big)/6$. Then

$$\sum_{j=1}^{n+1} j^2 = \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2 = \frac{2n^3 + 9n^2 + 13n + 6}{6} = \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6},$$

which establishes the <u>induction step</u>, and we're done.

(b) Take any finite set X. Suppose its power set, 2^X has N elements. Suppose we construct a new set X' that consists of the union of X and a new element x'. That is: $X' = X \cup \{x'\}$. Suppose we index the elements of the original power set 2^X as $\{z_1, z_2, \ldots, z_N\}$, where each z_i is a subset of X. The power set of X' will now consist of:

$$2^{X'} = 2^X \cup \big\{\{z_1 \cup x'\}, \{z_2 \cup x'\}, \ldots, \{z_N, x'\}\big\}.$$

The cardinality of $2^{X'}$ is therefore $2 \cdot N$. Therefore, if a set X with n elements has a power set 2^X with 2^n elements, any set X' with $n + 1$ elements will have a power set $2^{X'}$ with $2 \cdot 2^n = 2^{n+1}$ elements. This establishes the <u>induction step</u> of the proof. Now consider a set X that consists of one element, $\{x\}$ (i.e, $n = 1$). The power set of X is given by $2^X = \{\emptyset, x\}$, which has 2^1 elements. This establishes the <u>basis for induction</u>, and we're done.

(c) The <u>basis for induction</u> is immediate: $7^1 - 2^1 = 5$. Suppose $7^n - 2^n$ is divisible by 5. That is: $7^n - 2^n = k \cdot 5$ for some $k \in \mathbb{N}$. We have

$$7^{n+1} - 2^{n+1} = 7^{n+1} - 2^{n+1} + 2 \cdot 7^n - 2 \cdot 7^n = 2[7^n - 2^n] - 7^n[7 - 2] = 2 \cdot (k \cdot 5) - 7^n \cdot 5$$
$$= (2 \cdot k - 7^n) \cdot 5 \equiv k' \cdot 5; \quad \text{where } k' = 2 \cdot k - 7^n \in \mathbb{N}.$$

Therefore $7^{n+1} - 2^{n+1}$ is divisible by 5. This proves the <u>induction step</u> and we're done.

(d) Take $n = 1$. Then trivially $|\sin 1 \cdot x| = 1 \cdot |\sin x|$, which establishes the <u>basis for induction</u>. Now assume that $|\sin n \cdot x| \leq n \cdot |\sin x|$. Using the trigonometric identity mentioned in the statement of the problem along with the triangle inequality satisfied by the absolute-value function, we have

$$\begin{aligned}\big|\sin(n+1)x\big| &= \big|\sin(nx+x)\big| = \big|\sin n\cdot x\cos x + \cos n\cdot x\sin x\big| \le \big|\sin n\cdot x\cos x\big| + \big|\cos n\cdot x\sin x\big|\\ &\le \big|\sin n\cdot x\big| + \big|\sin x\big| \le n\cdot|\sin x| + \big|\sin x\big| = (n+1)\cdot|\sin x|\end{aligned}$$

where the last inequality incorporates the assumption $|\sin n\cdot x| \le n\cdot|\sin x|$. This establishes the induction step and we're done.

2. (a) We first verify if d satisfies nonnegativity and nondegeneracy: Trivially, by construction $d(x,y)\ge 0$ and $d(x,y)=0$ if and only if $x=y$ ✓. Next, we check symmetry: $d(y,x)=1$ if $y\ne x$ and $d(y,x)=0$ if $y=x$. Easily, $d(y,x)=d(x,y)$ ✓. Next, we verify the triangle inequality: Take any triple $x,y,z\in\mathbb{R}$. Suppose $x\ne z$. Then $d(x,z)=1$. Can we have $d(x,y)=0$ and $d(z,y)=0$? The answer is no: Suppose $d(x,y)=0$. Then $x=y$ and since $x\ne z$, we must have $y\ne z$ and consequently $d(z,y)=1$. Therefore if $d(x,z)=1$, we must have $d(x,y)+d(z,y)\ge d(x,z)=1$. If $d(x,z)=0$, we trivially have $d(x,y)+d(z,y)\ge d(x,z)=0$, since $\min\{d(x,y),d(z,y)\}\ge 0$. Therefore for any $x,y,z\in\mathbb{R}$, we must have $d(x,z)\le d(x,y)+d(z,y)$ ✓. Consequently, d is a well-defined metric on $\mathbb{R}$.

(b,c) Consider the metric space $(\mathbb{R},d)$, where d is the discrete metric. To characterize interior, boundary points, open and closed sets, all we really need is to see what an open ball of radius ε looks like. We have:

$$\forall\, x\in\mathbb{R}:\ B_\varepsilon(x) = \begin{cases} x & \text{if } \varepsilon\le 1\\ \mathbb{R} & \text{if } \varepsilon>1\end{cases}$$

Take any subset $\mathbb{X}\subset\mathbb{R}$

$$\begin{aligned}\operatorname{int}\mathbb{X} &= \big\{x\in\mathbb{R}:\ \exists\,\varepsilon>0 \text{ such that } B_\varepsilon(x)\subset\mathbb{X}\big\};\\ \partial\mathbb{X} &= \big\{x\in\mathbb{R}:\ \forall\,\varepsilon>0,\ B_\varepsilon(x)\cap\mathbb{X}\ne\emptyset,\ \text{and } B_\varepsilon(x)\cap\mathbb{R}\setminus\mathbb{X}\ne\emptyset\big\}\end{aligned}$$

This definition applies for either $\mathbb{X}=\mathbb{N}$ or $\mathbb{X}=\mathbb{Q}$. Take any $x\in\mathbb{R}\setminus N$. Then $\nexists\varepsilon>0$ such that $B_\varepsilon(x)\subset\mathbb{N}$. Therefore, no point outside of $\mathbb{N}$ can belong in int $\mathbb{N}$. Now, take any $x\in\mathbb{N}$. Then, if $\varepsilon\le 1$, $B_\varepsilon(x)=x\in\mathbb{N}$. Since this holds for any arbitrary $x\in\mathbb{N}$, we have int $\mathbb{N}=\mathbb{N}$. Replicating the same argument, we get int $\mathbb{Q}=\mathbb{Q}$. As we noted above, if $x\in\mathbb{R}\setminus\mathbb{N}$, then for any $\varepsilon\le 1$: $B_\varepsilon(x)=x\notin\mathbb{N}$. Consequently, no point outside of $\mathbb{N}$

can belong in $\partial\mathbb{N}$. On the other hand, if $x \in \mathbb{N}$ and $\varepsilon \leq 1$, then $B_\varepsilon(x) = x \notin \mathbb{R} \setminus \mathbb{N}$. This shows that $\nexists\, x \in \mathbb{R}$ such that $B_\varepsilon(x) \cap \mathbb{N} \neq \emptyset$ and $B_\varepsilon(x) \cap \mathbb{R} \setminus \mathbb{N} \neq \emptyset$ for all $\varepsilon > 0$. Consequently, $\underline{\partial\mathbb{N} = \emptyset}$. A replication of the same argument shows that $\underline{\partial\mathbb{Q} = \emptyset}$. From these observations, we get

$$\text{cl } \mathbb{N} = \text{int } \mathbb{N} \cup \partial\mathbb{N} = \mathbb{N}; \quad \text{ext } \mathbb{N} = \mathbb{R} \setminus \text{cl } \mathbb{N} = \mathbb{R} \setminus \mathbb{N}$$

$$\text{cl } \mathbb{Q} = \text{int } \mathbb{Q} \cup \partial\mathbb{Q} = \mathbb{Q}; \quad \text{ext } \mathbb{Q} = \mathbb{R} \setminus \text{cl } \mathbb{Q} = \mathbb{R} \setminus \mathbb{Q}.$$

Next we consider the absolute-value metric and the corresponding metric space $(\mathbb{R}, |\cdot|)$. An open ball $B_\varepsilon(x)$ is now simply an open interval of length ε centered at x:

$$B_\varepsilon(x) = \{x' \in \mathbb{R} : \ |x' - x| < \varepsilon\}.$$

Take any $x \in \mathbb{R}$. Then $\nexists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{N}$. Consequently, $\underline{\text{int } \mathbb{N} = \emptyset}$. From our previous knowledge about the field $\mathbb{Q}$, we know that every open set in $\mathbb{R}$ includes irrational numbers (and rational numbers too, but this is irrelevant in this case). Consequently, if we take any arbitrary $x \in \mathbb{R}$, $\nexists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathbb{Q}$. Therefore, $\underline{\text{int } \mathbb{Q} = \emptyset}$. Now take any $x \in \mathbb{N}$. Then for any $\varepsilon > 0$, $B_\varepsilon(x) \cap \mathbb{R} \setminus \mathbb{N} \neq \emptyset$. Trivially, we also have $B_\varepsilon(x) \cap \mathbb{N} \neq \emptyset$ (since $x \in B_\varepsilon(x)$ and $x \in \mathbb{N}$). This is not true however if $x \in \mathbb{R} \setminus \mathbb{N}$. Therefore $\underline{\partial\mathbb{N} = \mathbb{N}.}$ From our previous knowledge about $\mathbb{Q}$, we know that any open set in $\mathbb{R}$ includes irrational AND rational numbers (recall the denseness property of $\mathbb{Q}$). This means that for all $x \in \mathbb{R}$ and for all $\varepsilon > 0$, $B_\varepsilon(x) \cap \mathbb{Q} \neq \emptyset$ and $B_\varepsilon(x) \cap \mathbb{R} \setminus \mathbb{Q} \neq \emptyset$. Therefore $\underline{\partial\mathbb{Q} = \mathbb{R}.}$ Combining these results we have

$$\text{cl } \mathbb{N} = \text{int } \mathbb{N} \cup \partial\mathbb{N} = \mathbb{N}; \quad \text{ext } \mathbb{N} = \mathbb{R} \setminus \text{cl } \mathbb{N} = \mathbb{R} \setminus \mathbb{N}$$

$$\text{cl } \mathbb{Q} = \text{int } \mathbb{Q} \cup \partial\mathbb{Q} = \mathbb{R}; \quad \text{ext } \mathbb{Q} = \mathbb{R} \setminus \text{cl } \mathbb{Q} = \emptyset.$$

3. (a) Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, where $x_1, y_1 \in S_1$ and $x_2, y_2 \in S_2$. The candidate metric d satisfies:

$$d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

We first verify symmetry: Since d_1 and d_2 are well-defined metrics, we have $d_i(y_i, x_i) = d_i(x_i, y_i)$ for $i = 1, 2$. Therefore

$$d(y, x) = \max\{d_1(y_1, x_1), d_2(y_2, x_2)\} = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = d(x, y),$$

which establishes symmetry ✓. Nonnegativity and nondegeneracy: $d_i(x_i, y_i) \geq 0$ for $i = 1, 2$. Consequently, $d(x, y) \geq \max\{0, 0\} \geq 0$ ✓. Now suppose $d(x, y) = 0$.

Then $\max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = 0$, which can only be true if both $d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$, which in turn implies immediately that $x_1 = y_1$ and $x_2 = y_2$, or equivalently: $x = y$. Now suppose $x = y$. Then $x_1 = y_1$ and $x_2 = y_2$ and we must have $d_1(x_1, y_1) = d_2(x_2, y_2) = 0$. Therefore $\max\{d_1(x_1, y_1), d_2(x_2, y_2)\} = 0 = d(x, y)$. This establishes nondegeneracy of d ✓. Finally, the <u>triangle inequality:</u> Since $d_i(x_i, y_i)$ is a well-defined metric for $i = 1, 2$, we must have

$$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \quad \text{for any } x_1, y_1, z_1 \in S_1.$$
$$d_2(x_2, z_2) \leq d_2(x_2, y_2) + d_2(y_2, z_2) \quad \text{for any } x_2, y_2, z_2 \in S_2.$$

Then,

$$\begin{aligned} d(x, z) = \max\big\{d_1(x_1, z_1), d_2(x_2, z_2)\big\} &\leq \max\big\{d_1(x_1, y_1) + d_1(y_1, z_1), d_2(x_2, y_2) + d_2(y_2, z_2)\big\} \\ &\leq \max\big\{d_1(x_1, y_1), d_2(x_2, y_2)\big\} + \max\big\{d_1(y_1, z_1), d_2(y_2, z_2)\big\} \\ &= d(x, y) + d(y, z) \ \checkmark \end{aligned}$$

These results combined show that $d(x, y)$ is a well-defined metric on $S_1 \times S_2$.

(b) Take any pair of bounded sequences x and y. The candidate metric is

$$d(x, y) = \sup\big\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \ldots\big\}.$$

Equivalently, $d(x, y) = \sup_{\mathbb{N}}\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}}$. We first evaluate <u>symmetry</u>: By the properties of $|\cdot|$, we know that the sequences $\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}}$ and $\big\{|y_n - x_n|\big\}_{n\in\mathbb{N}}$ are equal to each other for all n. Therefore $\sup_{\mathbb{N}}\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}} = \sup_{\mathbb{N}}\big\{|y_n - x_n|\big\}_{n\in\mathbb{N}} \Rightarrow d(x, y) = d(y, x)$ ✓. We now verify <u>nonnegativity and nondegeneracy</u>: Take any pair of sequences x and y. By the properties of $|\cdot|$, we know that $\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}} \geq 0$ for all n. The definition of supremum therefore yields $\sup_{\mathbb{N}}\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}} \not< 0$ or equivalently, $d(x, y) \geq 0$ ✓. Now suppose $x = y$. This means that $x_n = y_n \ \forall\, n \in \mathbb{N} \Rightarrow |x_n - y_n| = 0$ for all n. In this case, we have $\sup_{\mathbb{N}}\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}} = \sup_{\mathbb{N}}\big\{0\big\}_{n\in\mathbb{N}} = 0 \therefore x = y \Rightarrow d(x, y) = 0$. Now suppose $d(x, y) = 0$. This is true only if $\sup_{\mathbb{N}}\big\{|x_n - y_n|\big\}_{n\in\mathbb{N}} = 0$. This is true only if $|x_n - y_n| \leq 0 \ \forall\, n \in \mathbb{N}$. At the same time, we know that $|x_n - y_n| \geq 0 \ \forall\, n \in \mathbb{N}$. These two conditions can be satisfied at the same time only if $x_n = y_n \ \forall\, n \in \mathbb{N}$ or equivalently $x = y$. Then $d(x, y) = 0 \Rightarrow x = y$ ✓. Now, the <u>triangle inequality</u>: We know that the absolute-value function $|\cdot|$ satisfies the triangle inequality. Take any triplet of bounded sequences x, y, z. Then

$$|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \quad \forall\, n \in \mathbb{N}.$$

Since all three sequences are bounded, we know that for all $n \in \mathbb{N}$: $|x_n - y_n| \leq M_1$ and $|y_n - z_n| \leq M_2$ for some positive constants M_1, M_2. Consequently,

$$\sup_{\mathbb{N}}\{|x_n - z_n|\} \leq \sup_{\mathbb{N}}\{|x_n - y_n| + |y_n - z_n|\} \leq M_1 + M_2 = \sup_{\mathbb{N}}\{|x_n - y_n|\} + \sup_{\mathbb{N}}\{|y_n - z_n|\},$$

In other words, $d(x, z) \leq d(x, y) + d(y, z)$ ✓.

4. (a) Let $s_n = 2^n/n^2$ and $t_n = 3^n/n!$. Then

$$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty} 2 \cdot \left[\frac{n}{n+1}\right]^2 = 2; \quad \lim_{n\to\infty}\left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Consequently, $\lim_{n\to\infty}|s_n| = \lim_{n\to\infty} s_n = +\infty$, and $\lim_{n\to\infty}|t_n| = \lim_{n\to\infty} t_n = 0$. The results follow from these facts, and the fact that $(-1)^n$ is a bounded sequence for all n.

(b) Let $n = 1$. Then $(1 - a^2)/(1 - a) = 1 + a$, which establishes the basis for induction. Now suppose $1 + a + \cdots + a^n = (1 - a^{n+1})/(1 - a)$. Then

$$1 + a + \cdots + a^n + a^{n+1} = \frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a} = \frac{1 - a^{n+2}}{1 - a},$$

which establishes the induction step, and we're done. If $|a| < 1$, then $\lim_{n\to} a^n \to 0$. Therefore, if $|a| < 1$, $\lim_{n\to\infty}\big(1 + a + \cdots + a^n\big) = 1/(1 - a)$.

5. (a) We have

$$(g \circ f)(x) = 4 - x,\ x \in (-\infty, 4]; \quad (f \circ g)(x) = \sqrt{4 - x^2},\ x \in [-2, 2];$$
$$(f + g)(x) = x^2 + \sqrt{4 - x},\ x \in (-\infty, 4]; \quad (f \cdot g)(x) = x^2\sqrt{4 - x},\ x \in (-\infty, 4].$$

We have

$$\text{dom }(g\circ f) = (-\infty, 4]; \quad \text{dom }(f\circ g) = [-2, 2]; \quad \text{dom }(g+f) = (-\infty, 4]; \quad \text{dom }(g\cdot f) = (-\infty, 4]$$

Notice that dom $(g \circ f) \neq (-\infty, \infty)$ even though the composition looks simply like $4 - x$. The reason is that if $x > 4$, then $f(x)$ is not well-defined, and $(g \circ f)(x)$ is not defined either. All this becomes clear when we realize that the domain of $(g \circ f)$ is given by

$$\text{dom }(g \circ f) = \big\{x :\ x \in \text{dom }(f);\ \text{and } f(x) \in \text{dom }(g)\big\}$$

(b) We now have

$$(f + g)(x) = \begin{cases} 13 - x^2 & \text{if } x \geq 0. \\ 9 - x^2 & \text{if } x < 0. \end{cases}; \quad (f \cdot g)(x) = \begin{cases} 4(9 - x^2) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

and

$$(f \circ g)(x) = \begin{cases} 4 & \text{if } 9 - x^2 \geq 0 \text{ (i.e, if } |x| \leq 3) \\ 0 & \text{if } 9 - x^2 < 0 \text{ (i.e, if } |x| > 3) \end{cases} \quad ; \quad (g \circ f)(x) = \begin{cases} -7 & \text{if } x \geq 0 \\ 9 & \text{if } x < 0 \end{cases}$$

The functions $(f+g)$, $(f \cdot g)$ and $(g \circ f)$ are discontinuous at $x = 0$ and continuous everywhere else. The function $(f \circ g)$ is discontinuous at $x = -3$ and $x = 3$.

(c) We have to show that $|x|$ is a continuous function for all $x \in \mathbb{R}$. We first prove the following claim: "Take any pair $x, x_0 \in \mathbb{R}$. Then $\big||x| - |x_0|\big| \leq \big|x - x_0\big|$". Proof: Using the triangle inequality property of $|\cdot|$, we have

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| \Rightarrow |x| - |x_0| \leq |x - x_0|;$$
$$|x_0| = |x_0 + x - x| \leq |x - x_0| + |x| \Rightarrow |x| - |x_0| \geq -|x - x_0|$$

where we exploit the fact that $|x - x_0| = |x_0 - x|$. If we combine these results, we obtain $-|x - x_0| \leq |x| - |x_0| \leq |x - x_0|$ or equivalently: $\big||x| - |x_0|\big| \leq |x - x_0|$ as claimed. Now we are ready to show that $|x|$ is a continuous function. Take any $x_0 \in \mathbb{R}$ and any $\varepsilon > 0$. Let $\delta = \varepsilon/2 > 0$. Suppose $|x - x_0| < \delta$. Then, the result we just proved yields:

$$\big||x| - |x_0|\big| \leq |x - x_0| < \frac{\varepsilon}{2} < \varepsilon$$

Consequently, for all $x_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $\big||x| - |x_0|\big| < \varepsilon$, and therefore $|x|$ is continuous everywhere in $\mathbb{R}$.

(d) First we show that in fact $\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f - g|$. Take any $x \in \text{dom}\ (f) \cap \text{dom}\ (g)$. Then:

$$\text{If } f(x) \leq g(x), \Rightarrow \frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(x)-g(x)) = f(x).$$
$$\text{If } f(x) \geq g(x), \Rightarrow \frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)| = \frac{1}{2}(f(x)+g(x)) - \frac{1}{2}(f(x)-g(x)) = g(x).$$

Therefore, $\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ as claimed. Now we show that if f and g are continuous at a point $x_0 \in \text{dom}\ (f) \cap \text{dom}\ (g)$, then $\min\{f, g\}$ is also continuous there. Using the result we showed in the proof of continuity of the function $|x|$ (above) and the triangle inequality, we have that for any $x, x_0 \in \text{dom }(f) \cap \text{dom }(g)$:

$$\begin{aligned}
&\big|\min\{f(x), g(x)\} - \min\{f(x_0), g(x_0)\}\big| \\
&\qquad = \frac{1}{2}\Big|f(x) + g(x) - \big|f(x) - g(x)\big| - \Big(f(x_0) + g(x_0) - \big|f(x_0) - g(x_0)\big|\Big)\Big| \\
&\qquad \leq \frac{1}{2}\big|f(x) - f(x_0)\big| + \frac{1}{2}\big|g(x) - g(x_0)\big| + \frac{1}{2}\Big|\big|f(x) - f(x_0)\big| - \big|f(x_0) - g(x_0)\big|\Big| \\
&\qquad \leq \frac{1}{2}\big|f(x) - f(x_0)\big| + \frac{1}{2}\big|g(x) - g(x_0)\big| + \frac{1}{2}\Big|f(x) - f(x_0) - \big(f(x_0) - g(x_0)\big)\Big| \\
&\qquad \leq \big|f(x) - f(x_0)\big| + \big|g(x) - g(x_0)\big|.
\end{aligned}$$

Take any $\varepsilon > 0$. Since f and g are continuous at x_0, there exist δ_f and δ_g such that: $|x - x_0| < \delta_f \Rightarrow |f(x) - f(x_0)| < \varepsilon/2$, and $|x - x_0| < \delta_g \Rightarrow |g(x) - g(x_0)| < \varepsilon/2$. Let $\delta = \min\{\delta_f, \delta_g\}$. Then using the result above, we get

$$|x - x_0| < \delta \Rightarrow \big|\min\{f(x), g(x)\} - \min\{f(x_0), g(x_0)\}\big| \leq \big|f(x) - f(x_0)\big| + \big|g(x) - g(x_0)\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and therefore $\min\{f, g\}$ is continuous at x_0.

(e) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by: $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Take any $x_0 \neq 0$ and take <u>any</u> $\delta > 0$. Then for any such δ, there exists x such that $|x - x_0| < \delta$ and $|f(x) - f(x_0)| = |x_0|$. Consequently, f is discontinuous at any $x_0 \neq 0$ (for any $\varepsilon \leq |x_0|$, $\nexists\, \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon \leq |x_0|$ for all x such that $|x - x_0| < \delta$). Now take $x_0 = 0$. We know that $|f(x) - f(x_0)| \leq |x|$ for any $x \in \mathbb{R}$. Take any $\varepsilon > 0$. Let $\delta = \varepsilon/2$. Then $|x| < \delta$ implies that $|f(x) - f(x_0)| \leq |x| < \varepsilon$. Therefore f is continuous at $x_0 = 0$ and only there.

6. (a) We will determine if the candidate inner product satisfies all the requirements. <u>Symmetry:</u> Take any pair of vectors $x, y \in \mathbb{R}^n$. Then $\langle y, x\rangle = c_1 \cdot (y_1 x_1) + \cdots + c_n \cdot (y_n x_n) = c_1 \cdot (x_1 y_1) + \cdots + c_n \cdot (x_n y_n) = \langle x, y\rangle$ ✓. Next, we explore <u>nonnegativity and non-degeneracy:</u> For any $x \in \mathbb{R}^n$, we have $\langle x, x\rangle = c_1 x_1^2 + \cdots + c_n x_n^2 \geq 0$ (since the constants $c_1, \dots c_n$ are positive). Therefore, $\langle x, x\rangle = 0$ if and only if $x_1 = \cdots = x_n = 0$ ✓. Now we verify <u>linearity</u>: take any pair of scalars α, β and any triplet of vectors x, y, z. Then

$$\begin{aligned}
\langle \alpha \cdot x + \beta \cdot y, z\rangle &= c_1\big((\alpha x_1 + \beta y_1) \cdot z_1\big) + \cdots + c_n\big((\alpha x_n + \beta y_n) \cdot z_n\big) \\
&= \alpha\big[c_1(x_1 z_1) + \cdots + c_n(x_n z_n)\big] + \beta\big[c_1(y_1 z_1) + \cdots + c_n(y_n z_n)\big] \\
&= \alpha\langle x, z\rangle + \beta\langle y, z\rangle \ ✓
\end{aligned}$$

(b) The corresponding norm $\|\cdot\|$ and metric $d(\cdot,\cdot)$ functions are:

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{c_1 x_1^2 + \cdots + c_n x_n^2}; \quad d(x,y) = \|x - y\| = \sqrt{c_1(x_1 - y_1)^2 + \cdots + c_n(x_n - y_n)^2}$$

7. (a) Hint: Use mathematical inductions.

   (b) Hint: You can prove that $\sum_{n=0}^{\infty} k^{-n} = \frac{k}{k-1}$

8. (a) $\forall \alpha \in \mathrm{R}$

$$0 \leq \langle x - \alpha y, x - \alpha y\rangle = \langle x, x\rangle - \alpha\langle y, x\rangle - \bar{\alpha}\langle x, y\rangle + |\alpha|^2\langle y, y\rangle.$$

   In particular, if

$$\alpha = \langle x, y\rangle / \langle y, y\rangle$$

   we have

$$0 \leq \langle x, x\rangle - |\langle x, x\rangle - |\langle x, y\rangle|^2 / \langle y, y\rangle$$

   or

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \qquad \blacksquare$$

   (b) To show that $\langle x, x\rangle^{\frac{1}{2}}$ is a norm, we need to prove that it satisfies the triangle inequality. Indeed we have :

$$\begin{aligned} 0 \leq \| x + y \|^2 &= \langle x + y, x + y\rangle \\ &= \| x \|^2 + 2\mathrm{Re}\langle x, y\rangle + \| y \|^2 \\ &\leq \| x \|^2 + 2|\langle x, y\rangle| + \| y \|^2 \\ &\leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \text{ (using Schwarz inequality)} \\ &= (\| x \| + \| y \|)^2 \qquad \blacksquare \end{aligned}$$

   (c) Let f(x) = $\langle x, y\rangle$

   f(.) is a continuous function iff $\forall \epsilon > 0, \exists \eta > 0$ such that

$$\forall z \in \mathcal{V} \; \| z - x \| \leq \eta \text{ we have } |f(z) - f(x)| \leq \epsilon$$

   Indeed

$$\begin{aligned} |f(z) - f(x)| &= |\langle z, y\rangle - \langle x, y\rangle| \\ &= |\langle z - x, y\rangle| \\ &\leq \| z - x \| \| y \| \text{ (using Schwarz inequality)} \end{aligned}$$

   So if $\eta = \frac{\epsilon}{\|y\|}$ then

$$|f(z) - f(x)| \leq \| z - x \| \| y \| \leq \eta \| y \| = \epsilon \qquad \blacksquare$$