Department of Economics

Princeton University

S500, First Half.

Homework 2

1. The purpose of this problem is to show that all norms $\|\cdot\|$ are Lipschitz-equivalent in $\mathbb{R}^n$ through the following steps:

   (a) Let (X, d) be a metric space. Use the properties of the metric function d to show that $d(x, y) \geq |d(x, z) - d(z, y)|$ for all $x, y, z \in X$.

   (b) Let $\|\cdot\| \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a norm. Prove that $\|\cdot\|$ is a continuous function (Hint: Use the answer to the previous part).

   (c) Let $\|x\|_E$ be the Euclidean norm in $\mathbb{R}^n$. The unit circle in $\mathbb{R}^n$ is defined as the set $\boldsymbol{C} = \{x \in \mathbb{R}^n : \|x\|_E = 1\}$. Show that $\|\cdot\|$ attains a minimum and a maximum in the set $\boldsymbol{C}$ (Hint: Recall the extreme-value theorem).

   (d) Show that there exist m and M such that $m\|x\|_E \leq \|x\| \leq M\|x\|_E$ for all $x \in \mathbb{R}^n$. (Hint: Use your answer to the previous part. Can we express any vector in $\mathbb{R}^n$ in terms of a vector in the unit circle $\boldsymbol{C}$?)

   (e) Conclude that all norms in $\mathbb{R}^n$ are Lipschitz-equivalent to the Euclidean norm, and therefore Lipschitz-equivalent to each other.

2. Let $X = [0, \infty)$. Consider the following function:

$$\overline{d}(x, y) = \begin{cases} \left|1/x - 1/y\right| & \text{if } y \neq 0 \text{ and } x \neq 0. \\ 1/x & \text{if } y = 0,\ x \neq 0. \\ 1/y & \text{if } y \neq 0,\ x = 0. \\ 0 & \text{if } x = 0,\ y = 0 \end{cases}$$

   (a) Show that $\overline{d}$ is a metric over the set X and therefore $(X, \overline{d})$ is a metric space.

   (b) Let $X = Y = [0, \infty)$, and consider the metric spaces $(X, \overline{d})$ and $(Y, |\cdot|)$. Now take the function $f : X \rightarrow Y$ given by: $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0. \\ 0 & \text{if } x = 0 \end{cases}$.

   Is the function f continuous at $x = 0$? (Notice that we use the norm $\overline{d}$ in the domain of f, and the absolute-value norm in the range of f).

3. Find the Taylor series ("infinite-order Taylor approximation") for the following functions, and indicate why the functions can be represented by their respective Taylor series:

$$f_1(x) = \cos x; \quad f_2(x) = \sinh x = \frac{e^x - e^{-x}}{2}; \quad f_3(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

4. Let $f(x) = \frac{\sin x}{x}$, where $x \in (0, a]$ for some $a > 0$. Is f uniformly continuous on the interval $(0, a]$? (Hint: Find $\lim_{x\to 0} f(x)$ and construct an extension $\overline{f}$ of f over the set $[0, a]$. Find out if $\overline{f}$ is continuous).

5. Find the following limits:

$$\lim_{x\to\infty} (1 + 2/x)^x; \quad \lim_{x\to\infty} \left(x + e^x\right)^{1/x}.$$

Is $\lim_{x\to\infty}(1 + 2/x)^x = \lim_{x\to 0}(1 + 2x)^{1/x}$?

6. Let

$$f(x) = \frac{e^x}{1 + e^x}$$

(a) Show that f has a fixed-point in $[0, 1]$, and that this fixed-point is unique. (Hint: Use Mean-Value Theorem).

(b) Using a computer, find the fixed-point (Hint: Use the Contraction Mapping Theorem. Verify that it works with an arbitrary starting value in $[0, 1]$).

7. (a) Consider the correspondence $\Psi : [0, 1] \rightarrow\!\!\rightarrow [0, 1]$ given by:

$$\Psi(x) = \begin{cases} \left\{\frac{1}{2} + \frac{1}{2}x\right\} \cup \left\{\frac{1}{2} - \frac{1}{2}x\right\} & \text{if } x \in [0, 1/2] \\ \left[\frac{1}{4}, \frac{3}{4}\right] & \text{if } x \in (1/2, 1] \end{cases}$$

Draw the graph of Ψ. Is this correspondence upper-hemicontinuous? Is it lower-hemicontinuous? Is it convex-valued? Does it have a fixed-point?

(b) Draw three examples of correspondences $\Psi : [0, 1] \rightarrow\!\!\rightarrow [0, 1]$ that are upper-hemicontinuous but not lower-hemicontinuous. Draw three examples of correspondences $\Psi : [0, 1] \rightarrow\!\!\rightarrow [0, 1]$ that are lower-hemicontinuous but not upper-hemicontinuous.

(c) Draw three examples of correspondences $\Psi : [0, 1] \rightarrow\!\!\rightarrow [0, 1]$ that are nonempty-valued, upper-hemicontinuous but do not have a fixed-point.

8. (a) Suppose f is a continuous function everywhere on $\mathbb{R}$. Define

$$F_1(x) = \int_{x^2-1}^{x^2+1} f(t)dt; \quad F_2(x) = \int_0^{\cos x^3} f(t)dt.$$

Determine if F_1 and F_2 are differentiable on $\mathbb{R}$, and if they are, compute F_1' and F_2'.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a <u>continuous</u> function on $[a, b]$. Show that if $\int_a^b f(x)g(x) = 0$ for every continuous function g on $[a, b]$, then we must have $f(x) = 0$ for all $x \in [a, b]$.

9. (a) Consider the step-function F given by $F(t) = z$ for $t \in [z, z + 1)$, where $z \in \mathbb{Z}$ (the set of real integers). Compute

$$\int_{-1/2}^{5} xdF(x); \quad \int_{-3/2}^{3} x^3dF(x); \quad \int_{-2\pi}^{2\pi} \sin(x^2)dF(x).$$

(b) Let $F(t) = 0$ if $t \in (-\infty, 0)$, $F(t) = t^2$ if $t \in [0, 2)$, $F(t) = t + 5$ if $t \in [2, \infty)$. Let g be a continuous function on the interval $[-4, 4]$. Compute $\int_{-4}^{4} g(x)dF(x)$.

10. Let T : E $\rightarrow$ E a linear operator

   (a) Show that if T is continuous at 0 then T is continuous at all points.

   (b) We say a linear operator is bounded if $\exists\ K > 0$ such that $\| Tx \| \leq K \| x \|$. Also $\| T \|$ is defined as the smallest K. Show that $\| T \| = \sup_{\|z\| \leq 1} \| Tz \|$.

11. Let δ a metric on E.

   (a) Show that $\theta(x, y) = min(1, \delta(x, y))$ is also a metric on E.

   (b) Show that $\gamma(x, y) = \frac{\delta(x,y)}{1+\delta(x,y)}$ is also a metric on E.