Department of Economics

Princeton University

S500, First Half.

Homework 3

1. Orthogonal projections onto col (X)

   (a) Let $k < n$. Take the $n \times k$ matrix X and assume that X has full rank (i.e, $rank(X) = k$). Take any vector $y \in \mathbb{R}^n$. Show that the orthogonal projection of y onto col (X), is a linear transformation. Characterize conditions under which $P_{\text{col}(X)}(y)$ is a diagonalizable linear transformation. (Hint: Recall the definition of a projection matrix.)

   (b) Let

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}; \quad y_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \quad y_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

   Find the projection matrix $P_{\text{col}(X)}$. Find the orthogonal projection of y_1 and y_2 onto col (X). Take an arbitrary $y \in \mathbb{R}^3$. Is the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(y) = P_{\text{col}(X)} y$ a diagonalizable transformation?

2. (a) Find the matrices S_{T_1}, S_{T_2} that represent the following transformations:

$$T_1 : \mathbb{R}^3 \to \mathbb{R}^2;\ T_1\big[(x, y, z)\big] = (x-y, y+2z); \quad T_2 : \mathbb{R}^3 \to \mathbb{R}^3;\ T_2\big[(x, y, z)\big] = (x-y+z, 2y+4z, z)$$

   are S_{T_1} and S_{T_2} diagonalizable by a <u>real-valued</u> matrix?

   (b) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$T\big[(x, y, z)\big] = \big(x - y + z, 3x - 5y + 2z, x - 3y, 1\big)$$

   Find the nullity of T (Hint: Recall the steps described in Example 58 in the class notes).

   (c) Let $T : P_2 \to P_3 = x \cdot p(x)$. Compute the matrix S_T described in Theorem 51 in the classnotes using the basis $B_{P_2} = \{1, x, x^2\}$ and $B_{P_3} = \{1, x, x^2, x^3\}$.

   (d) Take the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^k$ given by $T(x) = Ax$, for some $k \times k$ matrix A. Suppose we take an arbitrary $x_0 \in \mathbb{R}^k$, and we construct the sequence $x_n = T(x_{n-1}) = Ax_{n-1}$, for $n = 1, 2, \ldots$. Suppose the matrix A is diagonalizable. Provide necessary and sufficient conditions for the sequence x_n to converge to a finite vector. (Hint: Recall that if A is diagonalizable, then $A^n = P^{-1}D^nP$. What is D?)

3. More orthogonal projections

   (a) Let $P_n[a,b]$ denote the space of polynomials of order at most n defined on the interval $[a,b]$. Let $p = p(x)$ and $q = q(x)$ be two polynomials in $P_n[a,b]$. Define

   $$\langle p, q \rangle = \int_a^b p(x)q(x)dx$$

   As we claimed in lecture, $\langle p, q \rangle$ is an inner product on $P_n[a,b]$. Let $V = P_1[a,b]$ and $u = x^2 + x^3$. Find the orthogonal projection of u on to V.

   (b) Let P_2 be the space of polynomials of at most order 2. Take any pair of polynomials p q in the space P_2. Let $p(x)$ denote the value of the polynomial p evaluated at x. The same for $q(x)$. We showed in lecture that the following operation defined an inner product between p and q:

   $$\langle p, q \rangle = p(2)q(2) + p(3)q(3) + p(5)q(5)$$

   Let $p = 1 + x^2$. Using this inner product, find the orthogonal projection of p on to P_1, the space of polynomials of at most order 1 (note that P_1 is a subspace of P_2).

4. Consider the function

   $$f(x, y) = \sqrt{|xy|}$$

   Show that at the point $(0,0)$ the following is true: (i) $f(x,y)$ is continuous. (ii) The partial derivatives of f with respect to x and y exist. (iii) No other directional derivatives exist there.

5. Taylor series approximations

   (a) Compute the Taylor series approximation of order two of the function $f_1(x,y) = x^{1/4}y^{3/4}$ at $(1,1)$. Is the Hessian of $f_1(x,y)$ evaluated at $(1,1)$ diagonalizable by a real-valued matrix?

   (b) Compute the Taylor series approximation of order two of the function $f_2(x,y) = x^2 + y^2 + e^{xy} + \sin^2 x$ at $(0,0)$. Is the Hessian of $f_2(x,y)$ evaluated at $(0,0)$ diagonalizable by a real-valued matrix?

   (c) Compute the Taylor series approximation of order two of the function $f_3(x,y,z) = x \sin z - z \sin y$ at $(x,y,z) = (-1, \pi/2, 0)$. Is the Hessian of $f_3(x,y,z)$ evaluated at $(-1, \pi/2, 0)$ diagonalizable by a real-valued matrix?

6. Implicit Function Theorem

(a) Consider the expression $x^2 - xy^3 + y^5 - 17 = 0$. Does this expression define y as an implicit function of x in a neighborhood of the point $(x, y) = (5, 2)$? If so, compute $y'(x)$ there.

(b) Consider the expression $x^2 - 3xy + y^3 - 7 = 0$. Does this expression define y as an implicit function of x in a neighborhood of the point $(x, y) = (4, 3)$? If so, compute $y'(x)$ there.

(c) Consider the expression $x^3 + 3y^2 + 4xz^2 - 3z^2y - 1 = 0$. If $(x, y) = (1, 1)$, can you find z that satisfies this expression? Repeat for the points $(x, y) = (1, 0)$ and $(x, y) = (1/2, 0)$. In each of the three cases, state if z can be expressed as an implicit function of x and y in a neighborhood of each of these points. If it can, compute the corresponding values of $\partial z(x, y)/\partial x$ and $\partial z(x, y)/\partial y$.

(d) Repeat part (c) for the expression $x^2 - y^2 + z^2 = 0$ at the points $(x, y) = (1, 1)$ and $(x, y) = (6, 3)$.

7. Let $e_1, e_2, ..., e_n$ be an orthogonal basis of a vector space V, and $d_1, d_2, ..., d_n$ the corresponding lengths of these vectors.

   (a) Prove that if an m-dimensional subspace $W \in V$ such that the projections of $e_1, e_2, ...e_n$ on W are of equal length exists then

   $$d_i^2(d_1^{-2} + d_2^{-2} + ... + d_n^{-2}) \geq m, \text{for i=1,...,n} \tag{1}$$

   (b) Let $(\beta_i)_1^n$ be any series such that $0 \leq \beta_i \leq 1$ and $\beta_1 + \beta_2 + ... + \beta_n = m.$ Prove that if $m \leq \frac{n}{2}$ then there exists an orthogonal matrix $\parallel y_{ki} \parallel_1^n$ such that $y_{1i}^2 + y_{2i}^2 + ... + y_{mi}^2 = \beta_i, \forall i = 1, ...n$

   (c) Prove that is also true for any m.

   (d) Prove that if (1) satisfies then a subspace W as such defined in (a) exists.

8. Let I = (a,b) be an interval in R. We call C_0^∞(I) the class of indefinitely derivable functions which take value 0 outside I.

   $C_0^\infty(I) = \{f : R \to R : \frac{d^n f}{dx^n}$ exists for all n and f(x) = 0 if x ≤ a or x ≥ b$\}$

   For any function f that is locally integrable on I, i.e. $\int_K f(x)dx < \infty$ for any compact set K in I, we define a linear form T_f such that

   $$T_f : C_0^\infty(I) \to \mathrm{R}$$
   $$\phi \to \int \phi(\mathrm{x})\mathrm{f}(\mathrm{x})\mathrm{dx}$$

   So T_f is also a function whose domain is C_0^∞(I). The class of such distributions is D(I). The derivative is defined as:

   $T_f'(\phi) = T_f(-\phi') = \int -\phi'(\mathrm{x})\mathrm{f}(\mathrm{x})\mathrm{dx}$

   (a) Prove that if $T_f' = 0$ then T_f is a constant.

   (b) Prove that if $T_f' = T_g$ where g is a locally integrable function on I then

   $f(x) = C + \int_c^x g(y)dy$